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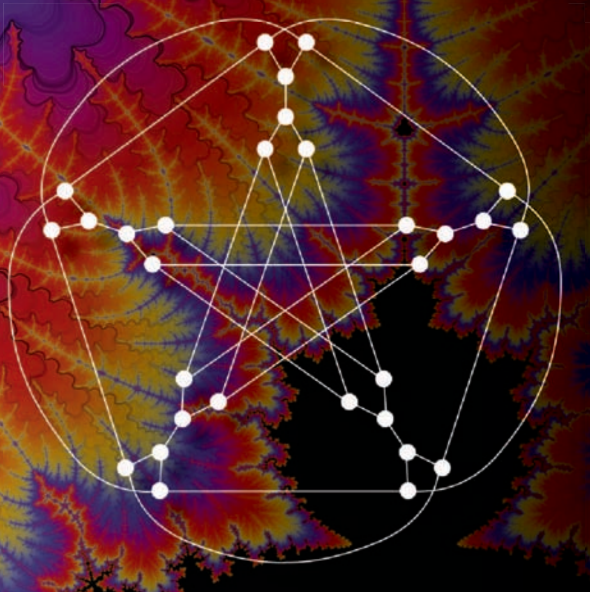

GARY CHARTRAND
LINDA LESNIAK
PING ZHANG

CRC Press
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A CHAPMAN & HALL BOOK

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To

the memory of my mother and father. G. C.

my mother and the memory of my father Stanley. L. L.

my mother and the memory of my father. P. Z.

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Preface to the Sixth Edition

Graph theory is an area of mathematics whose origin dates back to 1736 with the solution of the famous Königsberg Bridge Problem by the eminent Swiss mathematician Leonhard Euler. During the next several decades, topics in graph theory arose primarily through recreational mathematics. The development of graph theory received a substantial boost in 1852 when the young British mathematician Francis Guthrie introduced one of the best known problems in all of mathematics: the Four Color Problem. It wasn't until late in the 19th century, however, when graph theory became a theoretical area of mathematics through the research of the Danish mathematician Julius Petersen. Major progress in graph theory, however, didn't occur until World War II ended. Since then, though, the subject has developed into an area with a fascinating history, numerous interesting problems and applications in many diverse fields. It is the beauty of the subject, however, that has attracted so many to this field.

The goal of this sixth edition is, as with the previous editions, to describe much of the story that is graph theory – through its concepts, its theorems, its applications and its history. The audience for the sixth edition is beginning graduate students and advanced undergraduate students. The primary prerequisite required of students using this book is a knowledge of mathematical proofs. For some topics, an elementary knowledge of linear algebra and group theory is useful. For Chapter 21, an elementary knowledge of probability is needed. Proofs of some of the results that appear in this book have not been supplied because the techniques are beyond the scope of the book or are inordinately lengthy. Nevertheless, these results have been included due to their interest and since they provide a more complete description of what is known on a particular topic.

A one-semester course in graph theory using this textbook can be designed by selecting topics of greatest interest to the instructor and students. There is more than ample material available for a two-semester sequence in graph theory. Our goal has been to prepare a book that is interesting, carefully written, student-friendly and consisting of clear proofs. The sixth edition has been divided into shorter chapters as well as more sections and subsections to make reading and locating material easier for instructors and students. The following major additions have been made to the sixth edition:

- more than 160 new exercises
- several conjectures and open problems
- many new theorems and examples
- new material on graph decompositions
- a proof of the Perfect Graph Theorem

- material on Hamiltonian extension
- a new chapter on the probabilistic method in graph theory and random graphs.

At the end of the book is an index of mathematicians, an index of mathematical terms and an index of symbols. The references list research papers referred to in the book (indicating the page number(s) where the reference occurs) and some useful supplemental references. There is also a section giving hints and solutions to all odd-numbered exercises.

Over the years, there have been some changes in notation that a number of mathematicians now use. When certain notation appears to have been adopted by sufficiently many mathematicians working in graph theory so that this has become the norm, we have adhered to these changes. As with the fifth edition, the following notation is used in the sixth edition:

- a path is now expressed as $P = (v_1, v_2, \ldots, v_k)$ and a cycle as $C = (v_1, v_2, \ldots, v_k, v_1)$;
- the Cartesian product of two graphs G and H is expressed as $G \,\Box\, H$, rather than the previous $G \times H$;
- the union of G and H is expressed by $G + H$, rather than $G \cup H$;
- the join of two graphs G and H is expressed as $G \vee H$, rather than $G + H$.

We are most grateful to Bob Ross, senior editor of CRC Press, who has been a constant source of support and assistance throughout the entire writing process.

Gary Chartrand, Linda Lesniak and Ping Zhang

Chapter 1

Introduction

The theory of graphs is one of the few fields of mathematics with a definite birth date.

It is the subject of graph theory of course that we are about to describe. The statement above was made in 1963 by the mathematician Oystein Ore who will be encountered in Chapter 6. While graph theory was probably Ore's major mathematical area of interest during the latter part of his career, he is also known for his work and interest in number theory (the study of integers) and the history of mathematics.

Although awareness of integers can be traced back for many centuries, geometry has an even longer history. Early geometry concerned distance, lengths, angles, areas and volumes, which were used for surveying, construction and astronomy. While geometry dealt with magnitudes, the German mathematician Gottfried Leibniz introduced another branch of geometry called the geometry of position. This branch of geometry did not deal with measurements and calculations, but rather with the determination of position and its properties. The famous mathematician Leonhard Euler said that it hadn't been determined what kinds of problems could be studied with the aid of the geometry of position but in 1736 he believed that he had found one, which led to the origin of graph theory. It is this event to which Oystein Ore was referring in his quote above. We will visit Euler again, in Chapter 5 as well as in Chapters 10 and 11.

1.1 Graphs

Graphs arise in many different settings. Let's look at three of these.

Example 1.1 Eight students $s_1, s_2, \ldots, s_8$ have been invited to a dinner. Each student knows only some of the other students. The students that each student knows are listed below.

$s_1 :\ s_4, s_5, s_8$	$s_2 :\ s_3, s_4, s_6, s_8$
$s_3 :\ s_2, s_5, s_6, s_7, s_8$	$s_4 :\ s_1, s_2, s_5, s_6, s_7$
$s_5 :\ s_1, s_3, s_4, s_8$	$s_6 :\ s_2, s_3, s_4, s_7$
$s_7 :\ s_3, s_4, s_6$	$s_8 :\ s_1, s_2, s_3, s_5$

In order to determine if these eight students can be seated at a round table where each student sits next to two students he or she knows, it is useful to represent this situation by the diagram shown in Figure 1.1. Each point or small circle in the diagram represents a student and two points are joined by a line segment if the two students know each other. This diagram is referred to as a *graph*.

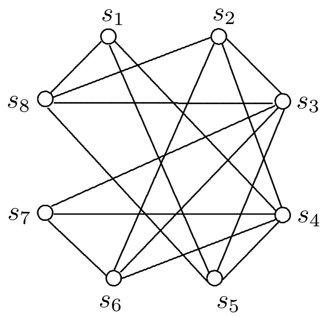


Figure 1.1: The diagram in Example 1.1

A related question is whether the students could be seated at a round table so that each student sits next to two students he or she does not know. ♦

Example 1.2 There are six special locations in a neighborhood park. Twelve trails are to be built between certain pairs of these locations, namely all pairs of locations except $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ (see Figure 1.2(a)). A trail can be straight or curved. Can this be done without any trails crossing? This situation can be represented by the diagram with six points (each point representing a location), where two points are joined by a line segment or a curve if the two points represent locations to be joined by a trail (see Figure 1.2(b)). Once again, this diagram is a graph. ♦

Example 1.3 A chemical company is to ship eight chemicals (denoted by $c_1, c_2, \ldots, c_8$) to a chemistry department in a university. Because some pairs of chemicals should not be shipped in the same container, more than one container needs to be used for this shipment. Each chemical is listed below together with the chemicals that should not be placed in the same container as this chemical.

$c_1 :\ c_2$	$c_2 :\ c_1, c_8$	$c_3 :\ c_5, c_6, c_7$	$c_4 :\ c_5, c_7$
$c_5 :\ c_3, c_4, c_8$	$c_6 :\ c_3, c_7$	$c_7 :\ c_3, c_4, c_6$	$c_8 :\ c_2, c_5$

It would be useful to know the minimum number of containers needed to ship these eight chemicals. This situation can be represented by the diagram

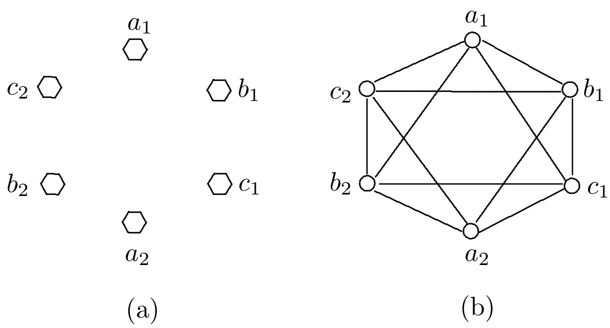


Figure 1.2: Constructing a graph in Example 1.2

in Figure 1.3, whose eight points represent the eight chemicals and where two points are joined by a line segment or curve if these chemicals cannot be shipped in the same container. Here too, this diagram is a graph. ♦

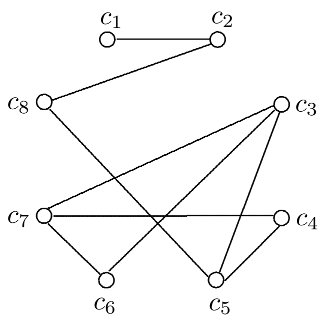


Figure 1.3: The graph in Example 1.3

We now give a formal definition of the term *graph.* A **graph** G is a finite nonempty set V of objects called **vertices** (the singular is **vertex**) together with a possibly empty set E of 2-element subsets of V called **edges**. Vertices are sometimes referred to as **points** or **nodes**, while edges are sometimes called **lines** or **links**. In fact, historically, graphs were referred to as *linkages* by some. Calling these structures graphs was evidently the idea of James Joseph Sylvester (1814–1897), a well-known British mathematician who became the first mathematics professor at Johns Hopkins University in Baltimore and who founded and became editor-in-chief of the first mathematics journal in the United States (the *American Journal of Mathematics*).

To indicate that a graph G has **vertex set** V and **edge set** E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. If $e = uv$ is an edge of G, then e is said to **join** u and v.

As the examples described above indicate, a graph G can be represented by a diagram, where each vertex of G is represented by a point or small circle and an edge joining two vertices is represented by a line segment or curve joining the corresponding points in the diagram. It is customary to refer to such a diagram as the graph G itself. In addition, the points in the diagram are referred to as the vertices of G and the line segments are referred to as the edges of G. For example, the graph G with vertex set $V(G) = \{u, v, w, x, y\}$ and edge set $E(G) = \{uv, uy, vx, vy, wy, xy\}$ is shown in Figure 1.4. Even though the edges vx and wy cross in Figure 1.4, their point of intersection is not a vertex of G.

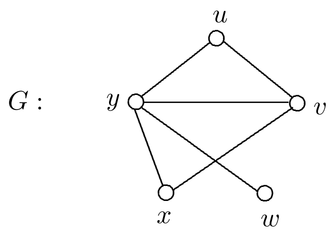


Figure 1.4: A graph

If uv is an edge of G, then u and v are **adjacent vertices**. Two adjacent vertices are referred to as **neighbors** of each other. The set of neighbors of a vertex v is called the **open neighborhood** of v (or simply the **neighborhood** of v) and is denoted by $N_G(v)$, or $N(v)$ if the graph G is understood. The set $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. If uv and vw are distinct edges in G, then uv and vw are **adjacent edges**. The vertex u and the edge uv are said to be **incident** with each other. Similarly, v and uv are incident.

For the graph G of Figure 1.4, the vertices u and v are therefore adjacent in G, while the vertices u and x are not adjacent. The edges uv and vx are adjacent in G, while the edges vx and wy are not adjacent. The vertex v is incident with the edge uv but is not incident with the edge wy.

The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. The order of the graph G of Figure 1.4 is 5 and its size is 6. We typically use n and m for the order and size, respectively, of a graph. A graph of order 1 is called a **trivial graph**. A **nontrivial graph** therefore has two or more vertices. A graph of size 0 is called an **empty graph**. A **nonempty graph** then has one or more edges. In any empty graph, no two vertices are adjacent. At the other extreme is a **complete graph** in which every two distinct vertices are adjacent. The size of a complete graph of order n is $\binom{n}{2} = n(n-1)/2$. Therefore, for every graph G of order n and size m, it follows that $0 \leq m \leq \binom{n}{2}$. The complete graph of order n is denoted by K_n. The complete graphs K_n for $1 \leq n \leq 5$ are shown in Figure 1.5.

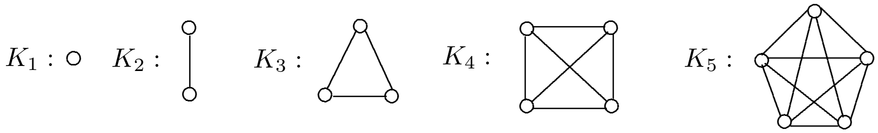


Figure 1.5: Some complete graphs

Two other classes of graphs that are often encountered are the paths and cycles. For an integer $n \geq 1$, the **path** P_n is a graph of order n and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. For an integer $n \geq 3$, the **cycle** C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. The cycle C_n is also referred to as an **n-cycle** and the **3-cycle** is also called a **triangle**. The paths and cycles of order 5 or less are shown in Figure 1.6. Observe that $P_1 = K_1$, $P_2 = K_2$ and $C_3 = K_3$.

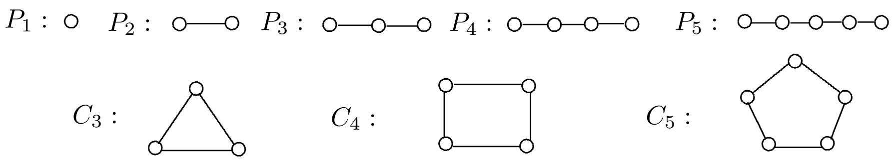


Figure 1.6: Paths and cycles of order 5 or less

1.2 The Degree of a Vertex

The **degree of a vertex** v in a graph G is the number of vertices in G that are adjacent to v. Thus, the degree of v is the number of vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of edges incident with v. The degree of a vertex v is denoted by $\deg_G v$ or, more simply, by $\deg v$ if the graph G under discussion is clear. Hence, $\deg v = |N(v)|$. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. An edge incident with an end-vertex is called a **pendant edge**. The largest degree among the vertices of G is called the **maximum degree** of G and is denoted by $\Delta(G)$. The **minimum degree** of G is denoted by $\delta(G)$. (The symbols Δ and δ are the upper case and lower case Greek letter delta, respectively.) Thus, if v is a vertex of a graph G of order n, then

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n-1.$$

For the graph G of Figure 1.4,

$$\deg w = 1,\ \deg u = \deg x = 2,\ \deg v = 3 \text{ and } \deg y = 4.$$

Thus, $\delta(G) = 1$ and $\Delta(G) = 4$.

The First Theorem of Graph Theory

A well-known theorem in graph theory dealing with the sum of the degrees of the vertices of a graph was observed (indirectly) by Leonhard Euler in a 1736 paper [86] of his that is now considered the first paper ever written on graph theory – even though graphs were never mentioned in the paper. This observation is often referred to as the **First Theorem of Graph Theory**. Some have called Theorem 1.4 the **Handshaking Lemma**, although Euler never used this name.

Theorem 1.4 (**The First Theorem of Graph Theory**) *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices. ■

The sum of the degrees of the vertices of the graph G of Figure 1.4 is 12, which is twice the size 6 of G, as expected from Theorem 1.4. The **average degree** of a graph G of order n and size m is

$$\frac{\sum_{v \in V(G)} \deg v}{n} = \frac{2m}{n}.$$

For example, the average degree of the graph G of Figure 1.4 (having order $n = 5$ and size $m = 6$) is $2m/n = 12/5$. Since the average degree of this graph is strictly between 2 and 3, it follows that G must have a vertex of degree 3 or more and a vertex of degree 2 or less. This graph actually has vertices of degrees 3 and 4 as well as vertices of degrees 1 and 2.

Even and Odd Vertices

A vertex in a graph G is **even** or **odd**, according to whether its degree in G is even or odd. Thus, the graph G of Figure 1.4 has three even vertices and two odd vertices. While a graph can have either an even or an odd number of even vertices, this is not the case for odd vertices.

Corollary 1.5 *Every graph has an even number of odd vertices.*

Proof. Suppose that G is a graph of size m. By Theorem 1.4,

$$\sum_{v \in V(G)} \deg v = 2m,$$

which, of course, is an even number. Since the sum of the degrees of the even vertices of G is even, the sum of the degrees of the odd vertices of G must be even as well, implying that G has an even number of odd vertices. ■

1.3 Isomorphic Graphs

There is only one graph of order 1, two graphs of order 2, four graphs of order 3 and eleven graphs of order 4. All 18 of these graphs are shown in Figure 1.7. This brings up the question of why every two graphs in Figure 1.7 are considered different. In fact, there is the related question of what it means for two graphs to be considered the same. The technical term for this is *isomorphic graphs* (graphs having the same structure).

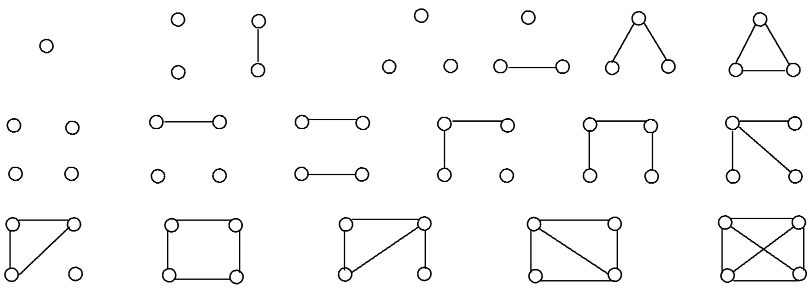


Figure 1.7: The (non-isomorphic) graphs of order 4 or less

Two graphs G and H are **isomorphic** if there exists a bijective function $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is called an **isomorphism** from G to H. If $\phi : V(G) \to V(H)$ is an isomorphism, then the inverse function $\phi^{-1} : V(H) \to V(G)$ is an isomorphism from H to G. If G and H are isomorphic, we write $G \cong H$. If there is no such function ϕ as described above, then G and H are **non-isomorphic graphs** and we write $G \not\cong H$.

The graphs G and H of Figure 1.8 (both of order 7 and size 8) are isomorphic and the function $\phi : V(G) \to V(H)$ defined by

$$\phi(u_1) = v_4,\ \phi(u_2) = v_5,\ \phi(u_3) = v_1,$$

$$\phi(u_4) = v_6,\ \phi(u_5) = v_2,\ \phi(u_6) = v_3,\ \phi(u_7) = v_7$$

is an isomorphism, although there are three other isomorphisms. The graphs F_1 and F_2 of Figure 1.8 (both of order 7 and size 10) are not isomorphic however. An explanation of this will be given shortly.

The graphs G_1 and G_2 in Figure 1.9 are isomorphic, while G_1 and G_3 are not isomorphic. For example, the function $\phi : V(G_1) \to V(G_2)$ defined by

$$\phi(u_1) = v_1,\ \phi(u_2) = v_3,\ \phi(u_3) = v_5,$$

$$\phi(u_4) = v_2,\ \phi(u_5) = v_4,\ \phi(u_6) = v_6$$

is an isomorphism. The graph G_3 of Figure 1.9 contains three mutually adjacent vertices w_1, w_2, w_6. If G_1 and G_3 were isomorphic, then for an isomorphism

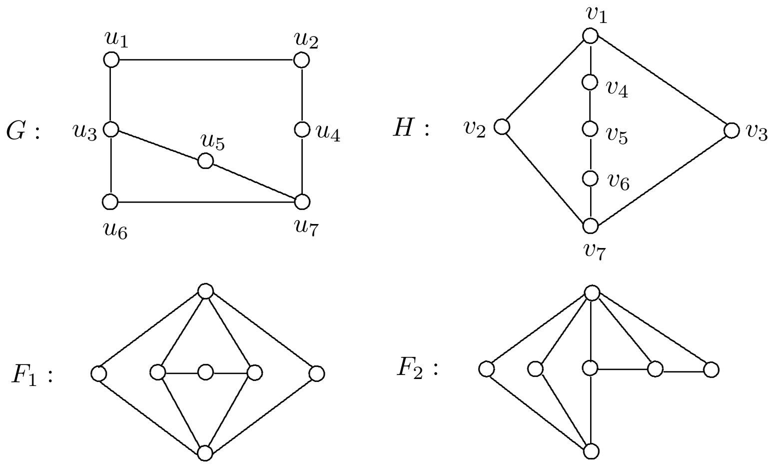


Figure 1.8: Isomorphic and non-isomorphic graphs

$\alpha : V(G_3) \to V(G_1)$, the vertices $\alpha(w_1), \alpha(w_2), \alpha(w_6)$ must also be mutually adjacent in G_1. Since G_1 does not contain three mutually adjacent vertices, there is no isomorphism from G_3 to G_1 and so $G_1 \not\cong G_3$. Furthermore, $G_2 \not\cong G_3$ as well.

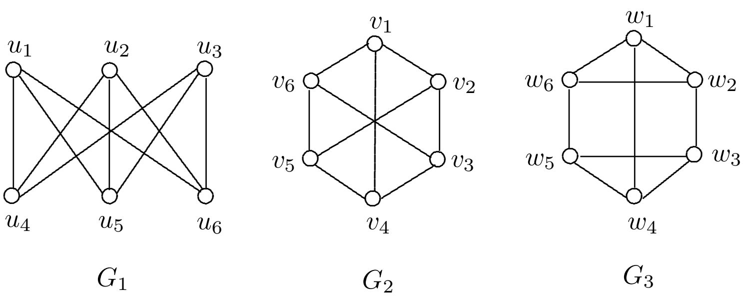


Figure 1.9: Isomorphic and non-isomorphic graphs

Suppose that two graphs G and H are isomorphic. Then there exists an isomorphism $\phi : V(G) \to V(H)$. Since ϕ is a bijective function, $|V(G)| = |V(H)|$. Furthermore, since two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H, it follows that $|E(G)| = |E(H)|$. These facts are summarized below, together with another necessary condition for two graphs to be isomorphic.

Theorem 1.6 *If two graphs G and H are isomorphic, then they have the same order and the same size, and the degrees of the vertices of G are the same as the degrees of the vertices of H.*

Proof. We have already observed that isomorphic graphs have the same order and the same size. Let v be a vertex of G and suppose that $\deg v = k$. Then v is adjacent to k vertices, say $v_1, v_2, \ldots, v_k$. Suppose that v is not adjacent to $u_1, u_2, \ldots, u_\ell$. If $\phi : V(G) \to V(H)$ is an isomorphism, then $\phi(v)$ is adjacent to $\phi(v_1), \phi(v_2), \ldots, \phi(v_k)$ while $\phi(v)$ is not adjacent to $\phi(u_1), \phi(u_2), \ldots, \phi(u_\ell)$. Hence, $\phi(v)$ has degree k in H. ∎

The proof of Theorem 1.6 also shows that every isomorphism from a graph G to a graph H maps every vertex of G to a vertex of the same degree in H.

It is therefore a consequence of Theorem 1.6 that if G and H are two graphs such that (1) the orders of G and H are different, or (2) the sizes of G and H are different or (3) the degrees of the vertices of G and those of the vertices of H are different, then G and H are not isomorphic. Since the graph F_2 in Figure 1.8 contains a vertex of degree 5 and no vertex of F_1 has degree 5, it follows by Theorem 1.6 that $F_1 \not\cong F_2$.

The conditions described in Theorem 1.6 are strictly necessary for two graphs to be isomorphic – they are not sufficient. Indeed, the graphs G_1 and G_3 of Figure 1.9 have the same order, the same size and the degrees of the vertices of G_1 and G_3 are the same; yet G_1 and G_3 are not isomorphic.

Next, consider the four graphs H_1, H_2, H_3 and H_4 shown in Figure 1.10. The graphs H_1 and H_2 have order 7, size 7 and the degrees of the vertices of these two graphs are the same. Furthermore, each graph contains a single triangle. Nevertheless, $H_1 \not\cong H_2$, for suppose that there is an isomorphism ϕ from H_1 to H_2. Since each graph has only one vertex of degree 3 and one vertex of degree 4, $\phi(u_1) = v_2$ and $\phi(y_1) = x_2$. Since v_1 is adjacent to u_1 and y_1, it follows that $\phi(v_1)$ is adjacent to v_2 and x_2. However, H_2 contains no vertex adjacent to v_2 and x_2. Thus, $H_1 \not\cong H_2$.

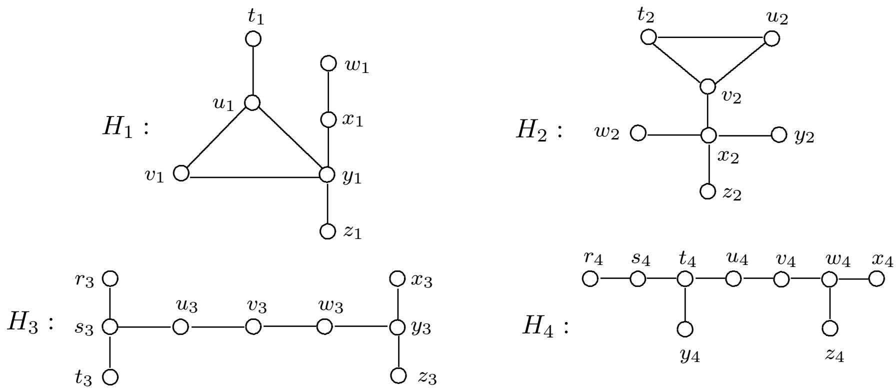


Figure 1.10: Non-isomorphic graphs

The graphs H_3 and H_4 of Figure 1.10 have order 9, size 8 and the degrees of the vertices of these two graphs are the same; yet these two graphs as well are *not* isomorphic. Suppose that there is an isomorphism ϕ from H_3 to H_4. Consider the vertex v_3. Since $\deg_{H_3} v_3 = 2$, it follows that $\phi(v_3)$ has degree 2 in H_4. Since v_3 is adjacent to u_3 and w_3, we must have $\phi(v_3)$ adjacent to $\phi(u_3)$ and $\phi(w_3)$. Since $\deg_{H_3} u_3 = \deg_{H_3} w_3 = 2$, it follows that $\phi(u_3)$ and $\phi(w_3)$ have degree 2 in H_4. But no vertex of degree 2 in H_4 is adjacent to two vertices of degree 2. Thus, $H_3 \not\cong H_4$.

Subgraphs

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. If H is a subgraph of G, then G is a **supergraph** of H. If $V(H) = V(G)$, then H is a **spanning subgraph** of G. If H is a subgraph of a graph G where $H \not\cong G$, then H is a **proper subgraph** of G. Therefore, if H is a proper subgraph of G, then either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$.

Figure 1.11 shows six graphs, namely G and the graphs G_i for $i = 1, 2, \ldots, 5$. All six of these graphs are proper subgraphs of G, except G itself and G_1. Although G is a subgraph of itself, it is not a proper subgraph of G. The graph G_1 contains the edge uz, which is not an edge of G and so G_1 is not even a subgraph of G. The graph G_3 is a spanning subgraph of G since $V(G_3) = V(G)$.

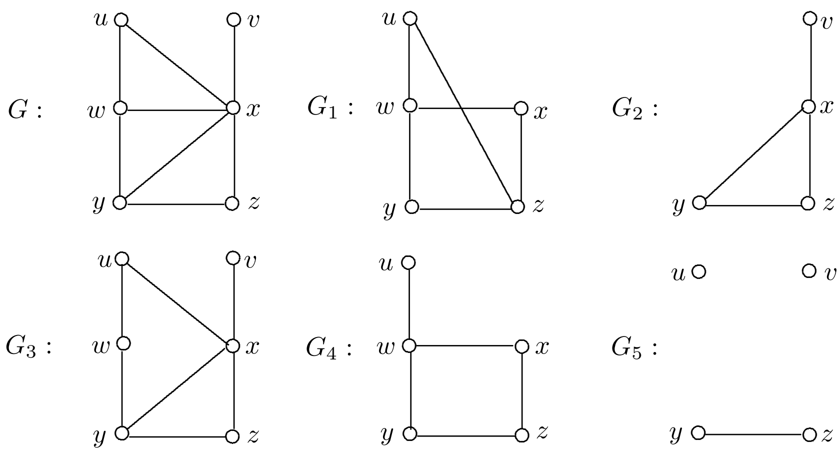


Figure 1.11: Graphs and subgraphs

Induced Subgraphs

For a nonempty subset S of $V(G)$, the **subgraph** $G[S]$ **of** G **induced by** S

has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an **induced subgraph** if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. Thus $G[V(G)] = G$. For a nonempty set X of edges of a graph G, the **subgraph** $G[X]$ **induced by** X has X as its edge set and a vertex v belongs to $G[X]$ if v is incident with at least one edge in X. A subgraph H of G is **edge-induced** if there is a nonempty subset X of $E(G)$ such that $H = G[X]$. Thus, $G[E(G)] = G$ if and only if G has no isolated vertices.

Once again, consider the graphs shown in Figure 1.11. Since $xy \in E(G)$ but $xy \notin E(G_4)$, the subgraph G_4 is not an induced subgraph of G. On the other hand, the subgraphs G_2 and G_5 are both induced subgraphs of G. Indeed, for $S_1 = \{v, x, y, z\}$ and $S_2 = \{u, v, y, z\}$, $G_2 = G[S_1]$ and $G_5 = G[S_2]$. The subgraph G_4 of G is edge-induced; in fact, $G_4 = G[X]$, where $X = \{uw, wx, wy, xz, yz\}$.

For a vertex v and an edge e in a nonempty graph $G = (V, E)$, the subgraph $G - v$, obtained by deleting v from G, is the induced subgraph $G[V - \{v\}]$ of G and the subgraph $G - e$, obtained by deleting e from G, is the spanning subgraph of G with edge set $E - \{e\}$. More generally, for a proper subset U of V, the graph $G - U$ is the induced subgraph $G[V - U]$ of G. For a subset X of E, the graph $G - X$ is the spanning subgraph of G with edge set $E - X$. If u and v are distinct nonadjacent vertices of G, then $G + uv$ is the graph with $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$. Thus, G is a spanning subgraph of $G + uv$. For the graph G of Figure 1.12, the set $U = \{t, x\}$ of vertices and the set $X = \{tw, ux, vx\}$ of edges, the subgraphs $G - u$, $G - wx$, $G - U$ and $G - X$ of G are also shown in that figure, as is the graph $G + uv$.

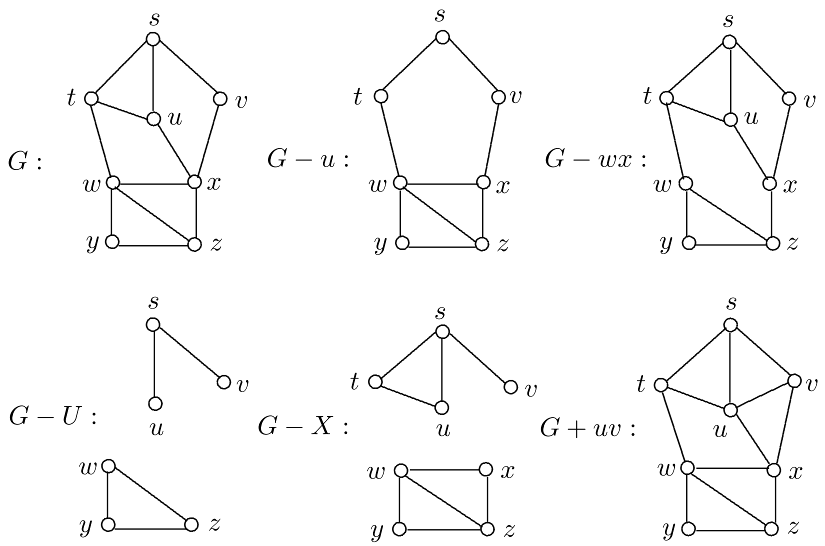


Figure 1.12: Deleting vertices and edges from and adding edges to a graph

If every two edges e_1 and e_2 of a graph G have the property that $G - e_1 \cong G - e_2$, then we write $G - e$ for the deletion of any edge from G. Hence, for $n \geq 2$, $K_n - e$ denotes the graph obtained by deleting any edge from K_n. If $G + uv \cong G + xy$ for any two pairs $\{u, v\}$ and $\{x, y\}$ of nonadjacent vertices of G, then we write $G + e$ for the addition of any edge to G. In particular, $C_4 + e = K_4 - e$.

1.4 Regular Graphs

There are certain classes of graphs that occur so often that they deserve special mention and, in some cases, special notation. We describe some of the most prominent of these now.

A graph G is **regular** if the vertices of G have the same degree and is **regular of degree** r if this degree is r. Such graphs are also called r**-regular**. The complete graph of order n is therefore a regular graph of degree $n - 1$ and every cycle is 2-regular. In Figure 1.13 are shown all (non-isomorphic) regular graphs of orders 4 and 5, including the cycles C_4 and C_5 and the complete graphs K_4 and K_5. Since no graph has an odd number of odd vertices, there is no 1-regular or 3-regular graph of order 5. Indeed, the pairs r, n of integers for which there exist r-regular graphs of order n are predictable.

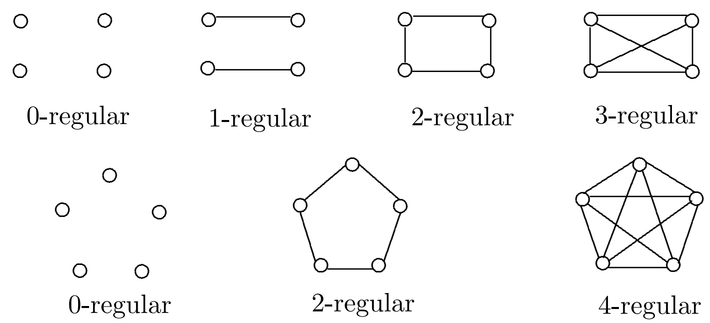


Figure 1.13: The regular graphs of orders 4 and 5

Theorem 1.7 *For integers r and n, there exists an r-regular graph of order n if and only if $0 \leq r \leq n - 1$ and r and n are not both odd.*

Proof. That the conditions are necessary is an immediate consequence of Corollary 1.5 and the fact that $0 \leq \deg v \leq n - 1$ for every vertex v in a graph of order n. For the converse, suppose that r and n are integers where $0 \leq r \leq n - 1$ and r and n are not both odd. Assume first that r is even. If $r = 0$, then the graph of order n consisting of n isolated vertices is r-regular. So we may assume that $r = 2k$ for some integer $k \geq 1$ and $n \geq 2k + 1$. Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that v_i $(1 \leq i \leq n)$ is adjacent to $v_{i\pm1}, v_{i\pm2}, \ldots, v_{i\pm k}$ (subscripts expressed modulo n). The resulting graph G is then an r-regular graph of order n. If r is odd, then $n = 2\ell$ is even. Hence $r = 2k + 1$ for some integer k with $0 \leq k \leq \ell - 1$. For the graph G in this case, the vertex v_i is adjacent to the $2k$ vertices described above and adjacent as well to $v_{i+\ell}$. Again, G is an r-regular graph of order n. ■

The 4-regular and 5-regular graphs of order 10 constructed in the proof of Theorem 1.7 are shown in Figure 1.14.

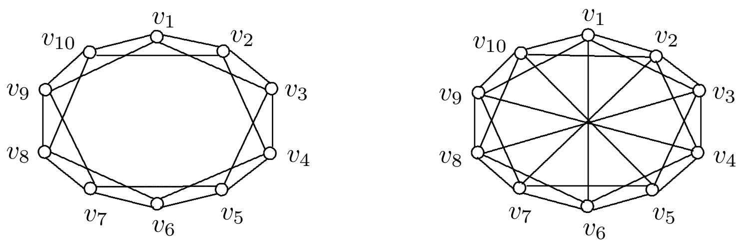


Figure 1.14: 4-regular and 5-regular graphs of order 10

The Petersen Graph

A 3-regular graph is also called a **cubic graph**. The graphs of Figure 1.9 are cubic as is the complete graph K_4. One of the best known cubic graphs is the **Petersen graph**, named for the Danish mathematician Julius Petersen whose 1891 research on regular graphs [186] is often credited as the beginning of the study of graphs as a theoretical subject. In fact, the Petersen graph is one of the best known graphs. Three different drawings of the Petersen graph are shown in Figure 1.15. We will have many occasions to encounter this graph.

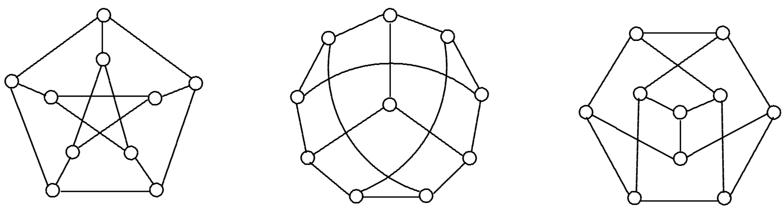


Figure 1.15: Three drawings of the Petersen graph

1.5 Bipartite Graphs

Another class of graphs that we often encounter are the bipartite graphs. A graph G is **bipartite** if $V(G)$ can be partitioned into two sets U and W (called

partite sets) so that every edge of G joins a vertex of U and a vertex of W. The graph G in Figure 1.16(a) is bipartite with partite sets $\{v_1, v_3, v_5, v_7\}$ and $\{v_2, v_4, v_6\}$. This graph is redrawn in Figure 1.16(b) to see more clearly that it is bipartite. If G is an r-regular bipartite graph, $r \geq 1$, with partite sets U and W, then $|U| = |W|$. This follows since the size of G is $r|U| = r|W|$. For two nonempty sets X and Y of vertices in a graph G, the set

$$[X, Y] = \{xy : x \in X, y \in Y\}$$

consists of those edges joining a vertex of X and a vertex of Y. Thus, if G is a bipartite graph with partite sets U and W, then $[U, W] = E(G)$.

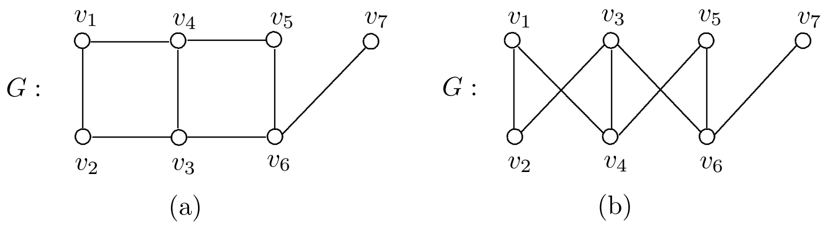


Figure 1.16: A bipartite graph

A graph G is a **complete bipartite graph** if $V(G)$ can be partitioned into two sets U and W (called **partite sets** again) so that uw is an edge of G if and only if $u \in U$ and $w \in W$. If $|U| = s$ and $|W| = t$, then this complete bipartite graph has order $s+t$ and size st and is denoted by $K_{s,t}$ (or $K_{t,s}$). The complete bipartite graph $K_{1,t}$ is called a **star**. The complete bipartite graphs $K_{1,3}$, $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$ are shown in Figure 1.17. Observe that $K_{2,2} = C_4$. The star $K_{1,3}$ is sometimes referred to as a **claw**.

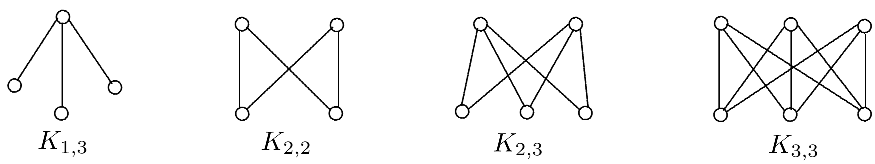


Figure 1.17: Complete bipartite graphs

Since the size of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$, there are bipartite graphs of order n and size $\lfloor n^2/4 \rfloor$. No bipartite graph of order n can have a larger size however.

Theorem 1.8 *The size of every bipartite graph of order n is at most $\lfloor n^2/4 \rfloor$.*

Proof. Let G be bipartite graph of order n with partite sets U and W. Then $|U| = x$ and $|W| = n - x$ for some integer x with $1 \leq x \leq n-1$. Hence the size of G is at most $x(n-x)$.

Since $(n-2x)^2 \geq 0$, it follows that

$$n^2 \geq 4nx - 4x^2 = 4x(n-x)$$

and so $x(n-x) \leq n^2/4$. Since $x(n-x)$ is an integer, $x(n-x) \leq \lfloor n^2/4 \rfloor$. ∎

No bipartite graph G can contain a triangle H, for otherwise, at least two vertices u and v of H must belong to the same partite set and so u and v are not adjacent. Hence if G is a graph of order $n \geq 3$ and size $m \leq \lfloor n^2/4 \rfloor$, then G need not contain a triangle. However, in 1907 the Dutch mathematician Willem Mantel [161] showed that any graph of order n with a larger size must contain a triangle.

Theorem 1.9 *Every graph of order $n \geq 3$ and size $m > \lfloor n^2/4 \rfloor$ contains a triangle.*

Proof. First, observe that the result is true for $n = 3$ and $n = 4$. Suppose, however, that the statement is false. Then there is a smallest integer $n \geq 5$ and a graph G of order n and size $m > \lfloor n^2/4 \rfloor$ not containing a triangle. Let uv be an edge of G. Since G contains no triangle, there is no vertex in G adjacent to both u and v. Hence, $(\deg u - 1) + (\deg v - 1) \leq n - 2$ and so $\deg u + \deg v \leq n$. Let $G' = G - u - v$. Since G' is a subgraph of G, it follows that G' also does not contain a triangle. Furthermore, G' has order $n-2$ and size

$$m' = m - (\deg u + \deg v) + 1 > \lfloor n^2/4 \rfloor - n + 1.$$

Thus,

$$m' > \frac{n^2}{4} - n + 1 = \frac{n^2 - 4n + 4}{4} = \frac{(n-2)^2}{4}.$$

From the defining property of the graph G, it follows that G' contains a triangle, producing a contradiction. ∎

Therefore, from Theorems 1.8 and 1.9, it follows that every graph of order $n \geq 3$ and size $m > \lfloor n^2/4 \rfloor$ not only fails to be bipartite, it must, in fact, contain a triangle.

Complete Multipartite Graphs

Bipartite graphs belong to a more general class of graphs. For an integer $k \geq 1$, a graph G is a **k-partite graph** if $V(G)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ (again called **partite sets**) such that every edge of G joins vertices in two different partite sets. A 1-partite graph is then an empty graph and a 2-partite graph is bipartite. A **complete k-partite graph** G is a k-partite graph with the property that two vertices are adjacent in G if and only if the vertices belong to different partite sets. If $|V_i| = n_i$ for $1 \leq i \leq k$, then G is denoted by $K_{n_1,n_2,\ldots,n_k}$ (the order in which the numbers $n_1, n_2, \ldots, n_k$

are written is not important). If $n_i = 1$ for all i $(1 \leq i \leq k)$, then G is the complete graph K_k. A **complete multipartite graph** is a complete k-partite graph for some integer $k \geq 2$. Some complete multipartite graphs are shown in Figure 1.18

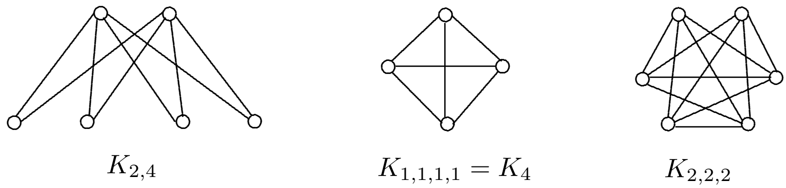


Figure 1.18: Some complete multipartite graphs

1.6 Operations on Graphs

There are many ways of producing a new graph from one or more given graphs. The most common of these is the complement of a graph.

The Complement of a Graph

The **complement** $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. Any isomorphism from a graph G to a graph H is also an isomorphism from $\overline{G}$ to $\overline{H}$. Consequently, $\overline{G} \cong \overline{H}$ if and only if $G \cong H$. If G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. A graph G and its complement are shown in Figure 1.19. The complement $\overline{K}_n$ of the complete graph K_n is the empty graph of order n.

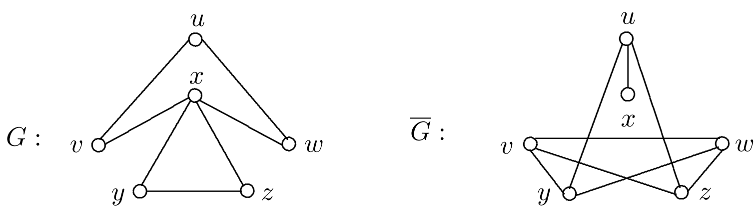


Figure 1.19: A graph and its complement

A graph G is **self-complementary** if G is isomorphic to $\overline{G}$. Certainly, if G is a self-complementary graph of order n, then its size is $m = \binom{n}{2}/2 = n(n-1)/4$. Since only one of n and $n-1$ is even, either $4 \mid n$ or $4 \mid (n-1)$; that is, if G is a self-complementary graph of order n, then either $n \equiv 0 \pmod 4$ or

$n \equiv 1 \pmod 4$. (See Exercise 29.) The self-complementary graphs of order 5 or less are shown in Figure 1.20.

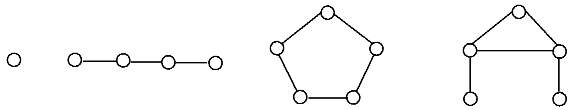


Figure 1.20: The self-complementary graphs of order 5 or less

The Union and Join of Graphs

We next describe some common binary operations defined on graphs. This discussion introduces notation that will be especially useful in giving examples. Over the years, different authors have used different notation for the operations we are about to describe. In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets.

The **union** $G = G_1 + G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. The union $G + G$ of two disjoint copies of G is denoted by $2G$. Indeed, if a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$. The graph $2K_1 + 3K_2 + K_{1,3}$ is shown in Figure 1.21(a). The **join** $G = G_1 \vee G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

Using the join operation, we see that $\overline{K}_s \vee \overline{K}_t = K_{s,t}$. Another illustration is given in Figure 1.21(b).

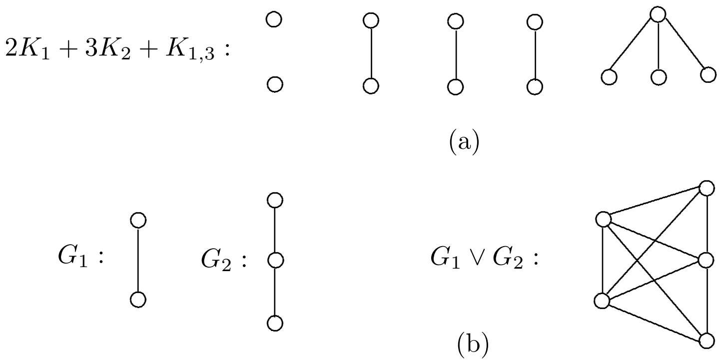


Figure 1.21: The union and join of graphs

The Cartesian Product of Graphs

The **Cartesian product** G of two graphs G_1 and G_2, commonly denoted by $G_1 \square G_2$ or $G_1 \times G_2$, has vertex set

$$V(G) = V(G_1) \times V(G_2),$$

where two distinct vertices (u, v) and (x, y) of $G_1 \square G_2$ are adjacent if either

$$(1)\ u = x \text{ and } vy \in E(G_2) \text{ or } (2)\ v = y \text{ and } ux \in E(G_1).$$

A convenient way of drawing $G_1 \square G_2$ is to first place a copy of G_2 at each vertex of G_1 (see Figure 1.22(b)) and then join corresponding vertices of G_2 in those copies of G_2 placed at adjacent vertices of G_1 (see Figure 1.22(c)). Equivalently, $G_1 \square G_2$ can be constructed by placing a copy of G_1 at each vertex of G_2 and adding the appropriate edges. As expected, $G_1 \square G_2 \cong G_2 \square G_1$ for all graphs G_1 and G_2.

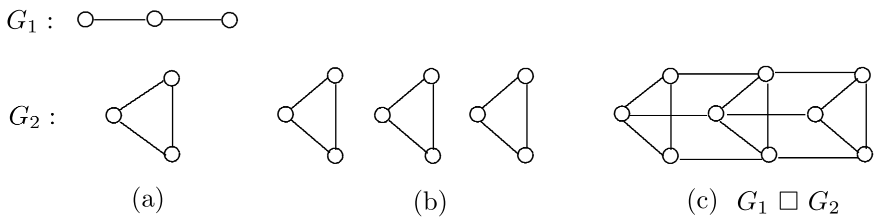


Figure 1.22: The Cartesian product of two graphs

Hypercubes

An important class of graphs is defined in terms of Cartesian products. The n**-cube** Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \square K_2$ of Q_{n-1} and K_2. The n-cube can also be defined as that graph whose vertex set is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ or $a_1 a_2 \cdots a_n$ where a_i is 0 or 1 for $1 \leq i \leq n$ (commonly called n**-bit strings**), such that two vertices are adjacent if and only if the corresponding ordered n-tuples differ at precisely one coordinate. The graph Q_n is an n-regular graph of order 2^n. The n-cubes for $n = 1, 2, 3$ are shown in Figure 1.23, where their vertices are labeled by n-bit strings. The graphs Q_n are also called **hypercubes**.

1.7 Degree Sequences

We saw in the First Theorem of Graph Theory (Theorem 1.4) that the sum of the degrees of the vertices of a graph G is twice the size of G and in Corollary 1.5

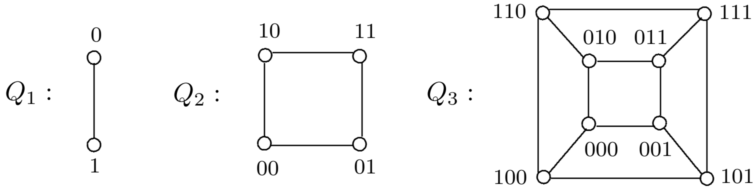


Figure 1.23: The n-cubes for $n = 1, 2, 3$

that G must have an even number of odd vertices. We have also described conditions under which a regular graph of order n can exist. We now consider the degrees of the vertices of a graph in more detail.

A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is called a **degree sequence** of a graph G of order n if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $\deg v_i = d_i$ for $1 \leq i \leq n$. For example, a degree sequence of the graph G of Figure 1.24 is $4, 3, 2, 2, 1$ (or $1, 2, 2, 3, 4$ or $2, 1, 4, 2, 3$, etc.). We commonly write the degree sequence of a graph as a nonincreasing sequence.

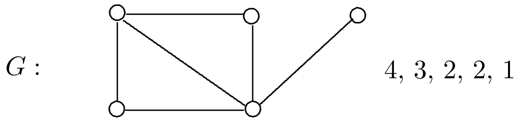


Figure 1.24: A degree sequence of a graph

A finite sequence s of nonnegative integers is a **graphical sequence** if s is a degree sequence of some graph. Thus, $4, 3, 2, 2, 1$ is graphical. There are some obvious necessary conditions for a sequence $s : d_1, d_2, \ldots, d_n$ of n nonnegative integers to be graphical. While the conditions that $d_i \leq n - 1$ for all i $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} \deg v_i$ is even are necessary for s to be graphical, they are not sufficient. For example, the sequence $3, 3, 3, 1$ satisfies both conditions but it is not graphical, for if three vertices of a graph of order 4 have degree 3 then the remaining vertex must have degree 3 as well.

It is not all that unusual for a graphical sequence to be the degree sequence of more than one graph. For example, the graphical sequence $3, 2, 2, 2, 1$ is the degree sequence of the two (non-isomorphic) graphs in Figure 1.25. On the other hand, each of the 18 graphs in Figure 1.7 has a degree sequence possessed by no other graph.

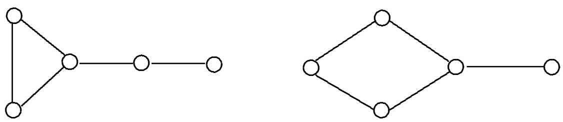


Figure 1.25: Two graphs with the same degree sequence

2-Switches

While two graphs with the same degree sequence need not be isomorphic, each can be obtained from the other by a sequence of edge shifts where at each step, two nonadjacent edges in some graph F are deleted and two nonadjacent edges in $\overline{F}$ are added to F such that the four edges involved are incident with the same four vertices.

Let H be a graph containing four distinct vertices u, v, w and x such that $uv, wx \in E(H)$ and $uw, vx \notin E(H)$. The process of deleting the edges uv and wx from H and adding uw and vx to H is referred to as a **2-switch** in H (see Figure 1.26, where a dashed line means no edge). This produces a new graph G having the same degree sequence as H. Of course, if G can be produced from H by a 2-switch, then H can be obtained from G by a 2-switch.

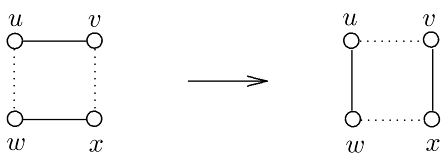


Figure 1.26: A 2-switch in a graph

Theorem 1.10 *Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{G}_s$ be the set of all graphs F with degree sequence s such that $V(F) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Then every graph $H \in \mathcal{G}_s$ can be transformed into a graph $G \in \mathcal{G}_s$ by a sequence of 2-switches such that $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$.*

Proof. Suppose that this statement is false. Let $W = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Among all graphs into which H can be transformed, let G be one for which the sum of the subscripts of the vertices in $N_G(v_1)$ is minimum. Since $N_G(v_1) \neq W$, the vertex v_1 is adjacent to a vertex v_k and is not adjacent to a vertex v_j with $j < k$ and so $d_j \geq d_k$. Consequently, there is a vertex v_ℓ such that $v_j v_\ell \in E(G)$ and $v_k v_\ell \notin E(G)$. Replacing the edges $v_1 v_k$ and $v_j v_\ell$ by $v_1 v_j$ and $v_k v_\ell$ is a 2-switch in G that produces a graph $G_1 \in \mathcal{G}_s$ for which the sum of the subscripts of the vertices in $N_{G_1}(v_1)$ is less than that of $N_G(v_1)$. Consequently, H can be transformed into G_1 and so this is a contradiction. ■

One consequence of Theorem 1.10 is that every two graphs with the same degree sequence are related in terms of 2-switches. The following theorem appeared in the book *Graphs and Hypergraphs* by Claude Berge [25].

Theorem 1.11 *If G and H are two graphs with the same degree sequence, then H can be transformed into G by a (possibly empty) sequence of 2-switches.*

Proof. We proceed by induction on the order n of G and H. If $n \leq 4$, then the result is immediate. For a given integer $n \geq 5$, assume that every two graphs of order $n-1$ with the same degree sequence can be transformed into each other by a sequence of 2-switches. Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{G}_s$ be the set of all graphs F with degree sequence s such that $V(F) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Let $W = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Let G and H be two graphs in $\mathcal{G}_s$. By Theorem 1.10, G can be transformed into a graph $G_1 \in \mathcal{G}_s$ such that $N_{G_1}(v_1) = W$ and H can be transformed into a graph $H_1 \in \mathcal{G}_s$ such that $N_{H_1}(v_1) = W$. Since $G_1 - v_1$ and $H_1 - v_1$ are two graphs of order $n-1$ with the same degree sequence, it follows by the induction hypothesis that $G_1 - v_1$ can be transformed into $H_1 - v_1$ by a sequence of 2-switches. Hence G_1 can be transformed into H_1 by a sequence of 2-switches and so G can be transformed into H by a sequence of 2-switches. ∎

The Havel–Hakimi Theorem

There are necessary and sufficient conditions for a finite sequence of nonnegative integers to be graphical. One of these is due to Václav Havel [124] and S. Louis Hakimi [117] and is a consequence of Theorem 1.10. This result is often referred to as the Havel-Hakimi Theorem, despite the fact that Havel and Hakimi gave independent proofs and wrote separate papers containing this theorem.

Theorem 1.12 **(Havel–Hakimi Theorem)** *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and $\Delta \geq 1$ is graphical if and only if the sequence*

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{\Delta+1} - 1, d_{\Delta+2}, \ldots, d_n$$

is graphical.

Proof. First, assume that s_1 is graphical. Then there exists a graph G_1 of order $n-1$ such that s_1 is a degree sequence of G_1. Thus, the vertices of G_1 can be labeled as $v_2, v_3, \ldots, v_n$ so that

$$\deg_{G_1} v_i = \begin{cases} d_i - 1 & \text{if } 2 \leq i \leq \Delta + 1 \\ d_i & \text{if } \Delta + 2 \leq i \leq n. \end{cases}$$

A new graph G can now be constructed by adding a new vertex v_1 to G_1 together with the Δ edges $v_1 v_i$ for $2 \leq i \leq \Delta + 1$. Since $\deg_G v_i = d_i$ for $1 \leq i \leq n$, it follows that $s : \Delta = d_1, d_2, \ldots, d_n$ is a degree sequence of G and so s is graphical.

Conversely, let s be a graphical sequence. By Theorem 1.10, there exists a graph G of order n having degree sequence s with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\deg v_i = d_i$ for all i $(1 \leq i \leq n)$ where $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Then $G - v_1$ has degree sequence s_1. ∎

Theorem 1.12 actually provides us with an algorithm for determining whether a given finite sequence of nonnegative integers is graphical. If, upon repeated application of Theorem 1.12, we arrive at a sequence every term of which is 0, then the original sequence is graphical. On the other hand, if we arrive at a sequence containing a negative integer, then the given sequence is not graphical.

We now illustrate Theorem 1.12 with the sequence

$$s : 5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1.$$

After one application of Theorem 1.12 (deleting 5 from s and subtracting 1 from the next five terms), we obtain

$$s_1' : 2, 2, 2, 2, 1, 2, 2, 1, 1, 1.$$

Reordering this sequence, we have

$$s_1 : 2, 2, 2, 2, 2, 2, 1, 1, 1, 1.$$

Continuing in this manner, we get

$$\begin{aligned}
&s_2' : 1, 1, 2, 2, 2, 1, 1, 1, 1\\
&s_2 : 2, 2, 2, 1, 1, 1, 1, 1, 1\\
&s_3' = s_3 : 1, 1, 1, 1, 1, 1, 1, 1\\
&s_4' : 0, 1, 1, 1, 1, 1, 1\\
&s_4 : 1, 1, 1, 1, 1, 1, 0\\
&s_5' : 0, 1, 1, 1, 1, 0\\
&s_5 : 1, 1, 1, 1, 0, 0\\
&s_6' : 0, 1, 1, 0, 0\\
&s_6 : 1, 1, 0, 0, 0\\
&s_7' = s_7 : 0, 0, 0, 0.
\end{aligned}$$

Therefore, s is graphical. Of course, if we observe that some sequence prior to s_7 is graphical, then we can conclude by Theorem 1.12 that s is graphical. For example, the sequence s_3 is clearly graphical since it is the degree sequence of the graph $G_3 = 4K_2$ in Figure 1.27. By Theorem 1.12, each of the sequences s_2, s_1 and s is also graphical. To construct a graph with degree sequence s_2, we proceed in reverse from $s_3' = s_3$ to s_2, observing that a vertex should be added to G_3 so that it is adjacent to two vertices of degree 1. We thus obtain a graph G_2 with degree sequence s_2 (or s_2'). Proceeding from s_2' to s_1, we again add a new vertex joining it to two vertices of degree 1 in G_2. This gives a graph G_1 with degree sequence s_1 (or s_1'). Finally, we obtain a graph G with degree sequence s by considering s_1'; that is, we add a new vertex to G_1 joining it to vertices of degrees $2, 2, 2, 2, 1$. This procedure is illustrated in Figure 1.27.

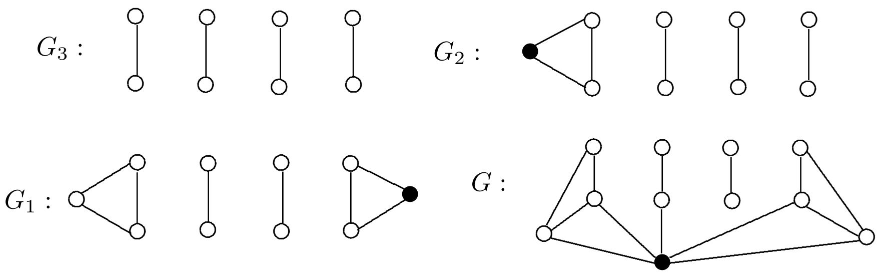


Figure 1.27: Construction of a graph G with a given degree sequence

It should be pointed out that the graph G in Figure 1.27 is not the only graph with degree sequence s. However, there are graphs that cannot be produced by the method used to construct the graph G in Figure 1.27. For example, the graph H of Figure 1.28 is such a graph.

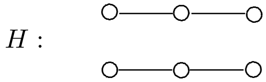


Figure 1.28: A graph that cannot be constructed by the method following Theorem 1.12

The Erdős–Gallai Theorem

Suppose that $s : d_1, d_2, \ldots, d_n$ is a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then there exists a graph G of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\deg v_i = d_i$ for all i $(1 \leq i \leq n)$. Of course, the sum $\sum_{i=1}^{n} d_i$ is even. Let k be an integer with $1 \leq k \leq n-1$. Suppose that $V_1 = \{v_1, v_2, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. Now the sum $\sum_{i=1}^{k} d_i$ counts every edge in $G[V_1]$ twice and counts each edge in $[V_1, V_2]$ once. The size of $G[V_1]$ is at most $\binom{k}{2} = k(k-1)/2$, while for each i $(k+1 \leq i \leq n)$ the number of edges joining v_i and V_1 is at most $\min\{k, d_i\}$. Thus,

$$\sum_{i=1}^{k} d_i \leq 2\binom{k}{2} + \sum_{i=k+1}^{n} \min\{k, d_i\} = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \qquad (1.1)$$

Hence, for every graphical sequence s, we must have both that $\sum_{i=1}^{n} d_i$ is even and (1.1) is satisfied for every integer k with $1 \leq k \leq n-1$. These conditions are not only necessary for a sequence of nonnegative integers to be graphical, they are sufficient as well. Since the proof of the sufficiency is technical, we omit this. This result was established by Paul Erdős and Tibor Gallai [81], who were introduced to graph theory as youngsters by Dénes König, author of the first book on graph theory. In fact, Gallai was König's only doctoral student.

Theorem 1.13 (**Erdős–Gallai Theorem**) *A sequence* $s : d_1, d_2, \ldots, d_n$ $(n \geq 2)$ *of nonnegative integers with* $d_1 \geq d_2 \geq \cdots \geq d_n$ *is graphical if and only if* $\sum_{i=1}^{n} d_i$ *is even and for each integer* k *with* $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Irregular Graphs

According to Theorem 1.7, there is an r-regular graph of order n if and only if $0 \leq r \leq n-1$ and rn is even. At the other extreme are nontrivial graphs, no two vertices of which have the same degree. A nontrivial graph G is **irregular** if $\deg u \neq \deg v$ for every two vertices u and v of G. Actually, no graph has this property.

Theorem 1.14 *No graph is irregular.*

Proof. Assume, to the contrary, that there exists an irregular graph G of order $n \geq 2$. Since the degree of every vertex of G is one of the n integers $0, 1, \ldots, n-1$, each of these integers is the degree of exactly one vertex. Thus, we may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = i-1$ for $1 \leq i \leq n$. Since $\deg v_1 = 0$, the vertex v_1 is isolated in G and since $\deg v_n = n-1$, it follows that v_n is adjacent to v_1. This is a contradiction. ■

By Theorem 1.14, for each integer $n \geq 2$, there is no graph of order n whose n vertices have distinct degrees. Since the degrees of the vertices of the graph G of Figure 1.24 are $4, 3, 2, 2, 1$, it is possible for $n-1$ vertices of a graph of order n to have distinct degrees. A graph G of order $n \geq 2$ is **nearly irregular** if exactly two vertices of G have the same degree.

Theorem 1.15 *For every integer* $n \geq 2$*, there are exactly two nearly irregular graphs of order* n*.*

Proof. First, observe that if G is a nearly irregular graph of order n, then G cannot contain a vertex of degree 0 *and* a vertex of degree $n-1$. Thus, either each vertex of G has one of the degrees $1, 2, \ldots, n-1$ or each vertex of G has one of the degrees $0, 1, \ldots, n-2$. Furthermore, if G is nearly irregular, then so is $\overline{G}$.

We show by induction that for each integer $n \geq 2$ there are exactly two nearly irregular graphs of order n. Since K_2 and $\overline{K_2}$ are nearly irregular, the result holds for $n = 2$. Assume for an integer $n \geq 3$ that there are exactly two nearly irregular graphs of order $n-1$. Necessarily, one of these graphs is a graph F with $\Delta(F) = n-2$ and $\delta(F) = 1$ while the other is $\overline{F}$ where $\Delta(\overline{F}) = n-3$ and $\delta(\overline{F}) = 0$. Then $H = F + K_1$ and $\overline{H}$ are nearly irregular graphs of order

n, where $\Delta(H) = n-2$ and $\delta(H) = 0$. We claim that these are the only nearly irregular graphs of order n. Assume, to the contrary, that there is a third graph G of order n that is nearly irregular. Then either $\Delta(G) = n-1$ or $\delta(G) = 0$, say the latter. Then $G = G_1 + K_1$, where G_1 is the nearly irregular graph of order $n-1$ with $\Delta(G_1) = n-2$. However then, $G_1 \cong H$, which is a contradiction. ■

It follows by Theorem 1.15 then that for each integer $n \geq 2$, there exist exactly two distinct graphical sequences of length n having exactly two equal terms.

1.8 Multigraphs

There are occasions when a graph is not the appropriate structure to model a particular situation. For example, suppose that we are considering various locations in a certain community and there are roads between some pairs of locations that do not pass through any other location. Although this situation may be represented by a graph, there may be some characteristics in this network of roads that are not captured by a graph. For example, suppose that there are pairs of locations connected by two or more roads (not passing through any other location) and this information is important to us.

In the definition of a graph G, every two distinct vertices are joined by either one edge or no edge of G. There will be occasions when we will want to permit more than one edge to join two vertices. A **multigraph** is a nonempty set of vertices, every two of which are joined by a finite number of edges. Hence a multigraph H may be expressed as $H = (V, E)$, where E is a multiset of 2-element subsets of V. Two or more edges that join the same pair of distinct vertices are called **parallel edges**. The **underlying graph** of a multigraph H is that graph G for which $V(G) = V(H)$ and $uv \in E(G)$ if u and v are joined by at least one edge in H.

An edge joining a vertex to itself is called a **loop**. Structures that permit both parallel edges and loops (including parallel loops) are sometimes called **pseudographs**. For emphasis then, every two vertices of a graph are joined by at most one edge and loops are not permitted. In a multigraph, every two vertices are permitted to be joined by more than one edge but this is not required. Also, no multigraph contains a loop. In a pseudograph, every two vertices are permitted to be joined by more than one edge and loops are permitted. However, parallel edges and loops are not required in pseudographs. There are authors who refer to multigraphs or pseudographs as graphs and those who refer to what we call graphs as **simple graphs**. Consequently, when reading any material written on graph theory, it is essential that there is a clear understanding of how the term *graph* is being used. According to the terminology introduced here then, every multigraph is a pseudograph and every graph is both a multigraph and a pseudograph.

In Figure 1.29, H_1 and H_4 are multigraphs while H_2 and H_3 are pseudographs. Of course, H_1 and H_4 are also pseudographs while H_4 is the only graph in Figure 1.29. For a vertex v in a multigraph G, the **degree** $\deg v$ of v in G is the number of edges of G incident with v. In a pseudograph, each loop at a vertex contributes 2 to its degree. For the pseudograph H_3 of Figure 1.29, $\deg u = 5$ and $\deg v = 2$.

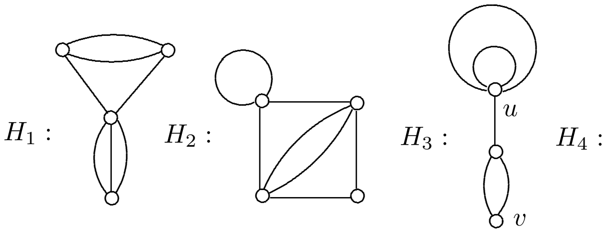


Figure 1.29: Multigraphs and pseudographs

The degree sequence of the multigraph G_1 in Figure 1.30 is $5, 4, 3$ and that of the multigraph G_2 is $4, 3, 2, 1$. That is, G_1 and G_2 are irregular multigraphs.

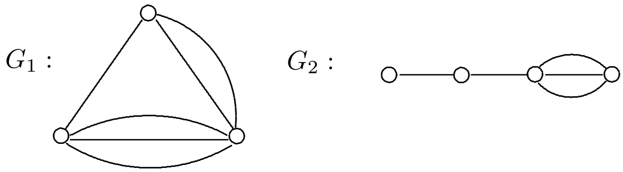


Figure 1.30: Two irregular multigraphs

The multigraphs G_1 and G_2 in Figure 1.30 illustrate the following result.

Theorem 1.16 *For every connected graph G of order at least 3, there exists an irregular multigraph whose underlying graph is G.*

Proof. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_i = u_i v_i$ for $i = 1, 2, \ldots, m$. Replacing e_i by 2^{i-1} parallel edges joining u_i and v_i produces a multigraph H. Since every two vertices of G have distinct sets of edges incident with them and every positive integer has a unique base 2 representation, their degrees in H are distinct and so H is irregular. ■

When describing walks in multigraphs or in pseudographs, it is often necessary to list edges in the sequence as well as vertices in order to specify the edges being used in the walk. For example,

$$W = (u, e_1, u, e_3, v, e_6, w, e_6, v, e_7, w)$$

is a $u - w$ walk in the pseudograph G of Figure 1.31.

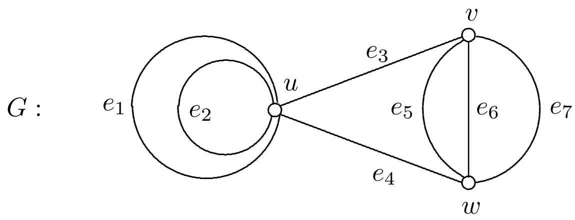


Figure 1.31: Walks in a pseudograph

Exercises for Chapter 1

Section 1.1. Graphs

1. An electronics company keeps on hand wire segments of a fixed length and of different colors for various purposes. Each wire is either colored blue (b), green (g), purple (p), red (r), silver (s), white (w) or yellow (y). The company has many wire segments of each color. All of the wire segments have been randomly stored in a large barrel. Eight handfuls of wires are removed from the barrel and each collection of wires is placed in a box. The boxes are denoted by B_i $(1 \le i \le 8)$. The colors of the wire segments in each box are:

$$B_1 = \{b, r\} \quad B_2 = \{p, r, s, w\} \quad B_3 = \{p, w, y\} \quad B_4 = \{g, r, y\}$$
$$B_5 = \{g\} \quad B_6 = \{b, g, y\} \quad B_7 = \{g, p, s, w, y\} \quad B_8 = \{s, w, y\}.$$

   We are interested in those pairs of boxes containing at least one wire segment of the same color. Model this situation by a graph.

2. A graph $G = (V, E)$ of order 8 has the power set of the set $S = \{1, 2, 3\}$ as its vertex set, that is, V is the set of all subsets of S. Two vertices A and B of V are adjacent if $A \cap B = \emptyset$. Draw the graph G, determine the degree of each vertex of G and determine the size of G.

Section 1.2. The Degree of a Vertex

3. A graph G of order 26 and size 58 has 5 vertices of degree 4, 6 vertices of degree 5 and 7 vertices of degree 6. The remaining vertices of G all have the same degree. What is this degree?

4. A graph G has order $n = 3k + 3$ for some positive integer k. Every vertex of G has degree $k + 1$, $k + 2$ or $k + 3$. Prove that G has at least $k + 3$ vertices of degree $k+1$ or at least $k+1$ vertices of degree $k+2$ or at least $k + 2$ vertices of degree $k + 3$.

5. The degree of every vertex of a graph G is one of three consecutive integers. For each degree x, the graph G contains exactly x vertices of degree x. Prove that for every graph G with this property, two-thirds of the vertices of G have odd degree.

6. Show for every positive integer k that there exists a graph G of order $2k$ containing two vertices of degree i for each $i = 1, 2, \ldots, k$.

Section 1.3. Isomorphic Graphs

7. Consider the pairs G_1, G_2 and H_1, H_2 of graphs in Figure 1.32.

(a) Determine whether $G_1 \cong G_2$.

(b) Determine whether $H_1 \cong H_2$.

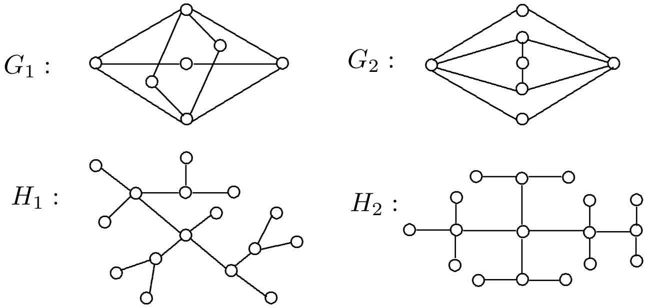


Figure 1.32: The graphs G_1, G_2, H_1, H_2 in Exercise 7

8. (a) Determine all non-isomorphic graphs of order 5.

   (b) Determine the minimum size of a graph G of order 5 such that every graph of order 5 and size 5 is isomorphic to some subgraph of G.

9. (a) Let G and H be two isomorphic graphs where one or more vertices of G (and of H) have degree r. Let S be the set of vertices of degree r in G and T be the set of vertices of degree r in H. Prove that $G[S] \cong H[T]$.

   (b) Use the result in (a) to show that the graphs G and H in Figure 1.33 are not isomorphic.

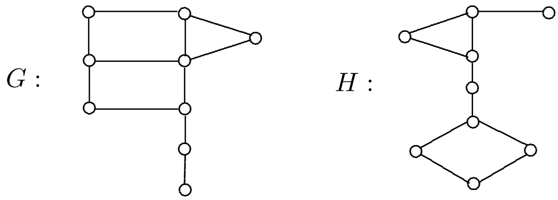


Figure 1.33: The graphs G and H in Exercise 9

10. (a) Give an example of three graphs of size 3, no two of which are isomorphic, such that in each graph, every two edge-induced subgraphs of the same size are isomorphic.

    (b) Give an example of two graphs H and G of the same order and two spanning subgraphs F_1 and F_2 of G such that $F_i \cong H$ for $i = 1, 2$ and $G - E(F_1) \not\cong G - E(F_2)$.

Section 1.4. Regular Graphs

11. Show that if G is a nonregular graph of order n and size $rn/2$ for some integer r with $1 \le r \le n-2$, then $\Delta(G) - \delta(G) \ge 2$.

12. For each integer $k \ge 2$, give an example of k non-isomorphic regular graphs, all of the same order and same size.

13. Give an example of a nonregular graph G containing an edge e and a vertex u such that $G - e$ and $G - u$ are both regular.

14. (a) Give an example of two non-isomorphic regular graphs G_1 and G_2 of the same order and same size such that (1) for every two vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, $G_1 - v_1 \not\cong G_2 - v_2$ and (2) there exist 2-element subsets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ such that $G_1 - S_1 \cong G_2 - S_2$.

    (b) Give an example of two non-isomorphic regular graphs H_1 and H_2 of the same order and same size such that (1) for every 2-element subsets $S_1 \subseteq V(H_1)$ and $S_2 \subseteq V(H_2)$, $H_1 - S_1 \not\cong H_2 - S_2$ and there exist 3-element subsets $S_1' \subseteq V(H_1)$ and $S_2' \subseteq V(H_2)$ such that $H_1 - S_1' \cong H_2 - S_2'$.

15. Prove for every graph G and every integer $r \ge \Delta(G)$ that there exists an r-regular graph containing G as an induced subgraph.

16. Let G be a graph of order n all of whose vertices have degree r, where r is a positive integer, except for exactly one vertex of each of the degrees $r-1, r-2, \ldots, r-j$, where $1 < j < r$. Show, in fact, that there exists an r-regular graph of order $2n$ containing G as an induced subgraph.

17. Let $S = \{1, 2, 3, 4, 5\}$. The vertex set of a graph G is the set of 2-element subsets of S. Two vertices of G are adjacent if the vertices are disjoint. What familiar graph is G?

18. For positive integers k and n with $n > 2k$, the graph $G_{n,k}$ is that graph whose vertices are the k-element subsets of an n-element set $S = \{1, 2, \ldots, n\}$ and where two vertices (k-element subsets) A and B are adjacent if A and B are disjoint. The graph $G_{n,k}$ is called the **Kneser graph**.

    (a) Determine the graphs $G_{6,1}$ and $G_{5,2}$.

    (b) Show that $G_{n,k}$ is an r-regular graph for some integer r.

Section 1.5 Bipartite Graphs

19. A bipartite graph G of order n has partite sets U and W where $|U| = 10$. Every vertex of U has degree 6. In W, there are four vertices of degree 2 and three vertices of degree 4. All other vertices of G have degree 8. What is n?

20. Show for each integer $n \geq 2$ that there is exactly one bipartite graph of order n having size $\lfloor n^2/4 \rfloor$.

21. Prove for a 3-partite graph of order $n = 3k$ and size m that $m \leq 3k^2$.

22. Let G be a nonempty graph with the property that whenever $uv \notin E(G)$ and $vw \notin E(G)$, then $uw \notin E(G)$. Prove that G is a complete multipartite graph.

Section 1.6. Operation on Graphs

23. Determine all bipartite graphs G such that $\overline{G}$ is bipartite.

24. Let G be a graph of odd order $n = 2k + 1 \geq 3$ for some positive integer k. Prove that if the vertices of G have exactly the same degrees as the vertices of $\overline{G}$, then G has an odd number of vertices of degree k.

25. (a) Show that there are exactly two 4-regular graphs G of order 7.

    (b) How many 6-regular graphs of order 9 are there?

26. Prove that there is no regular self-complementary graph of even order.

27. We have seen that C_5 is a self-complementary graph. Therefore, there is a regular self-complementary graph of order 5. Show that there is a regular self-complementary graph of order 5^n for every positive integer n.

28. Let G_1 and G_2 be self-complementary graphs, where G_2 has even order n. Let G be the graph obtained from G_1 and G_2 by joining each vertex of G_2 whose degree is less than $n/2$ to every vertex of G_1. Show that G is self-complementary.

29. Prove that there exists a self-complementary graph of order n for every positive integer n with $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

30. (a) Give an example of a graph G of order 6 and size 7 such that G is isomorphic to a subgraph H of $\overline{G}$.

    (b) Give an example of a graph G of order 7 and size 10 such that G is isomorphic to a subgraph H of $\overline{G}$.

31. Prove for every integer $n \geq 3$ that there exists a graph G of order n and size $\lfloor \binom{n}{2}/2 \rfloor$ that is isomorphic to a graph $H \subseteq \overline{G}$.

32. Let P be the Petersen graph. Show that $\overline{P}$ contains a subgraph H such that $H \cong P$.

33. For $i = 1, 2$, let u_i be a vertex in a graph G_i of order n_i and size m_i.

    (a) Determine the degree of u_1 in $G_1 + G_2$.

    (b) Determine the degree of u_1 in $G_1 \vee G_2$.

    (c) Determine the degree of (u_1, u_2) in $G_1 \square G_2$.

34. Determine the order and size of each of the graphs $P_3 \vee 2P_3$, $P_3 \square 2P_3$ and $Q_1 + Q_2 + Q_3$.

Section 1.7. Degree Sequences

35. Find a sequence of 2-switches that transforms the graph G of Figure 1.34 into the graph H.

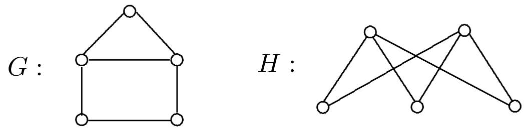


Figure 1.34: The graphs G and H in Exercise 35

36. For two pairs G_1, H_1 and G_2, H_2 of graphs shown in Figure 1.35, determine the minimum number of 2-switches required to transform
(a) G_1 into H_1 and (b) G_2 into H_2.

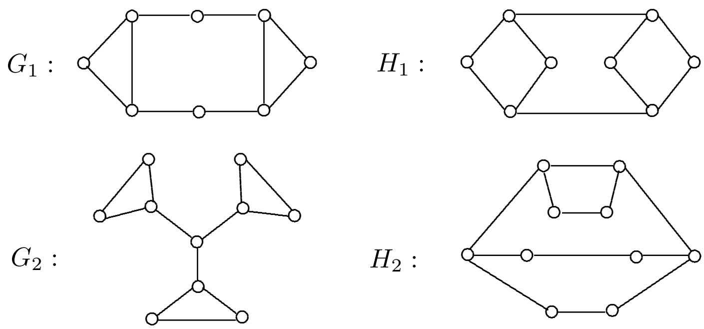


Figure 1.35: The graphs in Exercise 36

37. Let $s : 2, 2, 2, 2, 2, 2, 2, 2, 2, 2$ and let $\mathcal{G}_s$ be the set of all graphs with degree sequence s. Let G be a graph with $V(G) = \mathcal{G}_s$ where two vertices F and H in G are adjacent if F can be transformed into H by a single 2-switch. To which familiar graph is G isomorphic?

38. Give an example of a graphical sequence s (where $\mathcal{G}_s$ is the set of all graphs with degree sequence s) such that (1) the graph G has $V(G) = \mathcal{G}_s$, (2) two vertices F and H of G are adjacent if F can be transformed into H by a single 2-switch and (3) G contains a triangle.

39. Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$. Show, for each integer k with $1 \leq k \leq n$, that there exists a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$ having the property that v_k is adjacent to either (1) the vertices of $\{v_1, v_2, \ldots, v_{d_k}\}$ if $k > d_k$ or (2) the vertices of $\{v_1, v_2, \ldots, v_{d_k+1}\} - \{v_k\}$ if $1 \leq k \leq d_k$.

40. Let G and H be two graphs that are neither empty nor complete. The graph H is said to be obtained from G by an **edge rotation** if G contains three vertices u, v, and w where $uv \in E(G)$ and $uw \notin E(G)$ and $H \cong G - uv + uw$.

    (a) Show that the graph G_2 of Figure 1.36 is obtained from G_1 by an edge rotation.

    (b) Show that G_3 of Figure 1.36 cannot be obtained from G_1 by an edge rotation.

    (c) Show that for every two nonempty, noncomplete graphs G and H of the same order and same size, there exists a sequence $G = G_0$, G_1, …, $G_k = H$ of graphs such that G_{i+1} is obtained from G_i by an edge rotation for $i = 0, 1, \ldots, k-1$.

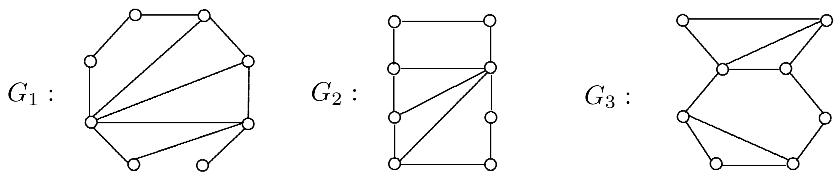


Figure 1.36: The graphs in Exercise 40

41. Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

    (a) $4, 4, 3, 2, 1$

    (b) $3, 3, 2, 2, 2, 2, 1, 1$

    (c) $7, 7, 6, 5, 4, 4, 3, 2$

    (d) $7, 6, 6, 5, 4, 3, 2, 1$

    (e) $7, 4, 3, 3, 2, 2, 2, 1, 1, 1$.

42. Prove that a sequence $d_1, d_2, \ldots, d_n$ is graphical if and only if $n - d_1 - 1, n - d_2 - 1, \ldots, n - d_n - 1$ is graphical.

43. Prove that for every integer x with $0 \leq x \leq 5$, the sequence $x, 1, 2, 3, 5, 5$ is not graphical.

44. For which integers x $(0 \leq x \leq 7)$, if any, is the sequence $7, 6, 5, 4, 3, 2, 1, x$ graphical?

45. Use Theorem 1.13 to determine whether the sequence $s : 6, 6, 5, 4, 3, 2, 2$ is graphical.

46. We have seen that there is only one graphical sequence d_1, d_2, d_3, d_4, d_5 with $4 = d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 = 1$ such that at least one term is 3 and at least one term is 2. How many graphical sequences d_1, d_2, d_3, d_4, d_5 are there with $4 = d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 = 2$ such that at least one term is 3?

47. Show that for every finite set S of positive integers, there exists a positive integer k such that the sequence obtained by listing each element of S a total of k times is graphical. Find the minimum such k for $S = \{2, 6, 7\}$.

48. According to Theorem 1.15, for each integer $n \geq 2$, there exist exactly two distinct graphical sequences of length n having exactly two equal terms. What terms are equal for these two sequences?

49. Two finite sequences s_1 and s_2 of nonnegative integers are called **bigraphical** if there exists a bipartite graph G with partite sets V_1 and V_2 such that s_i lists the degrees of the vertices of G in V_i for $i = 1, 2$. Prove that the sequences $s_1 : a_1, a_2, \ldots, a_r$ and $s_2 : b_1, b_2, \ldots, b_t$ of nonnegative integers with $r \geq 2$, $a_1 \geq a_2 \geq \cdots \geq a_r$, $b_1 \geq b_2 \geq \cdots \geq b_t$, $0 < a_1 \leq t$ and $0 < b_1 \leq r$ are bigraphical if and only if the sequences $s_1' : a_2, a_3, \cdots, a_r$ and $s_2' : b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_t$ are bigraphical.

50. The graphs G and H of order 10 have vertex sets $V(G) = \{u_1, u_2, \ldots, u_{10}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{10}\}$ and edge sets $E(G) = \{u_i u_j : i + j \geq 11\}$ and $E(H) = \{v_i v_j : i + j \geq 12\}$. How are G and H related?

51. (a) Let n be a given positive integer and let r and s be nonnegative integers such that $r + s = n$ and s is even. Give an example of a graph containing r even vertices and s odd vertices.

    (b) Determine the minimum size of a graph G containing r even vertices and s odd vertices and satisfying the properties in (a).

    (c) Determine the maximum size of a graph G containing r even vertices and s odd vertices and satisfying the properties in (a).

52. (a) Let G be a graph of order $n \geq 4$. Prove that if $\deg v \geq \frac{2n+1}{3}$ for every vertex v of G, then every edge of G belongs to a complete subgraph of order 4.

(b) Show that the result in (a) is best possible in general by showing that $\frac{2n+1}{3}$ cannot be replaced by $\frac{2n}{3}$.

Section 1.8. Multigraphs

53. We saw that the irregular multigraph G_2 in Figure 1.30 has degree sequence $4, 3, 2, 1$. Give an example of an irregular multigraph (if such a multigraph exists) having degree sequence

(a) $5, 4, 3, 2, 1$

(b) $6, 5, 4, 3, 2, 1$

(c) $7, 6, 5, 4, 3, 2, 1$.

54. Prove for every connected graph G of order $n = 3$ or $n = 4$ and size m that it is possible to label the edges of G by $e_1, e_2, \ldots, e_m$ and replace e_i by i parallel edges for each i $(1 \leq i \leq m)$ such that the degrees of the vertices of the resulting multigraph H are distinct.

55. Determine which of the following sequences are the degree sequences of a multigraph.

(a) $s_1 : 3, 2, 1$ (b) $s_2 : 5, 2, 1$

(c) $s_3 : 6, 4, 2$ (d) $s_4 : 3, 2, 2$

(e) $s_5 : 4, 4, 2, 2$ (f) $s_6 : 5, 3, 2, 1$

(g) $s_7 : 4, 4, 4, 4$ (h) $s_8 : 7, 5, 3, 1$.

56. Prove that a sequence $s : d_1, d_2, \ldots, d_n$ $(n \geq 1)$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a multigraph if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq \frac{1}{2} \sum_{i=1}^{n} d_i$.

57. Let G be a connected graph of order n where the vertices of G are labeled as $v_1, v_2, \ldots, v_n$ in some way. A multigraph H of size m with $V(H) = V(G)$ is obtained by replacing each edge v_iv_j of G by $\min\{i, j\}$ parallel edges.

(a) Find m if $G = K_5$.

(b) Find sharp upper and lower bounds for m if $G = C_5$.

(c) Find the minimum value of m if G is bipartite.

Chapter 2

Connected Graphs and Distance

There are many problems in graph theory that deal with whether it is possible to travel from one vertex in a graph to another vertex and the manner in which this can be done. In order to study problems of this type, we now introduce several new concepts.

2.1 Connected Graphs

Walks, Trails and Paths

For two (not necessarily distinct) vertices u and v in a graph G, a $u - v$ **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in W are adjacent in G. Such a walk W in G can be expressed as

$$W = (u = v_0, v_1, \ldots, v_k = v), \tag{2.1}$$

where $v_i v_{i+1} \in E(G)$ for $0 \leq i \leq k - 1$. (The walk W is also commonly denoted by $W : u = v_0, v_1, \ldots, v_k = v$.) Nonconsecutive vertices in W need not be distinct. The walk W is said to contain each vertex v_i $(0 \leq i \leq k)$ and each edge $v_i v_{i+1}$ $(0 \leq i \leq k - 1)$. The walk W can therefore be thought of as beginning at the vertex $u = v_0$, proceeding along the edge $v_0 v_1$ to the vertex v_1, then along the edge $v_1 v_2$ to the vertex v_2, and so forth, until finally arriving at the vertex $v = v_k$. The number of edges encountered in W (including multiplicities) is the **length** of W. Hence, the length of the walk W in (2.1) is k. In the graph G of Figure 2.1,

$$W_1 = (x, w, y, w, v, u, w) \tag{2.2}$$

is an $x - w$ walk of length 6. This walk encounters the vertex w three times and the edge wy twice.

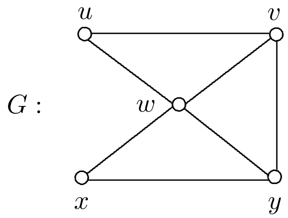


Figure 2.1: Walks in a graph

A walk whose initial and terminal vertices are distinct is an **open walk**; otherwise, it is a **closed walk**. Thus, the walk W_1 in (2.2) is an open walk. It is possible for a walk to consist of a single vertex, in which case it is a **trivial walk**. A trivial walk is therefore a closed walk.

A walk in a graph G in which no edge is repeated is a **trail** in G. For example, in the graph G of Figure 2.1, $T = (u, v, y, w, v)$ is a $u - v$ trail of length 4. While no edge of T is repeated, the vertex v is repeated, which is allowed. On the other hand, a walk in a graph G in which no vertex is repeated is called a **path**. Every nontrivial path is necessarily an open walk. Thus, $P' = (u, v, w, y)$ is $u - y$ path of length 3 in the graph G of Figure 2.1. Many proofs in graph theory make use of $u - v$ walks or $u - v$ paths of minimum length (or of maximum length) for some pair u, v of vertices of a graph. The proof of the following theorem illustrates this.

Theorem 2.1 *Let u and v be two vertices of a graph G. For every $u - v$ walk W in G, there exists a $u - v$ path P such that every edge of P belongs to W.*

Proof. Let W be a $u - v$ walk. Among all $u - v$ walks in G, every edge of which belongs to W, let

$$P = (u = u_0, u_1, \ldots, u_k = v)$$

be one of minimum length. Thus, the length of P is k. We claim that P is a $u - v$ path. Assume, to the contrary, that this is not the case. Then some vertex of G must be repeated in P, say $u_i = u_j$ for some i and j with $0 \leq i < j \leq k$. If we then delete the vertices $u_{i+1}, u_{i+2}, \ldots, u_j$ from P, we arrive at the $u - v$ walk

$$W' = (u = u_0, u_1, \ldots, u_{i-1}, u_i = u_j, u_{j+1}, \ldots, u_k = v)$$

whose length is less than k and such that every edge of W' belongs to W. This is a contradiction. ∎

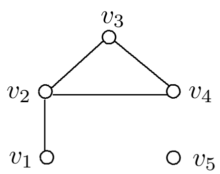
The Adjacency Matrix of a Graph

We have seen that a graph can be defined or described by means of sets (the definition) or diagrams. There are also matrix representations of graphs. Suppose that G is a graph of order n, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of G is the $n \times n$ zero-one matrix $A(G) = [a_{ij}]$, or simply $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E(G) \\ 0 & \text{if } v_iv_j \notin E(G). \end{cases}$$

Figure 2.2 shows the adjacency matrix of a graph G.

G :



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 2.2: A graph and its adjacency matrix

There are several observations that can be made about the adjacency matrix A of a graph G of order n. First, all entries along the main diagonal of A are 0 since no vertex of G is adjacent to itself. Second, A is a symmetric matrix, that is, row i of A is identical to column i of A for every integer i with $1 \le i \le n$. Also, if we were to add the entries in row i (or in column i), then we obtain the degree of v_i.

Whenever $a_{ij} = 1$, this means that G contains the edge v_iv_j and therefore a $v_i - v_j$ path of length 1 and, of course, a $v_i - v_j$ walk of length 1 as well. Not only can the adjacency matrix of G be used to identify whether G contains a $v_i - v_j$ walk of length 1, it can be used to determine whether G contains a $v_i - v_j$ walk of length k for an arbitrary positive integer k and, in fact, the number of such walks. Before stating a theorem that provides us with this information, we need to know when two $u - v$ walks in a graph G are considered to be the same.

Two $u-v$ walks $W = (u = u_0, u_1, \ldots, u_k = v)$ and $W' = (u = v_0, v_1, \ldots, v_\ell = v)$ in a graph are **equal** if $k = \ell$ and $u_i = v_i$ for all i with $0 \le i \le k$.

Theorem 2.2 *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix A. For each positive integer k, the number of different $v_i - v_j$ walks of length k in G is the (i, j)-entry in the matrix A^k.*

Proof. Let $a_{ij}^{(k)}$ denote the (i, j)-entry in the matrix A^k for a positive integer k. Thus, $A^1 = A$ and $a_{ij}^{(1)} = a_{ij}$. We proceed by induction on k. For vertices v_i and v_j of G, there can be only one $v_i - v_j$ walk of length 1 or no $v_i - v_j$

walks of length 1, and this occurs if $a_{ij} = 1$ or $a_{ij} = 0$, respectively. Therefore, the (i, j)-entry of the matrix A is the number of $v_i - v_j$ walks of length 1 in G. Thus, the basis step of the induction is established.

We now verify the inductive step. Assume, for a positive integer k, that $a_{ij}^{(k)}$ is the number of different $v_i - v_j$ walks of length k in G. We show that the (i, j)-entry $a_{ij}^{(k+1)}$ in A^{k+1} gives the number of different $v_i - v_j$ walks of length $k+1$ in G. First, observe that every $v_i - v_j$ walk of length $k+1$ in G is obtained from a $v_i - v_t$ walk of length k for some vertex v_t in G that is adjacent to v_j.

Since $A^{k+1} = A^k \cdot A$, it follows that the (i, j)-entry $a_{ij}^{(k+1)}$ in A^{k+1} can be obtained by taking the inner product of row i of A^k and column j of A. That is,

$$a_{ij}^{(k+1)} = a_{i1}^{(k)} a_{1j} + a_{i2}^{(k)} a_{2j} + \ldots + a_{in}^{(k)} a_{nj} = \sum_{t=1}^{n} a_{it}^{(k)} a_{tj}. \tag{2.3}$$

By the induction hypothesis, for each integer t with $1 \le t \le n$, the integer $a_{it}^{(k)}$ is the number of different $v_i - v_t$ walks of length k in G. If $a_{tj} = 1$, then v_t is adjacent to v_j and so there are $a_{it}^{(k)}$ different $v_i - v_j$ walks of length $k+1$ in G whose next-to-last vertex is v_t. On the other hand, if $a_{tj} = 0$, then v_t is not adjacent to v_j and there are no $v_i - v_j$ walks of length $k+1$ in G whose next-to-last vertex is v_t. In any case, $a_{it}^{(k)} \cdot a_{tj}$ gives the number of different $v_i - v_j$ walks of length $k+1$ in G whose next-to-last vertex is v_t. Consequently, the total number of different $v_i - v_j$ walks of length $k+1$ in G is the sum in (2.3), which is $a_{ij}^{(k+1)}$.

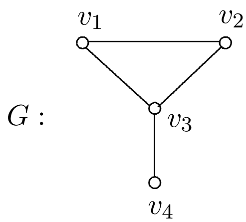
By the Principle of Mathematical Induction, $a_{ij}^{(k)}$ is the number of different $v_i - v_j$ walks of length k in G for every positive integer k. ■

Before giving an example to illustrate Theorem 2.2, we make a few observations. Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $A^k = \left[a_{ij}^{(k)}\right]$, where A is the adjacency matrix of G. By Theorem 2.2, $a_{ii}^{(2)}$ gives the number of different $v_i - v_i$ walks of length 2 in G. Since a $v_i - v_i$ walk of length 2 is (v_i, v_t, v_i) for some vertex v_t adjacent to v_i, it follows that $a_{ii}^{(2)} = \deg v_i$ for every vertex v_i of G. For $i \ne j$, $a_{ij}^{(2)}$ is the number of different $v_i - v_j$ *paths* of length 2 in G. Again, by Theorem 2.2, $a_{ii}^{(3)}$ gives the number of different $v_i - v_i$ walks of length 3 in G. Since a $v_i - v_i$ walk of length 3 is (v_i, v_s, v_t, v_i) for adjacent vertices v_s and v_t, each of which is adjacent to v_i, it follows that v_i must belong to a triangle. Not only is (v_i, v_s, v_t, v_i) a $v_i - v_i$ walk of length 3, so too is (v_i, v_t, v_s, v_i) a (different) $v_i - v_i$ walk of length 3 in G. Therefore, $a_{ii}^{(3)}$ is twice the number of triangles in G that contain v_i.

As an illustration, consider the graph G of Figure 2.3 having the adjacency matrix A. We can compute A^2 without matrix multiplication by observing that the (i, i) entry of A^2, $1 \le i \le 4$, is $\deg v_i$, and the (i, j) entry of A^2, $i \ne j$, is the number of different $v_i - v_j$ paths of length 2. We now turn to A^3. Since the different $v_1 - v_3$ walks of length 3 in G are

$$W_1 = (v_1, v_3, v_1, v_3),\ W_2 = (v_1, v_2, v_1, v_3),$$
$$W_3 = (v_1, v_3, v_2, v_3),\ W_4 = (v_1, v_3, v_4, v_3),$$

the $(1,3)$ entry of A^3 is 4. The entire matrix A^3 can be computed in this manner.



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Figure 2.3: A graph G and powers of its adjacency matrix

Circuits and Cycles

A nontrivial closed walk in a graph G in which no edge is repeated is a **circuit** in G. For example,

$$C = (u, w, x, y, w, v, u)$$

is a circuit in the graph G of Figure 2.1. In addition to the necessary repetition of u in this circuit, w is repeated as well. This is acceptable since no edge is repeated in C. A circuit

$$C = (v = v_0, v_1, \ldots, v_k = v), \tag{2.4}$$

$k \geq 2$, for which the vertices v_i, $0 \leq i \leq k-1$, are distinct is a **cycle** in G. Therefore,

$$C' = (u, v, y, x, w, u)$$

is a cycle of length 5 in the graph G of Figure 2.1. As with the class of graphs C_k, $k \geq 3$, called cycles in Chapter 1, the cycle C in (2.4) is called a k**-cycle**. Once again, a 3-cycle is referred to as a **triangle**. A cycle of even length is an **even cycle**, while a cycle of odd length is an **odd cycle**.

The subgraph induced by the edges in a path $(v_1, v_2, \cdots, v_k)$ or a cycle $(v_1, v_2, \cdots, v_k, v_1)$, $k \geq 3$, is itself called a **path** or **cycle**, respectively. Consequently, paths and cycles have more than one interpretation – a means to proceed between vertices in a graph, as subgraphs in a graph and as a class of graphs. This is also the case with trails and circuits. The path P' and the cycle C' described earlier in the graph G of Figure 2.1 correspond to the subgraphs shown in Figure 2.4. We saw in Section 1.1 that a graph of order n that is a path or a cycle is denoted by P_n and C_n, respectively.

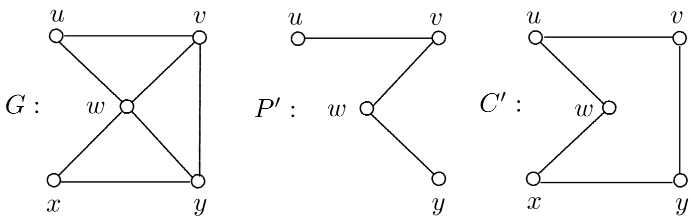


Figure 2.4: A path and cycle in a graph

The length of a smallest cycle in a graph G (containing cycles) is the **girth** of G, denoted by $g(G)$, and the length of a longest cycle is the **circumference** of G, denoted by $c(G)$. Thus, $g(K_n) = 3$ for $n \geq 3$ and $c(K_n) = n$; while $g(K_{s,t}) = 4$ for $2 \leq s \leq t$ and $c(K_{s,t}) = 2s$. The girth of the Petersen graph is 5 and its circumference is 9 (as we will see in Chapter 6).

Connected Graphs

Two vertices u and v in a graph G are **connected** if G contains a $u - v$ path. The graph G itself is **connected** if every two vertices of G are connected. By Theorem 2.1, a graph G is connected if G contains a $u - v$ walk for every two vertices u and v of G. A graph G that is not connected is a **disconnected graph**. The graph F of Figure 2.5 is connected since F contains a $u - v$ path (and a $u - v$ walk) for every two vertices u and v in F. On the other hand, the graph H is disconnected since, for example, H contains no $y_4 - y_5$ path.

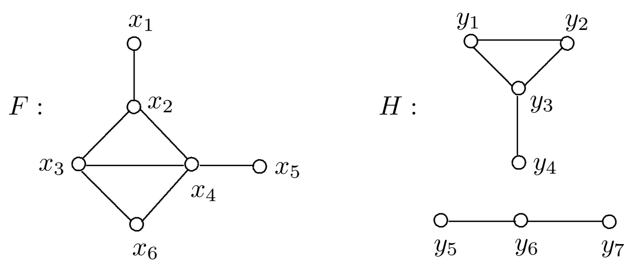


Figure 2.5: A connected graph and disconnected graph

A connected subgraph H of a graph G is a **component** of G if H is not a proper subgraph of any connected subgraph of G. Thus, every component of G is an induced subgraph of G. The number of components in a graph G

is denoted by $k(G)$. Therefore, G is connected if and only if $k(G) = 1$. For the sets $S_1 = \{y_1, y_2, y_3, y_4\}$ and $S_2 = \{y_5, y_6, y_7\}$ of vertices of the graph H of Figure 2.5, the induced subgraphs $H[S_1]$ and $H[S_2]$ are (the only) components of H. Therefore, $k(H) = 2$.

All of the familiar classes of graphs we've introduced are connected. This includes the paths P_n, cycles C_n, complete bipartite graphs $K_{s,t}$, hypercubes Q_n and complete graphs K_n. Because the complete graphs K_n are connected, every graph of order n where each vertex has degree $n-1$ is connected. This illustrates a fact that we will encounter often. Suppose that the complete graph K_n has a certain property (in this case, it is connected). Thus, if every vertex of a graph G of order n has a sufficiently high degree, then G also has this property. Since the disconnected graph $H = 2K_k$ of order $n = 2k$ in Figure 2.6 is $(k-1)$-regular, it is clear that $k - 1 = (n-2)/2$ is not sufficiently large for the degrees of the vertices of a graph G of order n to guarantee that it is connected. The number $(n-2)/2$ is close to having this property, however. Prior to showing this (in Corollary 2.4), we present an even more general result.

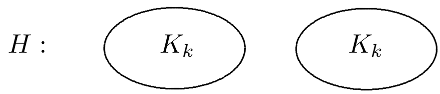


Figure 2.6: A $(k-1)$-regular disconnected graph of order $n = 2k$

Theorem 2.3 *If G is a nontrivial graph of order n such that* $\deg u + \deg v \geq n-1$ *for every two nonadjacent vertices u and v of G, then G is connected.*

Proof. Let x and y be distinct vertices of G. We show that G contains an $x-y$ path. This is obvious if x and y are adjacent, so suppose that x and y are not adjacent. Since $\deg x + \deg y \geq n-1$, there is a vertex z that is adjacent to both x and y. Therefore, (x, z, y) is an $x-y$ path and so G is connected. ■

The graph H in Figure 2.6 shows that the lower bound $n-1$ in Theorem 2.3 cannot be replaced by $n-2$. The following corollary is thus a consequence of Theorem 2.3.

Corollary 2.4 *If G is a graph of order n with $\delta(G) \geq (n-1)/2$, then G is connected.*

Proof. For every two nonadjacent vertices u and v of G,

$$\deg u + \deg v \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1.$$

It then follows by Theorem 2.3 that G is connected. ■

In addition, since the complete graphs are connected, every graph of order n and size $\binom{n}{2}$ is connected. Therefore, graphs of order n with sufficiently large size are connected.

Theorem 2.5 *If G is a graph of order $n \geq 2$ and size $m \geq \binom{n-1}{2} + 1$, then G is connected.*

Proof. The result is clear for $n = 2, 3, 4$, so we may assume that $n \geq 5$. Let u and v be two vertices of G. We show that G contains a $u - v$ path. Since this is clear if $uv \in E(G)$, we may assume that $uv \notin E(G)$. If G contains a vertex w that is adjacent to both u and v, then (u, w, v) is a $u - v$ path in G. Hence, we may assume that there is no such vertex and so $\deg u + \deg v \leq n - 2$. Let $G' = G - u - v$. Thus, G' has order $n' = n - 2$ and size

$$m' = m - \deg u - \deg v \geq \binom{n-1}{2} + 1 - (n-2) = \binom{n-2}{2} + 1.$$

Since $m' \leq \binom{n-2}{2}$, this is impossible. ■

The number $\binom{n-1}{2} + 1$, $n \geq 2$, cannot be lowered in the statement of Theorem 2.5 since the size of the disconnected graph $G = K_1 + K_{n-1}$ is $\binom{n-1}{2}$.

2.2 Distance in Graphs

If u and v are distinct vertices in a connected graph G, then there is a $u - v$ path in G. In fact, there may very well be several $u - v$ paths in G, possibly of varying lengths. This information can be used to provide a measure of how close u and v are to each other or how far from each other they are. The most common definition of distance between two vertices in a connected graph is the following.

The **distance** $d_G(u, v)$ from a vertex u to a vertex v in a connected graph G is the smallest length of a $u - v$ path in G. If the graph G being considered is understood, then this distance is written more simply as $d(u, v)$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ **geodesic**. In the graph G of Figure 2.7, the path $P = (v_1, v_5, v_6, v_{10})$ is a shortest $v_1 - v_{10}$ path; thus, P is a $v_1 - v_{10}$ geodesic and so $d(v_1, v_{10}) = 3$. In addition,

$$d(v_1, v_1) = 0,\ d(v_1, v_2) = 1,\ d(v_1, v_6) = 2,\ d(v_1, v_7) = 3 \text{ and } d(v_1, v_8) = 4.$$

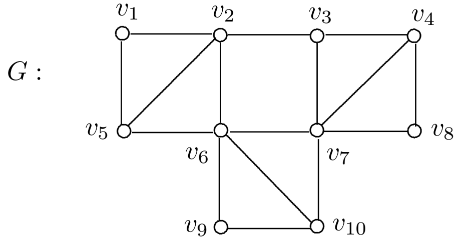


Figure 2.7: Distances in a graph

The distance d defined above satisfies the following properties in a connected graph G:

(1) $d(u, v) \geq 0$ for every two vertices u and v of G;

(2) $d(u, v) = 0$ if and only if $u = v$;

(3) $d(u, v) = d(v, u)$ for all $u, v \in V(G)$ (the **symmetric property**);

(4) $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V(G)$ (the **triangle inequality**).

Since d satisfies the four properties (1)–(4), d is a **metric** on $V(G)$ and $(V(G), d)$ is a **metric space**. Because d satisfies the symmetric property, we can speak of the distance *between* two vertices u and v rather than the distance *from* u to v.

With the aid of this distance we can present a useful characterization of bipartite graphs.

Theorem 2.6 *A nontrivial graph G is a bipartite graph if and only if G contains no odd cycles.*

Proof. Suppose first that G is bipartite. Then $V(G)$ can be partitioned into partite sets U and W (and so every edge of G joins a vertex of U and a vertex of W). Let $C = (v_1, v_2, \ldots, v_k, v_1)$ be a k-cycle of G. We may assume that $v_1 \in U$. Thus, $v_2 \in W$, $v_3 \in U$ and so forth. In particular, $v_i \in U$ for every odd integer i with $1 \leq i \leq k$ and $v_j \in W$ for every even integer j with $2 \leq j \leq k$. Since $v_1 \in U$, it follows that $v_k \in W$ and so k is even.

For the converse, let G be a nontrivial graph containing no odd cycles. If G is empty, then G is clearly bipartite. Hence it suffices to show that every nontrivial component of G is bipartite and so we may assume that G itself is connected. Let u be a vertex of G and let

$$U = \{x \in V(G) : \quad d(u, x) \text{ is even}\} \text{ and } W = \{x \in V(G) : \quad d(u, x) \text{ is odd}\}.$$

Thus, $u \in U$. We show that G is a bipartite graph having partite sets U and W. It suffices to show that no two vertices of U are adjacent and no two vertices of W are adjacent. Suppose that W contains two adjacent vertices w_1 and w_2. Let P_1 be a $u - w_1$ geodesic and P_2 a $u - w_2$ geodesic. Let z be the last vertex that P_1 and P_2 have in common (possibly $z = u$). Then the length of the $z - w_1$ subpath P_1' of P_1 and the length of the $z - w_2$ subpath P_2' of P_2 are of the same parity. Thus, the paths P_1' and P_2' together with the edge $w_1 w_2$ produce an odd cycle. This is a contradiction. The argument that no two vertices of U are adjacent is similar. ∎

The Eccentricity of a Vertex

It is often of value to know how far any vertex of a connected graph G is from a given vertex of G. The **eccentricity** $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G. The eccentricities of the vertices of the graph G in Figure 2.7 are shown in Figure 2.8.

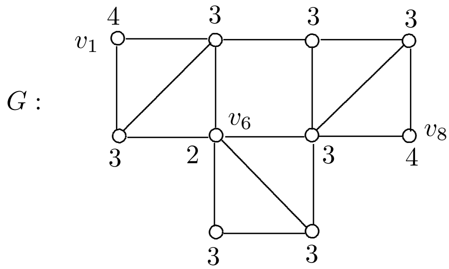


Figure 2.8: Eccentricities of vertices

The eccentricities of the vertices in the graph G of Figure 2.8 illustrate the following theorem.

Theorem 2.7 *If u and v are adjacent vertices in a connected graph G, then*

$$|e(u) - e(v)| \leq 1.$$

Proof. Assume, without loss of generality, that $e(u) \geq e(v)$. Moreover, let w be a vertex of G such that $e(u) = d(u, w)$. By the triangle inequality, $d(u, w) \leq d(u, v) + d(v, w)$. Since $d(u, v) = 1$ and $d(v, w) \leq e(v)$, it follows that

$$|e(u) - e(v)| = e(u) - e(v) \leq d(u, w) - d(v, w) \leq d(u, v) = 1. \blacksquare$$

The **diameter** $\text{diam}(G)$ of a connected graph G is the largest eccentricity among the vertices of G, while the **radius** $\text{rad}(G)$ is the smallest eccentricity among the vertices of G. The diameter of G is, therefore, the greatest distance between any two vertices of G. A vertex v with $e(v) = \text{rad}(G)$ is called a **central vertex** of G and a vertex v with $e(v) = \text{diam}(G)$ is called a **peripheral vertex** of G. Two vertices u and v of G with $d(u, v) = \text{diam}(G)$ are **antipodal vertices** of G. Necessarily, if u and v are antipodal vertices in G, then both u and v are peripheral vertices. For the graph G of Figure 2.8, $\text{diam}(G) = 4$ and $\text{rad}(G) = 2$. In particular, v_6 is the only central vertex of G and v_1 and v_8 are the only peripheral vertices of G. Since $d(v_1, v_8) = 4 = \text{diam}(G)$, it follows that v_1 and v_8 are antipodal vertices of G. It is certainly not always the case that $\text{diam}(G) = 2\,\text{rad}(G)$ as, for example, $\text{diam}(P_4) = 3$ and $\text{rad}(P_4) = 2$. Indeed, the following can be said about the radius and diameter of a connected graph.

Theorem 2.8 *For every nontrivial connected graph G,*

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\,\text{rad}(G).$$

Proof. The inequality $\text{rad}(G) \leq \text{diam}(G)$ is immediate from the definitions. Let u and w be two antipodal vertices of G and let v be a central vertex of G. Therefore, $d(u,w) = \text{diam}(G)$ and $e(v) = \text{rad}(G)$. By the triangle inequality,

$$\text{diam}(G) = d(u,w) \leq d(u,v) + d(v,w) \leq 2e(v) = 2\,\text{rad}(G),$$

as desired. ■

For a connected graph G of order $n \geq 2$, the **eccentricity sequence** of G is the nondecreasing sequence $e_1, e_2, \ldots, e_n$ of positive integers for which the vertices of G can be labeled as $v_1, v_2, \ldots, v_n$ such that $e(v_i) = e_i$ for $1 \leq i \leq n$. Thus, $e_1 = \text{rad}(G)$ and $e_n = \text{diam}(G)$. In fact, since G has at least two peripheral vertices, $e_{n-1} = \text{diam}(G)$ as well. The eccentricity sequence of the graph G of Figure 2.8 is

$$2, 3, 3, 3, 3, 3, 3, 3, 4, 4.$$

In fact, for every connected graph G of order $n \geq 2$, its eccentricity sequence $e_1, e_2, \ldots, e_n$ satisfies (1) $e_n \leq 2e_1$ by Theorem 2.8 and (2) $e_{i+1} - e_i \leq 1$ for $1 \leq i \leq n-1$. To see why (2) is true, suppose that $e(v_i) = e_i$ for $1 \leq i \leq n$. Let $P = (v_1 = u_1, u_2, \ldots, u_k = v_n)$ be a $v_1 - v_n$ path in G. By Theorem 2.7, $|e(u_{i+1}) - e(u_i)| \leq 1$ for $1 \leq i \leq k-1$ and so the eccentricities of consecutive vertices on P differ by at most 1. Therefore, every integer between e_1 and e_n is the eccentricity of some vertex of G.

Theorem 2.8 gives the lower bound $\text{rad}(G)$ for the diameter of a connected graph G as well as the upper bound $2\,\text{rad}(G)$. This is one of many results for which a question of "sharpness" is involved. These involve the question: Just how good is this result? Ordinarily, there are many interpretations of such a question. We shall consider some possible interpretations in the case of the upper bound.

Certainly, the upper bound in Theorem 2.8 would not be considered sharp if $\text{diam}(G) < 2\,\text{rad}(G)$ for every graph G; however, it would be considered sharp indeed if $\text{diam}(G) = 2\,\text{rad}(G)$ for every graph G. In the latter case, we would actually have a formula or an identity, not just a bound. Of course, we have seen that there are graphs G for which $\text{diam}(G) < 2\,\text{rad}(G)$ and there are graphs H for which $\text{diam}(H) = 2\,\text{rad}(H)$. This alone may be satisfactory to say that this bound is sharp. However, a preferred reason for calling this bound sharp is if there is an infinite class $\mathcal{H}$ of graphs such that $\text{diam}(H) = 2\,\text{rad}(H)$ for each graph H belonging to $\mathcal{H}$. Such a class exists; for example, let $\mathcal{H}$ consist of the graphs of the type $K_t \vee \overline{K}_2$. One disadvantage of this class is that for each $H \in \mathcal{H}$, $\text{diam}(H) = 2$ and $\text{rad}(H) = 1$. Perhaps a more satisfactory infinite class would be the class of paths P_{2k+1}, $k \geq 1$. In this case, $\text{diam}(P_{2k+1}) = 2k$ and $\text{rad}(P_{2k+1}) = k$; that is, for each positive integer k, there exists a connected graph G such that $\text{diam}(G) = 2\,\text{rad}(G) = 2k$.

Center and Periphery

The subgraph induced by the central vertices of a connected graph G is the **center** of G and is denoted by $\mathrm{Cen}(G)$. If every vertex of G is a central vertex, then $\mathrm{Cen}(G) = G$ and G is **self-centered**. The subgraph induced by the peripheral vertices of a connected graph G is the **periphery** of G and is denoted by $\mathrm{Per}(G)$.

For the graph G of Figure 2.7, the center of G consists of the isolated vertex v_6 and the periphery consists of the two isolated vertices v_1 and v_8. The graph H of Figure 2.9 has radius 2 and diameter 3. Therefore, every vertex of H is either a central vertex or a peripheral vertex. Indeed, the center of H is the triangle induced by the three "exterior" vertices of H, while the periphery of H is the 6-cycle induced by the six "interior" vertices of H.

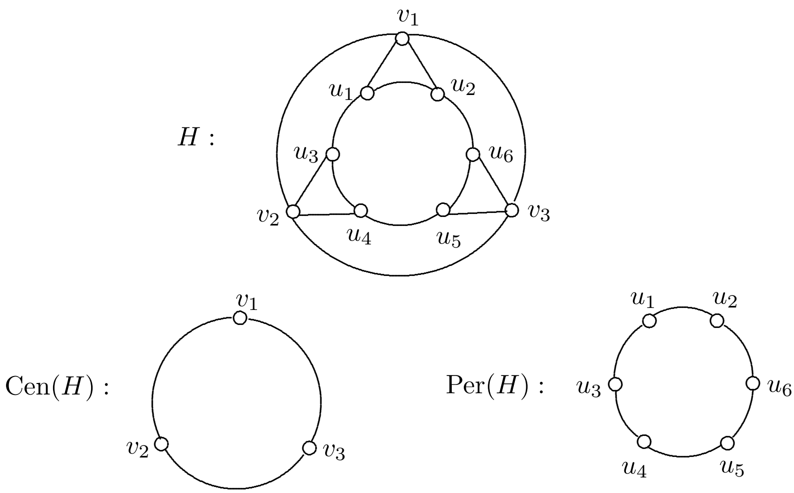


Figure 2.9: The center and periphery of a graph

It is not difficult to see that $\mathrm{Cen}(P_{2k+1}) = K_1$ and $\mathrm{Cen}(P_{2k}) = K_2$ for all $k \geq 1$. Also, $\mathrm{Cen}(C_n) = \mathrm{Per}(C_n) = C_n$ for all $n \geq 3$. In an observation first made by Stephen Hedetniemi (see [40]), there is no restriction on which graphs can be the center of some graph.

Theorem 2.9 *Every graph is the center of some graph.*

Proof. Let G be a graph. We construct a graph H from G by first adding two new vertices u and v to G and joining them to every vertex of G but not to each other. The construction of H is completed by adding two other vertices u_1 and v_1, where u_1 is joined to u and v_1 is joined to v. (This construction is illustrated in Figure 2.10.) Since $e(u_1) = e(v_1) = 4$, $e(u) = e(v) = 3$ and

$e_H(x) = 2$ for every vertex x in G, it follows that $V(G)$ is the set of central vertices of H and so $\text{Cen}(H) = H[V(G)] = G$. ■

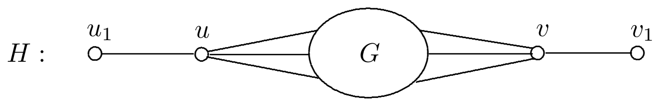


Figure 2.10: A graph with a given center

Example 2.10 Figure 2.11 shows a graph G that represents the street system of a community, where the edges are streets and vertices are street intersections s_i $(1 \leq i \leq 20)$. The community wants to build an emergency facility at one of the intersections so that the number of blocks needed to drive from the facility to the intersection farthest from it will be as small as possible. What are the possible locations for the emergency facility?

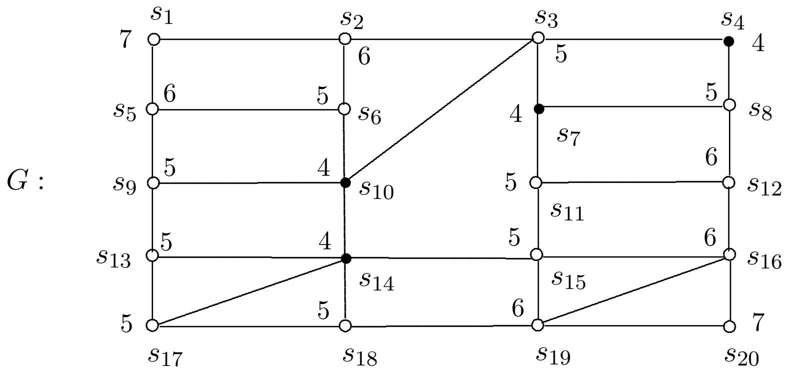


Figure 2.11: A graph representing a street system in Example 2.10

To answer this question, we need to place an emergency facility at a central vertex of the graph G. Consequently, the eccentricity of each vertex must be computed. These are shown in Figure 2.11. Since the minimum eccentricity (radius) of G is 4, the emergency facility should be placed at s_4, s_7, s_{10} or s_{14}. ♦

While every graph is the center of some graph, this is not true for the periphery, as Halina Bielak and Maciej Sysło [26] showed.

Theorem 2.11 *A nontrivial graph G is the periphery of some graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1.*

Proof. If every vertex of G has eccentricity 1, then G is complete and $\text{Per}(G) = G$; while if no vertex of G has eccentricity 1, then let F be the graph obtained from G by adding a new vertex w and joining w to each vertex of G. Since

$e_F(w) = 1$ and $e_F(x) = 2$ for every vertex x of G, it follows that every vertex of G is a peripheral vertex of F and so $\mathrm{Per}(F) = F[V(G)] = G$.

For the converse, let G be a graph that contains some vertices of eccentricity 1 and some vertices whose eccentricity is not 1 and suppose that there exists a graph H such that $\mathrm{Per}(H) = G$. Necessarily, G is a proper induced connected subgraph of H. Thus $\mathrm{diam}(H) = k \geq 2$. Furthermore, $e_H(v) = k \geq 2$ for each $v \in V(G)$ and $e_H(v) < k$ for $v \in V(H) - V(G)$. Let u be a vertex of G such that $e_G(u) = 1$ and let w be a vertex of H such that $d(u, w) = e_H(u) = k \geq 2$. Since w is not adjacent to u, it follows that $w \notin V(G)$. On the other hand, $d(u, w) = k$ and so $e_H(w) = k$. This implies that w is a peripheral vertex of H and so $w \in V(G)$, which is impossible. ∎

Exercises for Chapter 2

Section 2.1. Connected Graphs

1. Give an example of a graph G and two vertices u and v of G such that there is a $u-v$ trail containing all vertices of G but no $u-v$ path containing all vertices of G.

2. Give an example of a graph G with three vertices u, v and w such that (1) every $u-v$ path avoids w, (2) every $u-w$ path avoids v and no $v-w$ path avoids u.

3. Let G_1, G_2 and G_3 be three graphs of order n and size m having adjacency matrices A_1, A_2 and A_3, respectively.

   (a) Prove or disprove: If $A_1 = A_2$, then $G_1 \cong G_2$.

   (b) Prove or disprove: If $A_2 \neq A_3$, then $G_2 \not\cong G_3$.

4. (a) Use Theorem 2.2 to compute A^4 if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ without multiplying matrices.

   (b) Show that A^4 could be computed more easily by using Theorem 2.2 to first compute A^2.

5. Determine the adjacency matrix of the graph G_1 of Figure 2.12. Then determine A^2 and A^3 without multiplying matrices.

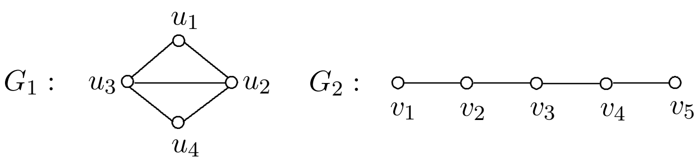


Figure 2.12: Graphs G_1 and G_2 in Exercises 5 and 6

6. Determine the adjacency matrix of the graph G_2 of Figure 2.12. Then determine A^2, A^3 and A^4 without multiplying matrices.

7. Determine the graph G with adjacency matrix A for which

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 2 & 2 & 3 & 1 & 1 \\ 2 & 2 & 3 & 1 & 1 \\ 3 & 3 & 2 & 4 & 0 \\ 1 & 1 & 4 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}.$$

8. For an $n \times n$ matrix A, it is common to define A^0 as $\mathbf{I}_n$ (the $n \times n$ identity matrix). If this is done, how is this related to Theorem 2.2?

9. For each positive integer k, show that there exists a graph G of order $2k+1$ such that every vertex of G lies on one or more triangles but on no larger cycles.

10. (a) Give an example of a cubic graph of order 10 containing a k-cycle for each integer k with $3 \leq k \leq 10$.

    (b) Give an example of a cubic graph G containing no k-cycle for some integer k with $g(G) < k < c(G)$.

11. Let G be a graph with $\delta(G) \geq 2$.

    (a) Prove that the circumference $c(G)$ of G satisfies $c(G) \geq \delta(G) + 1$.

    (b) Show that G has a path of length $\delta(G)$.

12. Determine $g(G)$ and $c(G)$ for $G = K_{k,2k,4k}$ for a positive integer k.

13. Prove that "is connected to" is an equivalence relation on the vertex set of a graph.

14. Prove that a graph G is connected if and only if for every partition $\{V_1, V_2\}$ of $V(G)$, there exists an edge of G joining a vertex of V_1 and a vertex of V_2.

15. Let G be a connected graph of order n and let k be an integer such that $2 \leq k \leq n-1$. Show that if $\deg u + \deg v \geq k$ for every pair u, v of nonadjacent vertices of G, then G contains a path of length k.

16. Let G be a disconnected graph of order $n \geq 6$ having three components. Prove that $\Delta(\overline{G}) \geq \frac{2n}{3}$.

17. Show, for every two vertices u and v in a connected graph G, that there exists a $u-v$ walk containing all vertices of G.

18. Characterize those graphs G having the property that every induced subgraph of G is a connected subgraph of G.

19. (a) Show that if G is a connected graph such that the degree of every vertex is one of three distinct numbers and each of these three numbers is the degree of at least one vertex of G, then there is a path in G containing three vertices whose degrees are distinct.

    (b) Is the statement in (a) true if "three" is replaced by "four"?

20. Let G be a nontrivial connected graph that is not bipartite. Show that G contains two adjacent vertices u and v such that $\deg u + \deg v$ is even.

21. Let k and n be integers with $2 \leq k < n$ and let G be a graph of order n. Prove that if every vertex of G has degree exceeding $(n-k)/k$, then G has fewer than k components.

22. Let $k \geq 2$ be an integer. Prove that if G is a graph of order $n \geq k+1$ and size $m \geq (k-1)(n-k-1)+\binom{k+1}{2}$, then G contains a subgraph having minimum degree k.

23. Prove that if G is a connected graph of order $n \geq 2$, then the vertices of G can be listed as $v_1, v_2, \ldots, v_n$ such that each vertex v_i $(2 \leq i \leq n)$ is adjacent to some vertex in the set $\{v_1, v_2, \ldots, v_{i-1}\}$.

24. Suppose that the vertices of a graph G of order $n \geq 2$ can be listed as $v_1, v_2, \ldots, v_n$ such that each vertex v_i $(2 \leq i \leq n)$ is adjacent to some vertex in the set $\{v_1, v_2, \ldots, v_{i-1}\}$. Prove that G is connected.

Section 2.2. Distance in Graphs

25. Let u, v and w be three vertices in a connected graph G. Prove that

$$d(u,v)+d(u,w)+d(v,w) \geq 2d(u,w).$$

26. Prove that a nontrivial graph G is bipartite if and only if G contains no induced odd cycle.

27. Let G be a graph of order $n \geq 6$ and size $m = \frac{(n-2)(n+2)}{4}$ containing no odd cycle. If G has three vertices u, v and w such that $\deg u < n/2$, $\deg v < n/2$ and $\deg w < n/2$, then what is G?

28. Let G be a connected graph such that the length of a longest path in G is ℓ.

    (a) Prove that no two paths of length ℓ in G are vertex-disjoint.

    (b) By (a), two paths of length ℓ cannot be vertex-disjoint. Prove that if P and Q are two paths of length ℓ that meet in a single vertex, then ℓ is even.

29. Let G be a connected graph of order n. For a vertex v of G and an integer k with $1 \leq k \leq n-1$, let $d_k(v)$ be the number of vertices at distance k from v.

    (a) What is $d_1(v)$?

    (b) Show that $\sum_{v \in V(G)} d_k(v)$ is even for every integer k with $1 \leq k \leq n-1$.

    (c) What is the value of $\sum_{v \in V(G)} \left(\sum_{k=1}^{n-1} d_k(v)\right)$?

30. Prove that if G is a disconnected graph, then $\overline{G}$ is connected and, in fact, $\text{diam}(\overline{G}) \leq 2$.

31. Let a and b be positive integers with $a \leq b \leq 2a$. Show that there exists a connected graph G with $\text{rad}(G) = a$ and $\text{diam}(G) = b$.

32. (a) Show that $2, 3, 3, 3$ is not the eccentricity sequence of any graph.
    (b) Determine all pairs a, b of positive integers with $a < b$ such that a, b, b, b is the eccentricity sequence of some graph.

33. For each integer $n \geq 4$, give an example of two nonisomorphic graphs of order n having the same eccentricity sequence.

34. (a) Give an example of a nontrivial connected graph G whose degree sequence is identical to its eccentricity sequence.
    (b) Give an example of a nontrivial connected graph G of order n with degree sequence $d_1, d_2, \ldots, d_n$ and eccentricity sequence $n - d_1, n - d_2, \ldots, n - d_n$.

35. Show, for every integer $k \geq 4$, that there exists a connected graph G containing exactly k vertices $v_1, v_2, \ldots, v_k$ with the property that $\deg v_i = e(v_i)$ for all i with $(1 \leq i \leq k)$.

36. Let G be a nontrivial connected graph. Show that if k is an integer with $\text{rad}(G) < k \leq \text{diam}(G)$, then there are at least two vertices of G with eccentricity k. (Hint: Let w be a vertex with $e(w) = k$ and let u be a vertex with $d(w, u) = e(w) = k$. For a central vertex v of G, let P be a $u - v$ path of length $d(u, v)$. Show that $e(v) < k \leq e(u)$. Then show that there is a vertex x (distinct from w) on P such that $e(x) = k$.)

37. Show that for every pair r, s of positive integers, there exists a positive integer n such that for every connected graph G of order n, either $\Delta(G) \geq r$ or $\text{diam}(G) \geq s$.

38. Every complete graph is the periphery of itself. Can a complete graph be the periphery of a connected graph G with $\text{diam}(G) \geq 2$?

39. A vertex w of a connected graph G is called an **eccentric vertex** of a vertex v of G if $d(v, w) = e(v)$. Show that there exists a connected graph containing four distinct vertices v_1, v_2, v_3, v_4 such that v_{i+1} is an eccentric vertex of v_i for $i = 1, 2, 3$ and the integers $e(v_1), e(v_2)$ and $e(v_3)$ are distinct.

40. The **total distance** $\text{td}(u)$ of a vertex u in a connected graph G is defined by
$$\text{td}(u) = \sum_{v \in V(G)} d(u, v).$$
A vertex v in G is called a **median vertex** if v has the minimum total distance among the vertices of G. Equivalently, v is a median vertex if v has the minimum average distance to all vertices of G. The **median** $\text{Med}(G)$ of G is the subgraph of G induced by its median vertices. Determine (i) the total distance $\text{td}(u)$ for each vertex u of the graph G of Figure 2.13 and (ii) the median $\text{Med}(G)$ of G.

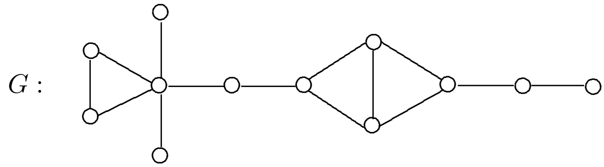


Figure 2.13: The graph in Exercise 40

41. Let F and H be two subgraphs in a connected graph G. Define the **distance** $d(F, H)$ between F and H as

$$d(F, H) = \min\{d(u, v) : u \in V(F), v \in V(H)\}.$$

Show that for every positive integer k, there exists a connected graph G such that $d(\text{Cen}(G), \text{Med}(G)) = k$ (see Exercise 40 for the definition of $\text{Med}(G)$).

42. Let G and its complement $\overline{G}$ both be connected graphs of order $n \geq 5$.

    (a) Prove that if the diameter of G is at least 3, then the diameter of its complement is at most 3.

    (b) What diameters are possible for self-complementary graphs with at least 4 vertices?

    (c) If the diameter of G is 2, what are the possible diameters of its complement $\overline{G}$?

43. For a graphical sequence s, let $\mathcal{G}_s$ be the set of all graphs with degree sequence s. For $G, H \in \mathcal{G}_s$, define the **distance** $d(G, H)$ from G to H as the minimum number of 2-switches required to transform G into H. Show that $(\mathcal{G}_s, d)$ is a metric space.

44. For two vertices u and v in a connected graph G, a $u - v$ **detour** is a longest $u - v$ path in G and the length of a $u - v$ detour is the **detour distance** $D(u, v)$ between u and v. Show that $(V(G), D)$ is a metric space.

Chapter 3

Trees

In nearly every concept, problem and theorem that we encounter, we are primarily concerned with connected graphs. In this chapter, we study graphs that are minimally connected in various senses.

3.1 Nonseparable Graphs

Some graphs are connected to such a small extent that the removal of a single vertex results in a disconnected graph. We now consider vertices having this property.

Cut-Vertices

A vertex v in a connected graph G is a **cut-vertex** if $G - v$ is disconnected. Therefore, if v is a cut-vertex of a connected graph G, then $G - v$ consists of components $G_1, G_2, \ldots, G_k$ for some integer $k \geq 2$. That is, $G - v = G_1 + G_2 + \cdots + G_k$. Necessarily, for every component G_i of $G - v$, at least one vertex of G_i is adjacent to v in G.

More generally, a vertex v is a **cut-vertex** of a graph G (connected or disconnected) if $k(G - v) > k(G)$. In the graph H_1 of Figure 3.1, the vertices u, v, w and x are cut-vertices; in H_2 every vertex of degree 4 is a cut-vertex; and in H_3 no vertex is a cut-vertex. For $n \geq 3$, no vertex of C_n is a cut-vertex, while at the other extreme, only two vertices of P_n are not cut-vertices, namely the two end-vertices of P_n. That this is the other extreme is verified in the following theorem.

Theorem 3.1 *Every nontrivial connected graph contains at least two vertices that are not cut-vertices.*

Proof. Let G be a nontrivial connected graph and let P be a longest path in G. Suppose that P is a $u - v$ path. We show that u and v are not cut-vertices.

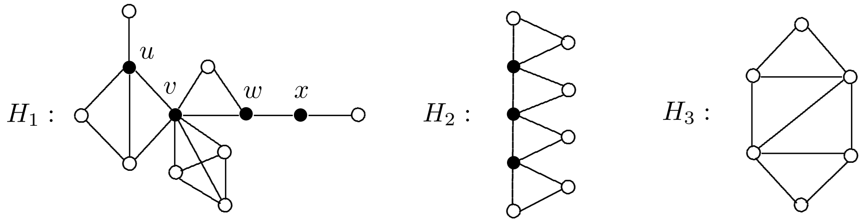


Figure 3.1: Cut-vertices in graphs

Assume, to the contrary, that u is a cut-vertex of G. Then $G-u$ is disconnected and so contains two or more components. Let w be the vertex on P that is adjacent to u and let P' be the $w-v$ subpath of P. Necessarily, P' belongs to a component, say G_1, of $G-u$. Let G_2 be another component of $G-u$. Then G_2 contains some vertex x that is adjacent to u. This produces an $x-v$ path P' that contains P. However, P' is longer than P, which is impossible. Similarly, v is not a cut-vertex of G. ■

If v is a cut-vertex of a graph G, then there exist paths in G that cannot avoid v.

Theorem 3.2 *A vertex v in a graph G is a cut-vertex of G if and only if there are two vertices u and w distinct from v such that v lies on every $u-w$ path in G.*

Proof. We may assume that G is connected, for otherwise we can consider a component of G containing v. If v is a cut-vertex in a connected graph G, then, of course, $G-v$ contains two or more components. If u and w are vertices in distinct components of $G-v$, then u and w are not connected in $G-v$. On the other hand, u and w are necessarily connected in G. Thus, v lies on every $u-w$ path in G.

For the converse, suppose that there are two vertices u and w distinct from v such that v lies on every $u-w$ path in G. Then there is no $u-w$ path in $G-v$. Thus, u and w are not connected in $G-v$, and so $G-v$ is disconnected. Therefore, v is a cut-vertex of G. ■

An Introduction to Nonseparable Graphs

We have seen then that some connected graphs may contain a vertex, the removal of which separates the graph into two or more connected graphs. These vertices are, of courses, cut-vertices. As we saw, a connected graph need not contain any cut-vertices.

A nontrivial connected graph containing no cut-vertices is a **nonseparable graph**. In particular, the cycles C_n, $n \geq 3$, and the complete graphs K_n, $n \geq 2$, are nonseparable graphs. In fact, K_2 and K_3 are the only nonseparable graphs of order 3 or less. Furthermore, if G is a nonseparable graph of order 3 or more, then $\delta(G) \geq 2$. Not only do nonseparable graphs of order 3 or more contain cycles, they contain cycles possessing a rather interesting property.

Theorem 3.3 *Let G be a graph of order 3 or more. Then G is nonseparable if and only if every two vertices of G lie on a common cycle of G.*

Proof. Suppose, first, that G is a nonseparable graph of order 3 or more and assume, to the contrary, that there are pairs of vertices of G that do not lie on a common cycle. Among all such pairs, let u, v be a pair for which $d(u, v)$ is minimum. Suppose first that $d(u, v) = 1$, that is, $uv \in E(G)$. Since G is a nonseparable graph of order 3 or more, $\deg u \geq 2$. Let w be a vertex different from v that is adjacent to u. Since $G - u$ is connected, $G - u$ contains a $w - v$ path P. Then the path P together with the path (w, u, v) produce a cycle containing u and v. Hence, we may assume that $d(u, v) = k \geq 2$.

Let $P = (u = v_0, v_1, \ldots, v_{k-1}, v_k = v)$ be a $u - v$ geodesic in G. Since $d(u, v_{k-1}) = k-1 < k$, there is a cycle C containing u and v_{k-1}. By assumption, v is not on C. Since v_{k-1} is not a cut-vertex of G and u and v are distinct from v_{k-1}, it follows from Theorem 3.2 that there is a $v - u$ path Q that does not contain v_{k-1}. Since u is on C, there is a first vertex x of Q that is on C (where possibly $x = u$). Let Q' be the $v - x$ subpath of Q and let P' be a $v_{k-1} - x$ path on C that contains u. (If $x \neq u$, then the path P' is unique.) However, the cycle C' produced by proceeding from v to its neighbor v_{k-1}, along P' to x, and then along Q' to v contains both u and v, a contradiction.

Conversely, suppose that every two vertices of G lie on a common cycle of G. Then G is connected. Assume, to the contrary, that G has a cut-vertex v. By Theorem 3.2, G contains two vertices u and w distinct from v such that v lies on every $u - w$ path in G. Since there is a cycle C containing u and w, there is a $u - w$ path on C not containing v. This is a contradiction. ■

This theorem has several consequences (see Exercises 3-5). For two distinct vertices u and v in a graph G, two $u - v$ paths are **internally disjoint** if they have only u and v in common.

Corollary 3.4 *A connected graph G of order 3 or more is nonseparable if and only if for every two distinct vertices u and v in G, there are two internally disjoint $u - v$ paths.*

Corollary 3.5 *Let u and w be two distinct vertices in a nonseparable graph G. If H is obtained from G by adding a new vertex v and joining v to u and w, then H is nonseparable.*

Corollary 3.6 *If U and W are disjoint sets of vertices in a nonseparable graph G of order 4 or more with $|U| = |W| = 2$, then G contains two disjoint paths connecting the vertices of U and the vertices of W.*

Blocks

Let G be a nontrivial connected graph. A **block** of G is a maximal nonseparable subgraph of G; that is, a block of G is a nonseparable subgraph of G that is not a proper subgraph of any nonseparable subgraph of G. Every two distinct blocks of G have at most one vertex in common; and if they have a vertex in common, then this vertex is a cut-vertex of G. A block of G containing exactly one cut-vertex of G is called an **end-block** of G. If B and B' are two blocks containing the cut-vertex v, while $u \in V(B)$ and $w \in V(B')$ for $u, w \neq v$, then every $u - w$ path in G must contain the vertex v. This is basically Theorem 3.2. A graph G and its five blocks B_i, $1 \leq i \leq 5$, are shown in Figure 3.2. The end-blocks of G are B_1, B_2 and B_5. A connected graph with cut-vertices must contain two or more end-blocks.

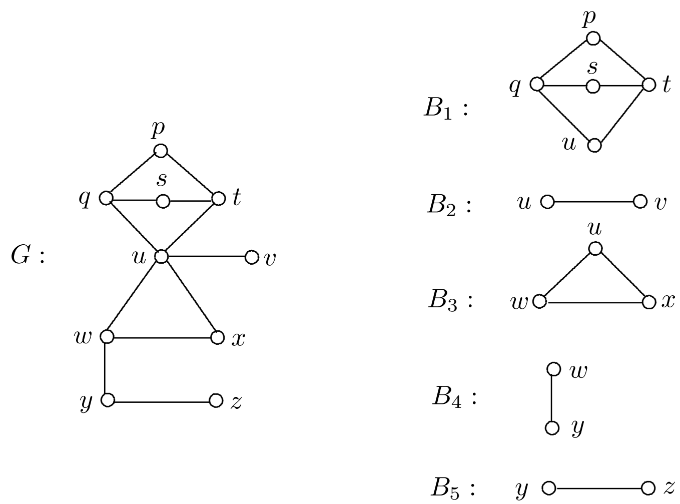


Figure 3.2: The blocks of a graph

Theorem 3.7 *Every connected graph containing cut-vertices has at least two end-blocks.*

Proof. If G contains only one cut-vertex, then every block of G is an end-block. Hence, we may assume that G contains two or more cut-vertices. Among all pairs of cut-vertices of G, let u, v be a pair for which $d(u, v)$ is maximum and let P be a $u - v$ geodesic, say

$$P = (u = u_0, u_1, \ldots, u_k = v), \text{ where } k \geq 1.$$

Then u_1 belongs to a block B and u_{k-1} belongs to a block B', where possibly $B = B'$. In fact, possibly $u_1 = v$. Since u is a cut-vertex of G, it follows that u belongs to one or more blocks different from B. Let B_0 be one of these. Similarly, let B_0' be a block different from B' that contains v. We claim that B_0 is an end-block of G. If B_0 is not an end-block, then B_0 contains a cut-vertex x different from u. Since every $x - u_1$ path must pass through u, it follows that

$$d(x, v) = d(x, u) + d(u, v) > d(u, v),$$

which is impossible. Similarly, B_0' is an end-block different from B_0. ■

The following result, which can be proved in a similar manner, is often useful as well.

Theorem 3.8 *Let G be a connected graph with at least one cut-vertex. Then G contains a cut-vertex v with the property that, with at most one exception, all blocks of G containing v are end-blocks.*

Proof. If G has only one cut-vertex, then every block of G is an end-block and contains the cut-vertex. Hence, we may assume that G contains two or more cut-vertices. Among the cut-vertices of G, let u and v be two for which $d(u, v)$ is maximum and let $P = (u = u_0, u_1, \ldots, u_k = v)$, $k \geq 1$, be a $u - v$ geodesic. Then u_{k-1} belongs to a block B containing v. Let B' be a block containing v that is different from B. If B' is not an end-block, then B' contains a cut-vertex w different from v. Let P' be a $v - w$ geodesic in G. Then the path P followed by P' produces a $u - w$ geodesic whose length exceeds that of P. This is a contradiction. Thus, every block containing v that is different from B is an end-block. ■

Another interesting property of blocks of graphs was observed by Frank Harary and Robert Z. Norman [123].

Theorem 3.9 *The center of every connected graph G lies in a single block of G.*

Proof. Suppose that G is a connected graph whose center $\mathrm{Cen}(G)$ does not lie within a single block of G. Then G has a cut-vertex v such that $G - v$ contains components G_1 and G_2, each of which contains vertices of $\mathrm{Cen}(G)$. Let u be a vertex such that $d(u, v) = e(v)$, and let P_1 be a $v - u$ geodesic. At least one of G_1 and G_2, say G_2, contains no vertices of P_1. Let w be a vertex of $\mathrm{Cen}(G)$ belonging to G_2, and let P_2 be a $w - v$ geodesic. The paths P_1 and P_2 together form a $u - w$ path P_3, which is necessarily a $u - w$ path of length $d(u, w)$. However, then $e(w) > e(v)$, which contradicts the fact that w is a central vertex. Thus, $\mathrm{Cen}(G)$ lies in a single block of G. ■

If a graph G has components $G_1, G_2, \ldots, G_k$ and a nonempty connected graph H has blocks $B_1, B_2, \ldots, B_\ell$, then $\{V(G_1), V(G_2), \ldots, V(G_k)\}$ is a partition of $V(G)$ and $\{E(B_1), E(B_2), \ldots, E(B_\ell)\}$ is a partition of $E(H)$.

Suppose, for a cut-vertex v of a connected graph G, that the disconnected graph $G - v$ has k components $G_1, G_2, \ldots, G_k$ ($k \geq 2$). The induced subgraphs

$$B_i = G[V(G_i) \cup \{v\}]$$

are connected and referred to as the **branches** of G at v. If a branch B_i contains no cut-vertices of G, then B_i is a block of G – in fact, an end-block of G.

A connected graph G containing three cut-vertices u, v and w and six blocks is shown in Figure 3.3. Four of these blocks are end-blocks. The graph G has four branches at v, all of which are shown in Figure 3.3. Two of the four branches at v are end-blocks of G.

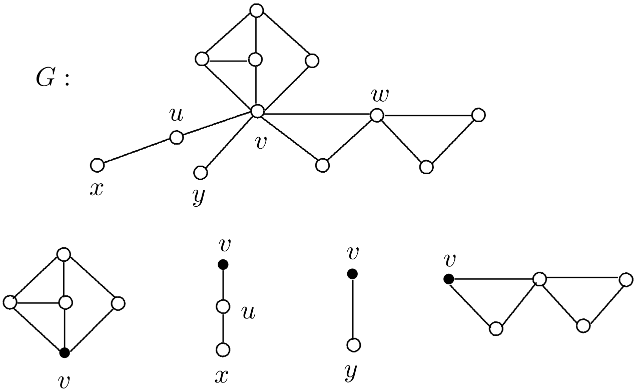


Figure 3.3: The four branches of a graph G at a cut-vertex v

3.2 Introduction to Trees

We have seen that a connected graph may contain a vertex whose removal results in a disconnected graph and that such a vertex is a cut-vertex. There are also edges possessing this property.

Bridges

An edge $e = uv$ in a connected graph G whose removal results in a disconnected graph is a **bridge**. Necessarily, $G - e$ consists of two components, one containing u and the other containing v. More generally, an edge e is a bridge in a graph G (connected or not) if $k(G-e) > k(G)$. Then $k(G-e) = k(G)+1$. In the graph G of Figure 3.4, v_5, v_6, v_7, v_8 and v_9 are cut-vertices, while v_4v_5, v_5v_6, v_7v_{11} and v_8v_9 are bridges.

While there are connected graphs no vertex of which is a cut-vertex (the nonseparable graphs), there is no connected graph in which every vertex is a cut-vertex (see Theorem 3.1). There are, however, connected graphs in which every edge is a bridge. It is this class of graphs that we study in the current section. First, we discuss an important property of bridges.

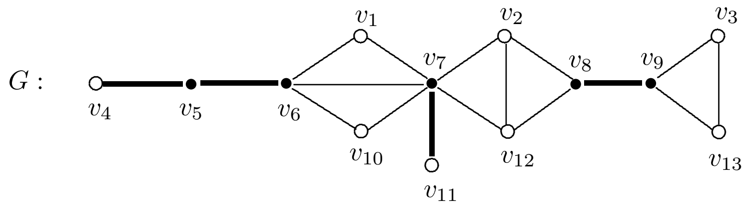


Figure 3.4: Bridges and cut-vertices in a graph

Theorem 3.10 *An edge e in a graph G is a bridge of G if and only if e lies on no cycle in G.*

Proof. We may assume that G is connected, for otherwise we can consider a component of G containing e. First, suppose that $e = uv$ is an edge of G that is not a bridge. Since $G - e$ is connected, there is a $u - v$ path P in $G - e$. Then P together with e produce a cycle in G containing e.

For the converse, assume that $e = uv$ is an edge of G belonging to a cycle of G. Since e lies on a cycle of G, there is a $u - v$ path P' in G not containing e. We show that $G - e$ is connected and, consequently, that e is not a bridge. Let x and y be two vertices of G. Since G is connected, G contains an $x - y$ path Q. If e does not lie on Q, then Q is an $x - y$ path in $G - e$ as well. If, on the other hand, e lies on Q, then replacing e in Q by the $u - v$ path P' produces an $x - y$ walk in $G - e$. By Theorem 2.1, $G - e$ contains an $x - y$ path. Thus $G - e$ is connected. ∎

We are now prepared to discuss one of the best known and most useful classes of graphs. An **acyclic graph** has no cycles. A **tree** is a connected acyclic graph. By Theorem 3.10, a tree is a connected graph, every edge of which is a bridge. Each graph in Figure 3.5 is a tree.

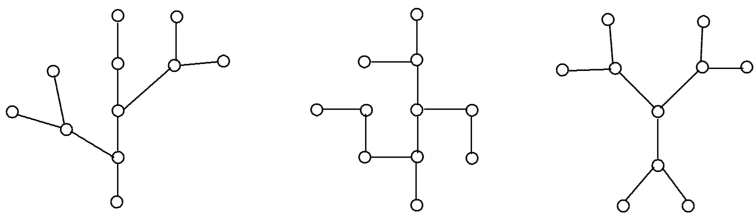


Figure 3.5: Three trees

Origin of Trees

Trees appeared implicitly in the 1847 work [142] of the German physicist Gustav Kirchhoff in his study of currents in electrical networks, while Arthur Cayley [43] used trees in 1857 to count certain types of chemical compounds. Trees are important to the understanding of the structure of graphs and are used to systematically visit the vertices of a graph. Trees are also widely used in computer science as a means to organize and utilize data.

The simplest organic chemical molecules are the *alkanes*. Alkanes are hydrocarbons and so their molecules consist only of carbon and hydrogen atoms, denoted by the symbols C and H, respectively. The valency of each carbon atom is 4 and of each hydrogen atom is 1, so if an alkane molecule has n carbon atoms, then there must be $2n+2$ hydrogen atoms, producing the formula C_nH_{2n+2} for each alkane molecule. The five simplest alkanes along with their chemical formulas are shown in Figure 3.6. In earlier days, the degree of a vertex v in a graph was sometimes referred to as the *valency* of v.

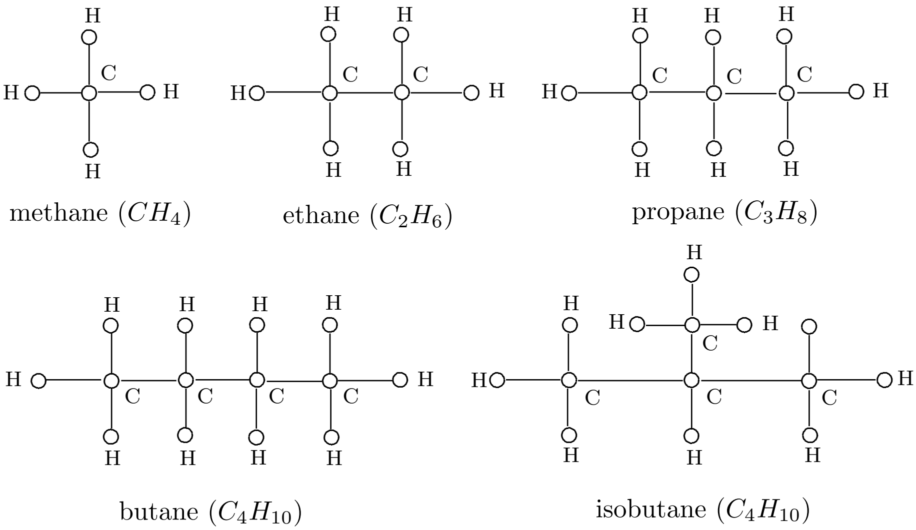


Figure 3.6: The five simplest alkanes

All of the structures in Figure 3.6 are trees. Observe that both butane and isobutane have the same chemical formula C_4H_{10} but they have a different structure. (They are not isomorphic.) So it is possible for two different alkanes to have the same chemical formula. In fact, there are three different alkanes having the formula C_5H_{12} and 1, 117, 743, 651, 746, 953, 270 (over one quintillion) different alkanes having the formula $C_{50}H_{102}$.

Special Trees

There are several well-known classes of trees. For example, the paths P_n and stars $K_{1,n-1}$ are trees of order $n \geq 2$. For $t \geq 2$, only one vertex of the star $K_{1,t}$ is not a leaf; the vertex of degree t in $K_{1,t}$ is the **central vertex** of $K_{1,t}$. A tree containing exactly two vertices that are not leaves (which are necessarily adjacent) is called a **double star**. Thus, a double star is a tree of diameter 3. A **caterpillar** is a tree T of order 3 or more, the removal of whose leaves produces a path (which is called the **spine** of T). Thus, every path, every star (of order at least 3) and every double star is a caterpillar. Figure 3.7 shows four trees T_1, T_2, T_3 and T_4. The tree T_1 is a star, T_2 is a double star and T_3 is a caterpillar that is not a double star, while T_4 is not a caterpillar.

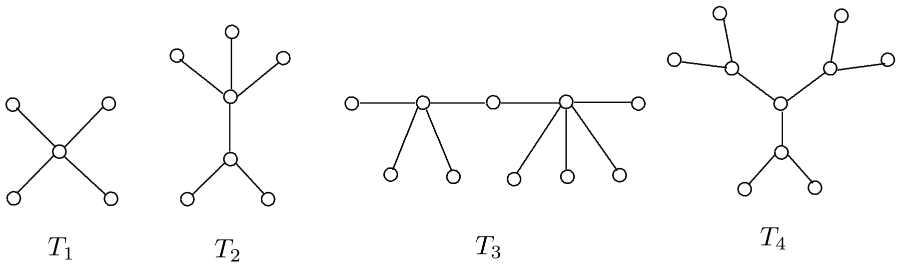


Figure 3.7: Four trees

Properties of Trees

Since every tree is connected, every two vertices are connected by a path. In fact, even more can be said.

Theorem 3.11 *A graph G is a tree if and only if every two vertices of G are connected by a unique path.*

Proof. First, suppose that G is a tree and that u and v are two vertices of G. Since G is connected, G contains at least one $u - v$ path. On the other hand, if G were to contain two $u - v$ paths, then G would contain a cycle, which is impossible. Therefore, G contains exactly one $u - v$ path.

Conversely, let G be a graph in which every two vertices are connected by a single path. Certainly then, G is connected. If G were to contain a cycle C, then every two vertices on C would be connected by two paths. Thus, G contains no cycle and so G is a tree. ■

While every vertex of degree 2 or more in a tree is a cut-vertex, the vertices of degree 1 (the leaves) are not. These observations provide a corollary of Theorem 3.1.

Corollary 3.12 *Every nontrivial tree contains at least two leaves.*

For a cut-vertex v of T, there are $\deg v$ branches of T at v, where, necessarily, each branch is a subtree of T. In the tree T of Figure 3.8, the four branches of T at v are shown in that figure.

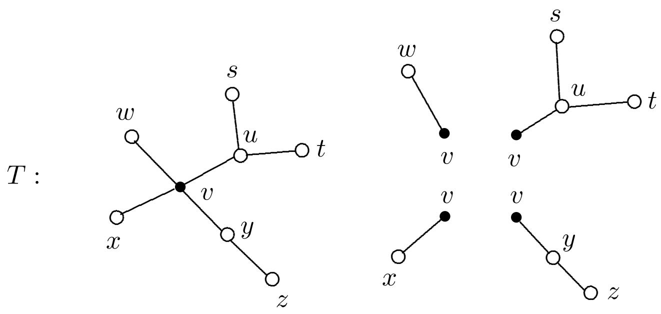


Figure 3.8: The branches of a tree at a vertex

There are several ways of constructing a new tree from a given tree. For example, if v is a leaf in a tree T, then $T - v$ is also a tree. If a new vertex is added to T and joined to any vertex of T, then another tree results.

In the tree T of Figure 3.8 and in each of the trees T_1, T_2, T_3 and T_4 of Figure 3.7, the size of the tree is one less than its order. This is not only true of these trees, it is true of all trees.

Theorem 3.13 *If T is a tree of order n and size m, then $m = n - 1$.*

Proof. We proceed by induction on the order of a tree. There is only one tree of order 1, namely K_1, and it has no edges. Thus, the basis step of the induction is established. Assume that the size of every tree of order $n - 1 \geq 1$ is $n - 2$ and let T be a tree of order n and size m. By Corollary 3.12, T has at least two leaves. Let v be one of them. As we observed, $T - v$ is a tree of order $n - 1$. By the induction hypothesis, the size of $T - v$ is $n - 2$. Thus, the size of T is $m = (n - 2) + 1 = n - 1$. ■

Suppose that a tree T of order $n \geq 3$, size m and maximum degree $\Delta(T) = \Delta$ has n_i vertices of degree i $(1 \leq i \leq \Delta)$. Then

$$\sum_{v \in V(T)} \deg v = \sum_{i=1}^{\Delta} i n_i = 2m = 2n - 2 = 2\sum_{i=1}^{\Delta} n_i - 2. \tag{3.1}$$

Solving (3.1) for n_1, we have the following.

Theorem 3.14 *Let T be a tree of order $n \geq 3$ having maximum degree Δ and containing n_i vertices of degree i $(1 \leq i \leq \Delta)$. Then the number n_1 of leaves of T is given by*

$$n_1 = 2 + n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta.$$

If $s : d_1, d_2, \ldots, d_n$ is the degree sequence of a tree of order $n \geq 2$, then, necessarily, $\sum_{i=1}^{n} d_i = 2n - 2$. Here too, more can be said.

Theorem 3.15 *A sequence $s : d_1, d_2, \ldots, d_n$ of $n \geq 2$ positive integers is the degree sequence of a tree of order n if and only if*

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

Proof. First, let T be a tree of order n and size m with degree sequence $s : d_1, d_2, \ldots, d_n$. Then, as observed,

$$\sum_{i=1}^{n} d_i = 2m = 2(n-1) = 2n - 2.$$

We verify the converse by induction on n. When $n = 2$, the only sequence of two positive integers whose sum is $2n - 2 = 2$ is $1, 1$, which is the degree sequence of the tree K_2.

Assume for a given integer $n \geq 3$ that any sequence of $n-1$ positive integers whose sum is $2(n-2) = 2n-4$ is the degree sequence of a tree of order $n-1$. Now let $d_1, d_2, \ldots, d_n$ be a sequence of n positive integers whose sum is $2n-2$. We show that this is the degree sequence of a tree of order n. Suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$. Since each term is positive and $\sum_{i=1}^{n} d_i = 2n-2$, it follows that at least two terms in the sequence must be 1, and so $d_{n-1} = d_n = 1$. Furthermore, $2 \leq d_1 \leq n-1$. Hence, $d_1 - 1, d_2, \ldots, d_{n-1}$ is a sequence of $n-1$ positive integers whose sum is $2(n-1) - 2 = 2n - 4$. By the induction hypothesis, there is a tree T' of order $n-1$ with $V(T') = \{v_1, v_2, \ldots, v_{n-1}\}$ such that $\deg_{T'} v_1 = d_1 - 1$ and $\deg_{T'} v_i = d_i$ for $2 \leq i \leq n-1$. Let T be the tree of order n obtained from T' by adding a new vertex v_n and joining it to v_1. The tree T then has the degree sequence $d_1, d_2, \ldots, d_n$. ■

A graph without cycles is a **forest**. That is, a forest is an acyclic graph. Thus, each tree is a forest and every component of a forest is a tree. All of the graphs F_1, F_2 and F_3 in Figure 3.9 are forests but none are trees.

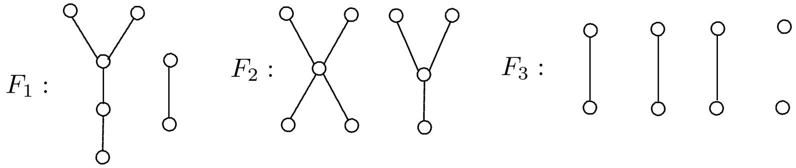


Figure 3.9: Forests

The following is an immediate corollary of Theorem 3.13.

Corollary 3.16 *The size of a forest of order n having k components is $n - k$.*

By Theorem 3.13, if G is a graph of order n and size m such that G is connected and has no cycles (that is, G is a tree), then $m = n-1$. It is easy to see that the converse of this statement is not true. However, if we were to add to the hypothesis of the converse either of the two defining properties of a tree, then the converse would be true.

Theorem 3.17 *Let G be a graph of order n and size m. If G has no cycles and $m = n-1$, then G is a tree.*

Proof. It remains only to show that G is connected. Suppose that the components of G are $G_1, G_2, \ldots, G_k$, where $k \geq 1$. Let n_i be the order of G_i $(1 \leq i \leq k)$ and m_i the size of G_i. Since each graph G_i is a tree, it follows by Theorem 3.13 that $m_i = n_i - 1$ and by Corollary 3.16 that $m = n-k$. Hence,

$$n - 1 = m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = n - k.$$

Thus, $k = 1$ and so G is connected. Therefore, G is a tree. ■

Theorem 3.18 *Let G be a graph of order n and size m. If G is connected and $m = n-1$, then G is a tree.*

Proof. Assume, to the contrary, that there exists some connected graph of order n and size $m = n-1$ that is not a tree. Necessarily then, G contains one or more cycles. By successively deleting an edge from a cycle in each resulting subgraph, a tree of order n and size less than $n-1$ is obtained. This contradicts Theorem 3.13. ■

Combining Theorems 3.13, 3.17 and 3.18, we have the following.

Theorem 3.19 *Let G be a graph of order n and size m. If G satisfies any two of the following three properties, then G is a tree:*

(1) *G is connected,* (2) *G has no cycles,* (3) $m = n-1$.

As one would anticipate, graphs often contain many subgraphs that are trees. In fact, for each tree of a fixed order, every graph whose vertices have sufficiently large degree contains a subgraph that is isomorphic to the tree.

Theorem 3.20 *Let T be a tree of order k. If G is a graph for which $\delta(G) \geq k-1$, then G contains a subgraph that is isomorphic to T.*

Proof. We proceed by induction on k. The result is obvious for $k = 1$ since K_1 is a subgraph of every graph and for $k = 2$ since K_2 is a subgraph of every nonempty graph.

Assume for every tree T' of order $k-1$ with $k \geq 3$ and for every graph G' with $\delta(G') \geq k-2$, that G' contains a subgraph that is isomorphic to T'. Now, let T be a tree of order k and let G be a graph with $\delta(G) \geq k - 1$. We show that G contains a subgraph that is isomorphic to T.

Let v be an end-vertex of T and let u be the vertex of T that is adjacent to v. Then $T - v$ is a tree of order $k - 1$. Since $\delta(G) \geq k - 1 > k - 2$, it follows by the induction hypothesis that G contains a subgraph T' that is isomorphic to $T - v$. Let u' denote the vertex of T' corresponding to u in $T - v$. Since $\deg_G u' \geq k - 1$ and the order of T' is $k - 1$, the vertex u' is adjacent to a vertex v' that does not belong to T'. The tree obtained by adding v' to T' and joining it to u' is isomorphic to T, completing the proof. ■

3.3 Spanning Trees

Although there is no closed formula for the number of non-isomorphic trees of a fixed order, a formula does exist for the number of distinct labeled trees of order n (whose vertices are labeled from a fixed set of cardinality n).

Two labelings of the same graph from the same set of labels are considered **distinct labelings** if they produce different edge sets. Figure 3.10 shows three labelings of a graph of order 9 from the set $\{1, 2, \ldots, 9\}$. Since the first two labelings produce the same edge set, these two labelings are considered the same. The third labeling is different from the first two, however, since 26 is an edge there while 26 is not an edge in either of the first two labelings.

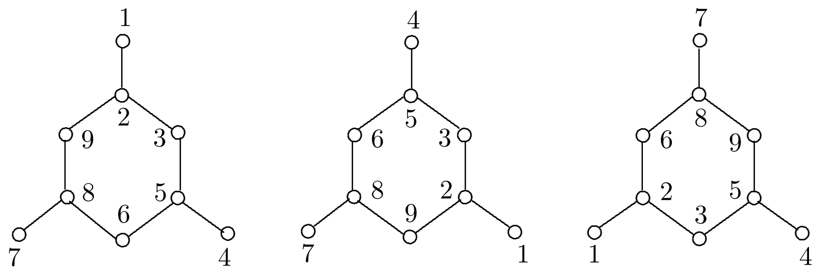


Figure 3.10: Labelings of a graph

There are three labeled trees of order 3 (whose vertices are labeled from the same set of three labels) and there are 16 labeled trees of order 4 (whose vertices are labeled from the set of four labels). These 19 trees are shown in Figure 3.11, where the vertex sets are $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$.

As we will see, the number of distinct labeled trees of order n whose vertices are labeled from the same set of n labels is n^{n-2}. Thus, the number of labeled trees of order 3 is $3^1 = 3$ and the number of labeled trees of order 4 is $4^2 = 16$, as mentioned above. This formula for the number of labeled trees of order n is due to the famous British mathematician Arthur Cayley, whom we have encountered earlier and will encounter again. While Cayley discovered this result in 1889, the proof we present was given in 1918 by Heinz Prüfer [191], a German mathematician known for his work on abelian groups.

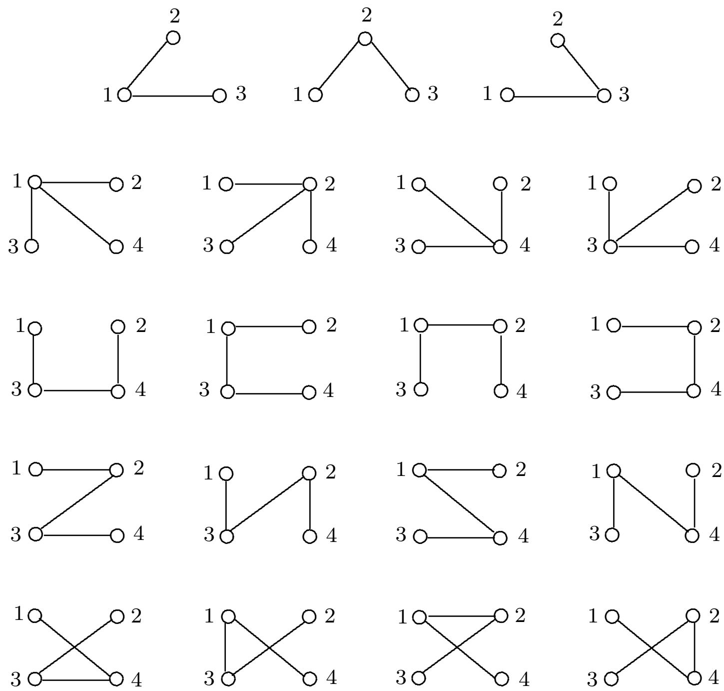
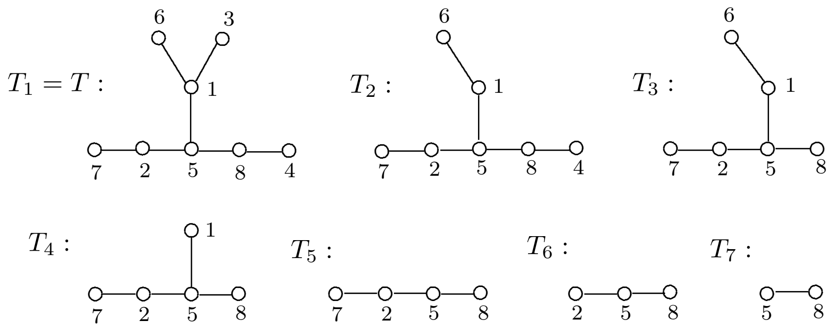


Figure 3.11: Labeled trees of orders 3 and 4

Prüfer Codes

Prüfer's proof of Cayley's result (often called *Cayley's Tree Formula*) consists of establishing a one-to-one correspondence between the trees of order n with the same vertex set, say $\{1, 2, \ldots, n\}$, and the sequences (called **Prüfer codes**) of length $n-2$ whose entries come from the set $\{1, 2, \ldots, n\}$. Since the number of such sequences is n^{n-2}, the proof is complete once the one-to-one correspondence has been verified.

Before giving a proof of Cayley's Tree Formula, we present an example to illustrate the technique employed. Consider the tree T of order $n = 8$ in Figure 3.12 whose vertices are labeled with elements of the set $\{1, 2, \ldots, 8\}$. The first term of the Prüfer code of $T_1 = T$ is the neighbor of the end-vertex of T_1 having the smallest label. This end-vertex is then deleted, producing a new tree T_2. The second term of the Prüfer code for T is the neighbor of the end-vertex of T_2 having the smallest label in T_2. We continue in this manner until we arrive at $T_{n-2} = K_2$. The resulting sequence of length $n-2$ is the Prüfer code for T, which in this case is $(1, 8, 1, 5, 2, 5)$.



Prüfer code for T : $(1, 8, 1, 5, 2, 5)$

Figure 3.12: Determining the Prüfer codes for a tree

In this example, every vertex v of T appears in its Prüfer code $\deg v - 1$ times. This is true in general. Therefore, no end-vertex of T appears in the Prüfer code for T. So if T is a tree of order n and size m, then the number of terms in its Prüfer code is

$$\sum_{v \in V(T)} (\deg v - 1) = 2m - n = 2(n-1) - n = n - 2.$$

We now consider the converse question. Suppose that $s = (a_1, a_2, \ldots, a_{n-2})$ is a sequence of length $n - 2$ where $a_i \in \{1, 2, \ldots, n\}$ for each i $(1 \leq i \leq n-2)$. We construct a labeled tree T of order n with vertex set $\{1, 2, \ldots, n\}$ such that the given sequence s is the Prüfer code of T. For example, suppose that the given sequence is $s = (1, 8, 1, 5, 2, 5)$. This sequence has length $n - 2 = 6$ and so $n = 8$. The smallest element of $\{1, 2, \ldots, 8\}$ not appearing in this sequence is 3. We join vertex 3 to the first term (vertex) of the sequence, that is, vertex 3 is joined to vertex 1. The first term of the sequence is then deleted, producing the reduced sequence $(8, 1, 5, 2, 5)$. Also, the element 3 is removed from the set $\{1, 2, \ldots, 8\}$. The smallest element of the reduced set $\{1, 2, 4, 5, 6, 7, 8\}$ not appearing in $(8, 1, 5, 2, 5)$ is 4, which is joined to vertex 8, the first term of the sequence $(8, 1, 5, 2, 5)$. This procedure is continued until only two elements of the final reduced set remain. These two vertices (5 and 8 in this case) are joined and a tree T has been constructed whose Prüfer code is s. This is illustrated in Figure 3.13.

In general then, to construct the Prüfer code $(a_1, a_2, \ldots, a_{n-2})$ of a tree $T = T_1$ of order n whose vertices are labeled with elements of the set $\{1, 2, \ldots, n\}$, we apply the following algorithm.

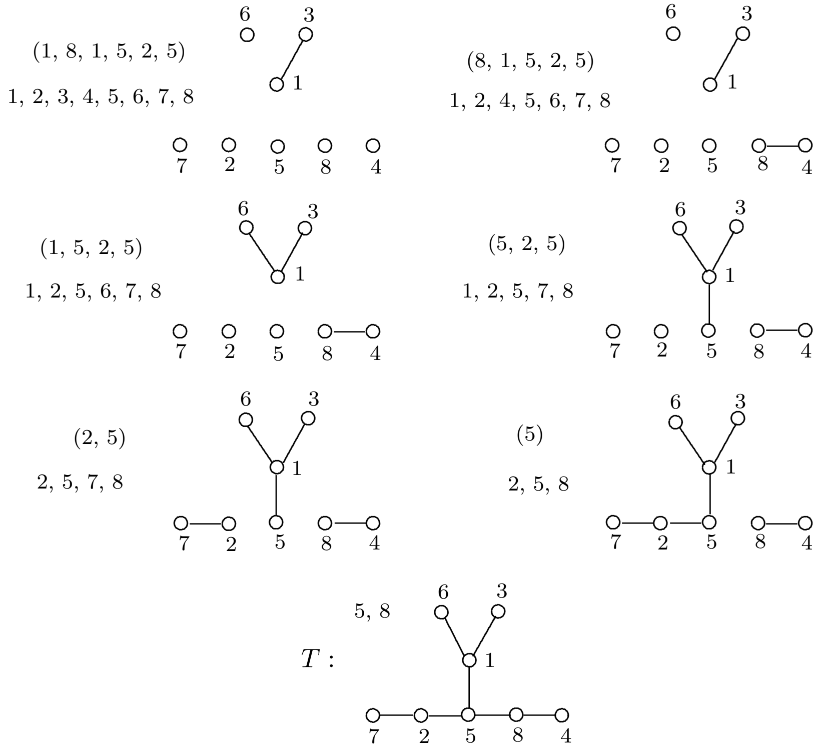


Figure 3.13: Constructing a tree with a given Prüfer code

Algorithm 3.21 *Construct the Prüfer code of a labeled tree with vertex set* $\{1, 2, \ldots, n\}$.

Input: An integer $n \geq 3$ and a tree $T = T_1$ with vertex set $\{1, 2, \ldots, n\}$.

Output: The Prüfer code $(a_1, a_2, \ldots, a_{n-2})$ of T.

1. For $i = 1$ to $n - 2$
   1.1. Locate the smallest leaf v_i of T_i and let a_i be the neighbor of v_i in T_i.
   1.2. Let $T_{i+1} = T_i - v_i$.
2. Output $(a_1, a_2, \ldots, a_{n-2})$.

For a fixed set $S = S_1$ of n positive integers and for a sequence $s = s_1$ of length $n - 2$ whose terms belong to S_1, a graph G of order n with $V(G) = S_1$ is constructed by following the algorithm below. We may assume that $S_1 = \{1, 2, \ldots, n\}$ and $s_1 = (a_1, a_2, \ldots, a_{n-2})$, where $a_i \in S_1$ for $1 \leq i \leq n - 2$.

Algorithm 3.22 *Given a sequence s_1 of length $n-2$ whose terms belong to the set $S_1 = \{1, 2, \ldots, n\}$, construct a tree with vertex set S_1 whose Prüfer code is s_1.*

Input: An integer $n \geq 3$, $S_1 = \{1, 2, \ldots, n\}$ and a sequence $s_1 = (a_1, a_2, \ldots, a_{n-2})$, where $a_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq n-2$.

Output: A graph G with vertex set $\{1, 2, \ldots, n\}$.

1. For $i = 1$ to $n-2$,
   1.1. Let k_i be the smallest element of S_i not appearing in s_i and let $e_i = k_i a_i$.
   1.2. Let $S_{i+1} = S_i - \{k_i\}$.
   1.3. For $i \leq n-3$, let $s_{i+1} = (a_{i+1}, a_{i+2}, \ldots, a_{n-2})$.
2. For the two elements $x, y \in S_{n-1}$, let $e_{n-1} = xy$.
3. Output $E(G) = \{e_1, e_2, \ldots, e_{n-2}\}$.

Cayley's Tree Formula

Algorithm 3.21 shows that the Prüfer code of a tree with vertex set $\{1, 2, \ldots, n\}$ is a sequence of length $n-2$ whose terms belong to $\{1, 2, \ldots, n\}$, while Algorithm 3.22 constructs a graph of order n (which will be shown to be a tree) whose Prüfer code is a given sequence. We now use Prüfer codes to verify Cayley's Tree Formula [44].

Theorem 3.23 (**Cayley's Tree Formula**) *For each positive integer n, there are n^{n-2} distinct labeled trees of order n having the same vertex set.*

Proof. The result is obvious for $n = 1$ and $n = 2$. For $n \geq 3$, we show by induction that there is a one-to-one correspondence between the set of distinct labeled trees of order n having a fixed vertex set S of n positive integers and the set of sequences of length $n-2$ whose terms belong to S. By Algorithm 3.21, the Prüfer code of every tree of order n with vertex set S is a sequence of length $n-2$ whose terms belong to S. The proof will be complete once we show that for each such sequence only one tree of order n with vertex set S has this sequence as its Prüfer code.

For $n = 3$, let S be a set of three positive integers, say $S = \{1, 2, 3\}$. There are three trees with vertex set S, which are shown in Figure 3.11. These trees have Prüfer codes (1), (2) and (3), respectively. Thus, the result is true for $n = 3$.

Assume for $n \geq 4$, that for each sequence s_0 of length $n-3$ whose terms belong to a fixed $(n-1)$-element set S_0 of positive integers that Algorithm 3.22

constructs a unique tree of order $n-1$ with vertex set S_0 whose Prüfer code is s_0.

Now let $S = \{1, 2, \ldots, n\}$ and let s be a sequence of length $n-2$ whose terms belong to S. Let G be the graph constructed by Algorithm 3.22. The goal then is to show that G is the unique tree with Prüfer code s.

Let k be the smallest element of S that does not belong to s. By Algorithm 3.22, ka_1 is an edge of the graph G constructed. By the induction hypothesis, there is a unique tree T' with vertex set $S' = S - \{k\}$ having Prüfer code $s' = (a_2, a_3, \ldots, a_{n-2})$. Since G is obtained from T' by adding a new vertex k to T' and joining k to a_1, the graph G is a tree that is uniquely determined. Furthermore, the vertex set of G is S and its Prüfer code is s. ■

A **spanning tree** of a graph G is a spanning subgraph of G that is a tree. A graph G can only contain a spanning tree if G is connected. Since the size of every tree of order n is $n-1$, it follows that if G is a connected graph of order n and size m, then $m \geq n-1$.

Spanning trees of a connected graph G can be obtained in a variety of ways. One possibility is to begin with $V(G)$. To construct the edge set $E(T)$ of a spanning tree T of G, we begin by selecting an edge e_1 of G. Next select an edge e_2 of G that is distinct from e_1. This is followed by selecting an edge e_3 distinct from those previously selected and that does not produce a cycle with those previously selected. We continue in this manner until an edge e_{n-1} is selected. Then $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$.

Another way to produce a spanning tree T of a connected graph G is to begin with G. If G is a tree, then G itself is a spanning tree. Otherwise, let f_1 be an edge on a cycle of G and remove it. Let $G_1 = G - f_1$. If G_1 is a tree, then G_1 is a spanning tree of G. Otherwise, let f_2 be an edge on a cycle of G_1 and remove it. Let $G_2 = G_1 - f_2$ and continue in this manner until no cycle remains. The spanning subgraph T resulting in this manner is a spanning tree of G. Hence, $m - (n-1) = m - n + 1$ edges must be deleted from G to obtain T. The number $m - n + 1$ is referred to as the **cycle rank** of G. Thus, a tree has cycle rank 0.

If G is a connected graph of order n and size m having cycle rank 1, then $m - n + 1 = 1$ and so $n = m$. Such a graph is therefore a connected graph with exactly one cycle. These graphs are called **unicyclic graphs**. All of the graphs in Figure 3.14 are unicyclic.

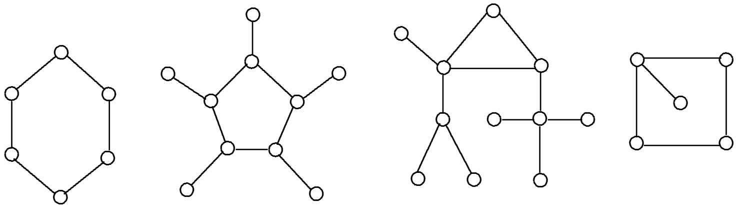


Figure 3.14: Unicyclic graphs

Still another way to construct a spanning tree T of G is to begin with a vertex v of G. Suppose that the eccentricity $e(v)$ of v is k. For $i = 0, 1, \ldots, k$, let

$$A_i = \{u \in V(G) : d(v, u) = i\}.$$

Each vertex $w \in A_i$, where $1 \leq i \leq k$, is adjacent to one or more vertices in A_{i-1}. Select exactly one such vertex $x \in A_{i-1}$ and let $wx \in E(T)$. Then T is a spanning tree with the property that $d_T(v, u) = d_G(v, u)$ for each $u \in V(G)$. Such a spanning tree T is then **distance-preserving** from v.

For the graph G of Figure 3.15, the trees T_1 and T_2 are spanning trees of G. The tree T_2 is distance-preserving from u. The proof of the next result of Lesniak [155] uses the concept of a distance-preserving tree to obtain an upper bound for the radius of a connected graph.

Theorem 3.24 *If G is a connected graph of order n, then* $\text{rad}(G) \leq n/2$.

Proof. Let u be a central vertex of G and let T be a spanning tree of T that is distance-preserving from u. Then $e_T(u) = e_G(u)$. Furthermore, for every vertex z of G, we have $e_G(z) \leq e_T(z)$. Thus, $\text{rad}(T) = \text{rad}(G)$ and so it suffices to show that $\text{rad}(T) = e_T(u) \leq n/2$.

Suppose, to the contrary, that $e_T(u) \geq (n + 1)/2$. Necessarily, $n \geq 3$. Moreover, u is a cut-vertex of T. Since $e_T(u) \geq (n+1)/2$, the forest $T - u$ has a component T' of order at least $(n+1)/2$. Let v be the unique vertex of T' that is adjacent to u in T. For each vertex w in T', we have $d_T(v, w) = d_T(u, w) - 1$, which implies that $d_T(v, w) < e_T(u)$. For each vertex w of T not in T', we have $d_T(v, w) = d_T(u, w) + 1$. Because there are at most $(n - 1)/2$ vertices of T that are not in T', it follows that $d_T(u, w) \leq (n - 3)/2$. Therefore, for every vertex v of T, we have

$$d_T(v, w) \leq (n - 1)/2 < e_T(u).$$

However then, $e_T(v) < e_T(u)$, contradicting the fact that u is a central vertex of T. ■

Edge Exchanges

Returning to the graph G in Figure 3.15, we note that the spanning tree T_1 of G has three leaves and the spanning tree T_2 has five leaves. Furthermore, no spanning tree of G has fewer than three or more than five leaves. According to a result due to Seymour Schuster [217], there must be a spanning tree of G with exactly four leaves.

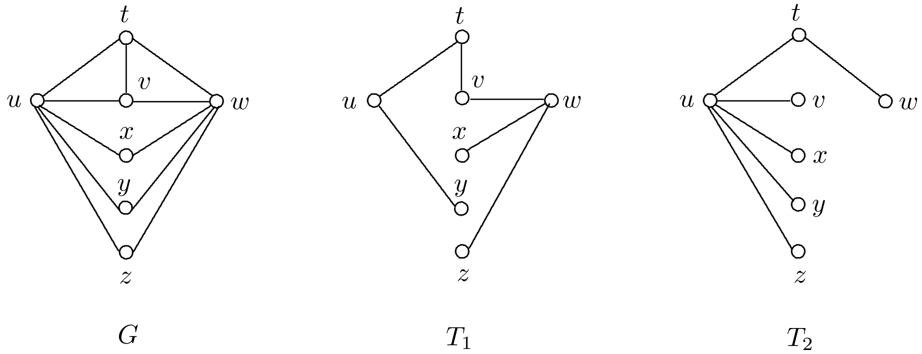


Figure 3.15: Two spanning trees T_1 and T_2 in a graph

If u and v are two nonadjacent vertices of a tree T_0, then $T_0 + uv$ contains exactly one cycle C and the cycle C contains the edge uv. For any edge xy belonging to C, the graph $T_1 = T_0 + uv - xy$ is again a tree. The tree T_0 is said to be transformed into T_1 by an **edge exchange**. This transformation from T_0 to T_1 is illustrated in Figure 3.16.

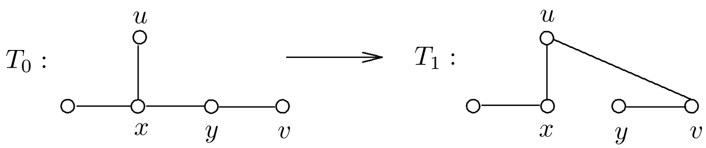


Figure 3.16: Transforming a tree T_0 into a tree T_1 by an edge exchange

Theorem 3.25 *Let G be a connected graph. If G contains a spanning tree with exactly r leaves and a spanning tree with exactly t leaves where $r < t$, then for every integer s with $r < s < t$, there is a spanning tree of G with exactly s leaves.*

Proof. Let T_0 be a spanning tree of G with r leaves and let T be a spanning tree of G with t leaves. Clearly, $E(T_0) \neq E(T)$. Let e be an edge of T not belonging to T_0. Then $T_0 + e$ is a unicyclic graph whose cycle C contains an edge f not belonging to T. Thus T_0 is transformed into the spanning tree $T_1 = T_0 + e - f$ by an edge exchange where T_1 has one more edge in common with T than T_0 does. If $T_1 = T$, then the transformation is complete; otherwise, T_1 can be transformed into a spanning tree T_2 having one more edge in common with T than T_1 does. We continue this until we arrive at T, that is, we have a sequence

$$T_0, T_1, T_2, \ldots, T_k = T$$

of spanning trees of G where for each i $(1 \leq i \leq k)$, T_{i-1} is transformed into T_i by an edge exchange and T_i has one more edge in common with T than T_{i-1}

does. Suppose that the tree T_i $(0 \leq i \leq k)$ has a_i leaves. Thus $a_0 = r$ and $a_k = t$.

In the process of transforming a tree T_{j-1} into T_j $(1 \leq j \leq k)$ by an edge exchange, some edge uv is added to T_{j-1} and some edge xy is removed from $T_{j-1} + uv$, that is, $T_j = T_{j-1} + uv - xy$. (It is possible that one of u and v is the same as one of x and y.) Depending on the degrees of u and v in T_{j-1} and the degrees of x and y in $T_{j-1} + uv$, the number $a_j - a_{j-1}$ can be any of the integers $-2, -1, 0, 1$ or 2.

If the number s does not appear in the sequence $r = a_0, a_1, \ldots, a_k = t$, then it must occur that $a_{j-1} = s - 1$ and $a_j = s + 1$ for some integer j with $1 \leq j \leq k$. Suppose that $T_j = T_{j-1} + uv - xy$. Since $a_j = a_{j-1} + 2$, it follows that the vertices u, v, x and y are distinct and that

$$\deg_{T_{j-1}} u \geq 2,\ \deg_{T_{j-1}} v \geq 2,\ \deg_{T_{j-1}} x = 2 \text{ and } \deg_{T_{j-1}} y = 2.$$

Observe that the unicyclic graph $T_{j-1} + uv$ has a_{j-1} leaves. Now both u and x lie on the cycle C in $H = T_{j-1} + uv$ and $\deg_H u \geq 3$ and $\deg_H x = 2$. Hence, on a $u-x$ path on C in H, there are adjacent vertices w and z with $\deg_H w \geq 3$ and $\deg_H z = 2$. Deleting wz from H produces a spanning tree T' with $a_{j-1} + 1 = s$ leaves. ■

The Matrix-Tree Theorem

Cayley's Tree Formula, which gives the number of labeled trees of a given order, has another interpretation. As a consequence of this formula, there are n^{n-2} distinct spanning trees of the labeled graph K_n. This brings up the question of determining the number of distinct spanning trees of labeled graphs in general. An answer to this question has been given as a determinant of a matrix. This result, implicit in the work [142] of Gustav Kirchhoff, is known as the *Matrix-Tree Theorem.*

Kirchhoff is well known for his research on electrical currents, which he announced in 1845. This led to Kirchhoff's laws, the first of which states that the sum of the currents into a vertex equals the sum of the currents out of the vertex. Two years later, in 1847, he graduated from the University of Königsberg. It was during that year that he published the paper that led to his theorem on counting spanning trees. Kirchhoff spent much of his life working on experimental physics.

The proof we give of the Matrix-Tree Theorem will employ several results from matrix theory. Let M be an $r \times s$ matrix and M' an $s \times r$ matrix with $r \leq s$. The product $M \cdot M'$ is therefore an $r \times r$ matrix. Since $M \cdot M'$ is a square matrix, its determinant $\det(M \cdot M')$ exists. An $r \times r$ submatrix M_0 of M is said to correspond to the $r \times r$ submatrix M'_0 of M' if the column numbers of M determining M_0 are the same as the row numbers of M' determining M'_0.

For example, suppose that $r = 2$ and $s = 3$ and

$$M = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \text{ and } M' = \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & 2 \end{bmatrix}. \tag{3.2}$$

Then the following 2×2 matrices M_0 and M_0' of M and M', respectively, correspond to each other:

$$M_0 = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \text{ and } M_0' = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}.$$

A result from matrix theory states that

$$\det(M \cdot M') = \sum (\det M_0)(\det M_0'), \tag{3.3}$$

where the sum is taken over all $r \times r$ submatrices M_0 of M and where M_0' is the $r \times r$ submatrix corresponding to M_0. The numbers $\det(M_0)$ and $\det(M_0')$ are referred to as **major determinants** of M and M', respectively.

For the matrices M and M' in (3.2),

$$M \cdot M' = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 4 & 6 \end{bmatrix}.$$

Thus $\det(M \cdot M') = (-4) \cdot 6 - 4 \cdot 3 = -36$. From the result in matrix theory mentioned above, we also have

$$\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 3 \\ 0 & 4 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = -36.$$

Suppose that A is an $n \times n$ matrix for some $n \geq 3$. Let A' be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting row i and column j from A, where $1 \leq i, j \leq n$. Then $(-1)^{i+j} \det(A')$ is a **cofactor** of A. For the 3×3 matrix A in Figure 3.17, A' is the 2×2 submatrix obtained by deleting row 3 and column 3 of A, while A'' is the 2×2 submatrix obtained by deleting row 1 and column 2 of A. Then these two cofactors of A are $(-1)^{3+3} \det(A')$ and $(-1)^{1+2} \det(A'')$.

Observe that both cofactors of A in Figure 3.17 have the value 7. Also, observe that every row sum and column sum of A is 0. It is a theorem of matrix theory that whenever each row sum and column sum of a square matrix M is 0, then all cofactors of M have the same value. If M is a square matrix whose rows (or columns) are linearly dependent, then $\det(M) = 0$.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **degree matrix** $D(G) = [d_{ij}]$ is the $n \times n$ matrix with

$$d_{ij} = \begin{cases} \deg v_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 2 & -3 \\ -4 & -1 & 5 \end{bmatrix}$$

$$A' = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \qquad (-1)^{3+3}\det(A') = 3(2) - (-1)1 = 7$$

$$A'' = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix} \qquad (-1)^{1+2}\det(A'') = -[1(5) - (-3)(-4)] = 7$$

Figure 3.17: Computing cofactors

Theorem 3.26 **(The Matrix-Tree Theorem)** *If G is a nontrivial labeled graph with adjacency matrix A and degree matrix D, then the number of distinct spanning trees of G is the value of any cofactor of the matrix $D - A$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. First, observe that the row sum of row i (or column sum of column i) in A is $\deg v_i$, so every row sum or column sum of $D - A$ is 0. Consequently, the cofactors of $D - A$ have the same value.

Assume first that G is a disconnected graph. Of course in this case, G has no spanning trees. Let G_1 be a component of G and suppose that $V(G_1) = \{v_1, v_2, \ldots, v_r\}$, where $1 \le r < n$. Let M be the $(n-1) \times (n-1)$ submatrix of $D - A$ obtained by deleting row n and column n from $D - A$. Since the sum of the first r rows of M is the zero vector of length $n - 1$, the rows of M are linearly dependent and so $\det(M) = 0$, as desired.

Henceforth, we assume that G is a connected graph of order n and size m where $E(G) = \{e_1, e_2, \ldots, e_m\}$. Thus $m \ge n - 1$. Let $C = [c_{ij}]$ be an $n \times m$ matrix where $c_{ij} = 1$ or $c_{ij} = -1$ if v_i is incident with e_j and such that each column has one entry that is 1 and one entry that is -1, while all other entries in the column are 0. We show that for the transpose C^t of C, we have $C \cdot C^t = D - A$. The (i, j)-entry of $C \cdot C^t$ is

$$\sum_{k=1}^{m} c_{ik} c_{jk},$$

which has the value $\deg v_i$ if $i = j$, the value -1 if $i \ne j$ and $v_i v_j \in E(G)$ and the value 0 if $i \ne j$ and $v_i v_j \notin E(G)$. Hence, as claimed, $C \cdot C^t = D - A$.

Consider a spanning subgraph H of G containing $n - 1$ edges of G. Let C' be the $(n-1) \times (n-1)$ submatrix of C determined by the columns associated with the edges of H and by all rows of C with one exception, say row k.

We now determine the absolute value $|\det(C')|$ of the determinant of C'. If H is disconnected, then H has a component H_1 not containing v_k. The sum of the row vectors of C' corresponding to the vertices of H_1 is the zero vector of length $n - 1$. Hence the row vectors in C' are linearly dependent and so $|\det(C')| = 0$.

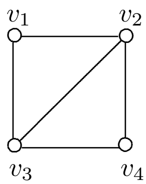
Next, assume that H is connected. Thus, H is a spanning tree of G. Let u_1 be an end-vertex of H that is distinct from v_k and let f_1 be the edge of H that is incident with u_1. In the tree $H - u_1$, let u_2 be an end-vertex distinct from v_k and let f_2 be the edge of $H - u_1$ that is incident with u_2. This procedure is continued until only the vertex v_k remains.

A matrix $C'' = \left[c''_{ij}\right]$ can now be obtained by a permutation of the rows and columns of C' such that $|c''_{ij}| = 1$ if and only if u_i and f_j are incident. From the manner in which C'' is defined, any vertex u_i is incident only with edges f_j with $j \le i$. This, however, implies that C'' is a lower triangular matrix and since $|c''_{ii}| = 1$ for all i, we conclude that $|\det(C'')| = 1$. Consequently, $|\det(C')| = |\det(C'')| = 1$.

Since every cofactor of $D - A$ has the same value, it suffices to evaluate the determinant of the matrix obtained by deleting both row i and column i from $D - A$ for some i ($1 \le i \le n$). Let C_i denote the matrix obtained from C by removing row i. Then the cofactor mentioned above equals $\det(C_i \cdot C_i^t)$, which implies by (3.3) that this number is the sum of the products of the corresponding major determinants of C_i and C_i^t. However, these corresponding major determinants have the same value and so their product is 1 if the defining columns correspond to a spanning tree and 0 otherwise. ■

Cayley's Tree Formula (Theorem 3.23) is then a corollary of the Matrix-Tree Theorem (see Exercise 58). We illustrate the Matrix-Tree Theorem for the graph G of Figure 3.18 where the matrices D and $D - A$ are also shown.

G :



$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad D - A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Figure 3.18: Illustrating the Matrix-Tree Theorem

To calculate a cofactor of $D - A$, we delete the entries in row i and column j for some i and j with $1 \le i, j \le 4$ and compute the product of $(-1)^{i+j}$ and the determinant of the resulting submatrix. For example, the cofactor of the $(2,3)$-entry in the matrix $D - A$ in Figure 3.18 is

$$(-1)^{2+3} \begin{vmatrix} 2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 2 \end{vmatrix}.$$

Expanding this determinant along the first row, we obtain

$$\begin{aligned} & -\left(2\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}\right) \\ = \; & -[2(-3) + 1(-2) + 0] = 8. \end{aligned}$$

Consequently, there are eight distinct spanning trees of the labeled graph G of Figure 3.18, all of which are shown in Figure 3.19.

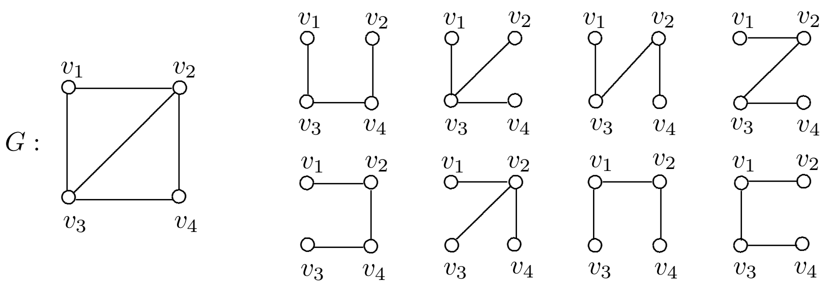


Figure 3.19: The distinct spanning trees of a graph

3.4 The Minimum Spanning Tree Problem

Let G be a connected graph each of whose edges is assigned a real number (called the **cost** or **weight** of the edge). We denote the weight of an edge e of G by $w(e)$. Such a graph is called a **weighted graph**. For each subgraph H of G, the **weight** $w(H)$ of H is defined as the sum of the weights of its edges, that is,

$$w(H) = \sum_{e \in E(H)} w(e).$$

We seek a spanning tree of G whose weight is minimum among all spanning trees of G. Such a spanning tree is called a **minimum spanning tree**. The problem of finding a minimum spanning tree in a connected weighted graph is called the **Minimum Spanning Tree Problem**.

The importance of the Minimum Spanning Tree Problem is due to its applications in the design of computer, communications and transportation networks. The history of this problem was researched by Ronald L. Graham and Pavol Hell [107] in 1985. They concluded that the Minimum Spanning Tree Problem was initially formulated by Otakar Borůvka [35] in 1926 because of his interest in the most economical layout of a power-line network. He also gave the first solution of the problem. Prior to Borůvka, however, the anthropologist Jan Czekanowski's work on classification schemes led him to consider ideas closely related to the Minimum Spanning Tree Problem.

Kruskal's Algorithm

Over the years, this problem has been solved in a variety of ways using a number of algorithms. One of the best known was discovered by Joseph Bernard Kruskal [150] in 1956. It was only two years after completing his doctoral degree that the paper containing the algorithm that bears his name was published.

Kruskal's Algorithm is an example of what is often called a **greedy algorithm**. Such an algorithm is, informally, a procedure that selects the best current option at each step without regard to future consequences.

Algorithm 3.27 (**Kruskal's Algorithm**) *For a connected weighted graph G of order n, a spanning tree T of G is constructed as follows: For the first edge e_1 of T, we select any edge of G of minimum weight. For subsequent edges $e_2, e_3, \ldots, e_{n-1}$, we select an edge of minimum weight among the remaining edges of G that does not produce a cycle with the previously selected edges.*

Figure 3.20 shows how a spanning tree T of a connected weighted graph G is constructed using Kruskal's algorithm. We now show that Kruskal's algorithm produces a minimum spanning tree in every connected weighted graph.

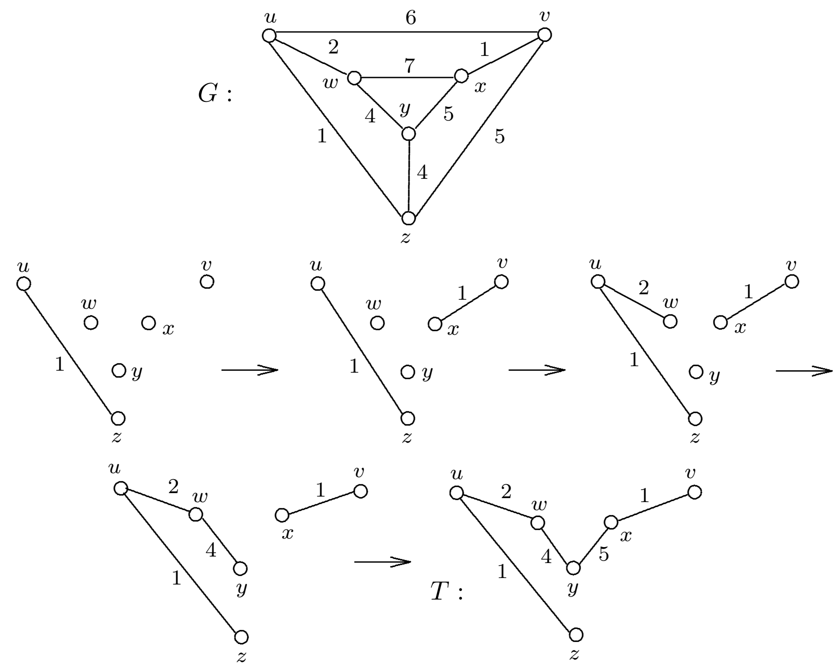


Figure 3.20: Constructing a spanning tree by Kruskal's algorithm

Theorem 3.28 *Kruskal's algorithm produces a minimum spanning tree in a connected weighted graph.*

Proof. Let G be a connected weighted graph of order n and let T be a spanning tree obtained by Kruskal's algorithm, where the edges of T are selected in the order e_1, e_2, ..., e_{n-1}. Necessarily then, $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_{n-1})$ and the weight of T is

$$w(T) = \sum_{i=1}^{n-1} w(e_i).$$

We show that T is a minimum spanning tree of G. Assume, to the contrary, that T is not a minimum spanning tree. Among all minimum spanning trees of G, let H be one that has a maximum number of edges in common with T. Since $H \neq T$, there is at least one edge of T that is not in H. Let e_i be the first edge of T that is not in H. Therefore, if $i > 1$, then the edges e_1, e_2, ..., e_{i-1} belong to both H and T. Now define $G_0 = H + e_i$. Then G_0 has a cycle C. Since T has no cycle, there is an edge e_0 on C that is not in T. The graph $T_0 = G_0 - e_0$ is therefore a spanning tree of G and

$$w(T_0) = w(H) + w(e_i) - w(e_0).$$

Since H is a minimum spanning tree of G, it follows that $w(H) \leq w(T_0)$. Thus, $w(H) \leq w(H) + w(e_i) - w(e_0)$ and so $w(e_0) \leq w(e_i)$. By Kruskal's algorithm, certainly $w(e_0) = w(e_i)$ if $i = 1$. Suppose then that $i > 1$. By Kruskal's algorithm, e_i is an edge of minimum weight that can be added to the edges $e_1, e_2, \ldots, e_{i-1}$ without producing a cycle. However, e_0 can also be added to $e_1, e_2, \ldots, e_{i-1}$ without producing a cycle. Thus $w(e_i) \leq w(e_0)$, which implies that $w(e_i) = w(e_0)$ when $i > 1$ as well. Therefore, $w(T_0) = w(H)$ and so T_0 is also a minimum spanning tree of G. However, T_0 has more edges in common with T than H does, which is a contradiction. ■

In fact, every minimum spanning tree in a connected weighted graph can be obtained by Kruskal's algorithm (see Exercise 70).

Prim's Algorithm

Another well-known algorithm for finding a minimum spanning tree in a connected weighted graph was developed by Robert Clay Prim [190] in 1957, although it had essentially been discovered by Vojtěch Jarník [136] in 1930.

Algorithm 3.29 (**Prim's Algorithm**) *For a connected weighted graph G of order n, a spanning tree T of G is constructed as follows: For an arbitrary vertex u for G, an edge of minimum weight incident with u is selected as the first edge e_1 of T. For subsequent edges $e_2, e_3, \ldots, e_{n-1}$, we select an edge of minimum weight among those edges having exactly one of its vertices incident with an edge already selected.*

Figure 3.21 illustrates how to construct a spanning tree T of a connected weighted graph G by Prim's algorithm. Again, a tree obtained by Prim's algorithm is also a minimum spanning tree, as we show next.

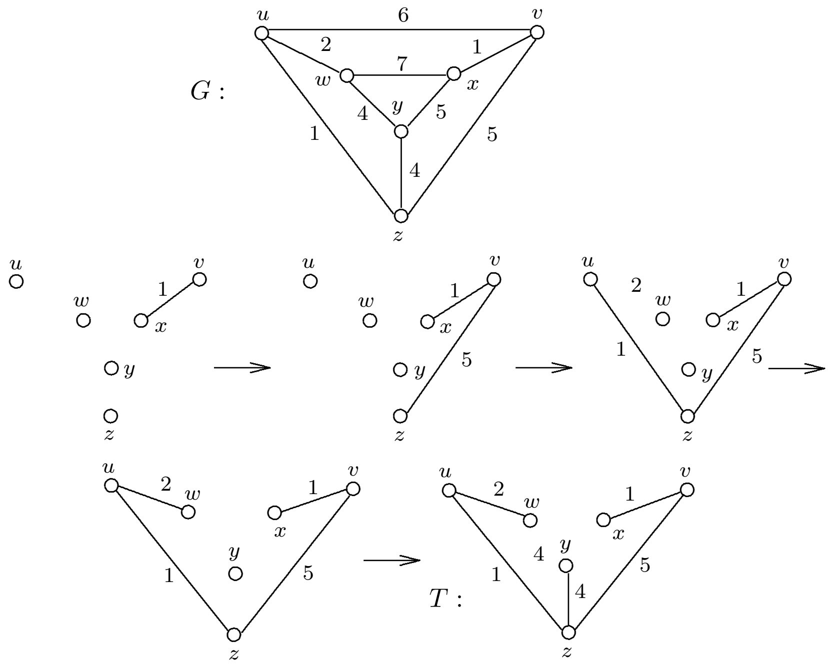


Figure 3.21: Constructing a spanning tree by Prim's algorithm

Theorem 3.30 *Prim's algorithm produces a minimum spanning tree in a connected weighted graph.*

Proof. Let G be a nontrivial connected weighted graph of order n and let T be a spanning tree obtained by Prim's Algorithm, where the edges of T are selected in the order e_1, e_2, …, e_{n-1}, and where e_1 is incident with a given vertex u. Thus the weight of T is

$$w(T) = \sum_{i=1}^{n-1} w(e_i).$$

Assume, to the contrary, that T is not a minimum spanning tree. Let $\mathcal{H}$ be the set of all minimum spanning trees of G having a maximum number of edges in common with T. By assumption, $T \notin \mathcal{H}$. If no tree in $\mathcal{H}$ contains e_1, let H be any tree in $\mathcal{H}$; otherwise, let H be a tree in $\mathcal{H}$ that contains the edges e_1, e_2, …, e_k for some integer k with $1 \le k < n-1$ but for which no tree in $\mathcal{H}$ contains all of the edges e_1, e_2, …, e_{k+1}. Hence no tree in $\mathcal{H}$ contains all of the edges e_1, e_2, …, e_{k+1}, where $0 \leq k < n-1$. If $k = 0$, let $U = \{u\}$; while if $k \geq 1$, let U be the set of $k+1$ vertices that are incident with one or more of the edges $e_1, e_2, \ldots, e_k$. By Prim's Algorithm, e_{k+1} joins a vertex of U and a vertex of $V(T) - U$.

The subgraph $H + e_{k+1}$ therefore contains a cycle C and e_{k+1} is on C. Necessarily, C contains an edge e_0 distinct from e_{k+1} such that e_0 also joins a vertex of U and a vertex of $V(T) - U$. By the construction of T from Prim's Algorithm, $w(e_{k+1}) \leq w(e_0)$. Now $T' = H + e_{k+1} - e_0$ is a spanning tree of G whose weight is

$$w(T') = w(H) + w(e_{k+1}) - w(e_0).$$

Since H is a minimum spanning tree, $w(H) \leq w(T')$ and so $w(H) \leq w(H) + w(e_{k+1}) - w(e_0)$, which implies that $w(e_0) \leq w(e_{k+1})$. Consequently, $w(e_0) = w(e_{k+1})$ and $w(H) = w(T')$. Therefore, T' is also a minimum spanning tree of G. If e_0 does not belong to T, then T' is a minimum spanning tree having more edges in common with T than H does, which is impossible since $H \in \mathcal{H}$. Hence e_0 belongs to T, which implies that T' has the same number of edges in common with T as H does and so $T' \in \mathcal{H}$. Necessarily, $e_0 = e_j$ for some $j > k+1$. Since T' contains all of the edges e_1, e_2, …, e_{k+1}, this contradicts the defining property of H. ■

Complexity of an Algorithm

While it is essential that an algorithm solve the problem under consideration, another important characteristic of an algorithm concerns its efficiency. The **complexity** of an algorithm refers to the number of basic computational steps (ordinarily arithmetic operations and comparisons) required to execute the algorithm. This number depends on the nature of its input as well as the size of the input. For a graph of order n and size m, the complexity typically depends on n and/or m. If the complexity is bounded above by a polynomial in its input size, then it is called a **polynomial-time algorithm**. For example, if the input is a graph of order n and the complexity of an algorithm is at most some constant times n^p for some positive integer p, then the algorithm is a polynomial-time algorithm and we denote the complexity by $O(n^p)$. An algorithm of complexity $O(n)$ is a **linear-time algorithm**, while an algorithm of complexity $O(n^2)$ is a **quadratic-time algorithm**.

Polynomial-time algorithms are considered efficient as these algorithms are typically computationally feasible even when applied to graphs of large order. The complexity of Kruskal's algorithm is $O(m^2)$ and that of Prim's algorithm is $O(n^3)$.

Exercises for Chapter 3

Section 3.1. Nonseparable Graphs

1. Let G be a nontrivial connected graph and let $u \in V(G)$. If v is a vertex that is farthest from u in G, then v is not a cut-vertex of G.

2. Let G be a nonseparable graph of order $n \geq 4$. Prove that if $u, v \in V(G)$ such that u is a cut-vertex of $G - v$, then v is a cut-vertex of $G - u$.

3. Prove Corollary 3.4: *A connected graph G of order 3 or more is nonseparable if and only if for every two distinct vertices u and v in G, there are two internally disjoint $u - v$ paths.*

4. Prove Corollary 3.5: *Let u and w be two distinct vertices in a nonseparable graph G. If H is obtained from G by adding a new vertex v and joining v to u and w, then H is nonseparable.*

5. Prove Corollary 3.6: *If U and W are disjoint sets of vertices in a nonseparable graph G of order 4 or more with $|U| = |W| = 2$, then G contains two disjoint paths connecting the vertices of U and the vertices of W.*

6. (a) Let k be the maximum length of a cycle in a nonseparable graph G. Prove that if C and C' are any two k-cycles in G, then C and C' have at least two vertices in common.

   (b) Show that the result in (a) cannot be improved by giving an example of an infinite class of nonseparable graphs for which there exist two cycles of maximum length having exactly two vertices in common.

7. (a) Show that no graph has a cut-vertex of degree 1.

   (b) Show that if G is a graph with $\delta(G) \geq 2$ containing a cut-vertex of degree 2, then G has at least three cut-vertices.

   (c) Show, for every integer $k \geq 2$, that there is a graph containing a cut-vertex of degree k.

8. Prove or disprove each of the following.

   (a) If v is a cut-vertex of a connected graph G and H is a proper connected subgraph of order at least 3 containing v, then v is a cut-vertex of H.

   (b) Let G be a connected graph of order at least 4. If every proper connected induced subgraph of G having order at least 3 contains a cut-vertex, then G also contains a cut-vertex.

9. Prove that if v is a cut-vertex of a connected graph G, then v is *not* a cut-vertex of $\overline{G}$.

10. Let u and v be distinct vertices of a nonseparable graph G of order $n \geq 3$. If P is a $u - v$ path of G, does there always exist a $u - v$ path Q in G such that P and Q are internally disjoint $u - v$ paths?

11. (a) An **element** of a graph G is a vertex or an edge of G. Prove that a connected graph G of order at least 3 is nonseparable if and only if every pair of elements of G lie on a common cycle of G.

    (b) Let G and H be graphs with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$, $n \geq 3$. Two vertices u_i and u_j are adjacent in H if and only if v_i and v_j belong to a common cycle in G. Characterize those graphs G for which H is complete.

12. Prove that if G is a graph of order $n \geq 3$ with the property that $\deg u + \deg v \geq n$ for every pair u, v of nonadjacent vertices of G, then G is nonseparable.

13. Prove or disprove: If B is a block of order 3 or more in a connected graph G, then there is a cycle in B that contains all the vertices of B.

14. A connected graph G contains k blocks and ℓ cut-vertices. What is the relationship between k and ℓ?

15. Prove or disprove: If G is a connected graph with cut-vertices and u and v are antipodal vertices of G, then no block of G contains both u and v.

16. By Theorem 3.9, the center of every connected graph G lies in a single block of G. Prove or disprove: If G is a connected graph with two or more cut-vertices, then the center of G lies in a block of G that is not an end-block.

17. Let G be a nontrivial connected graph.

    (a) Prove that no cut-vertex of G is a peripheral vertex of G.

    (b) Prove or disprove: Every peripheral vertex of G belongs to an end-block of G.

18. (a) Show that for every positive integer k, there exists a connected graph G and a non-cut-vertex u of G such that $\text{rad}(G - u) = \text{rad}(G) + k$.

    (b) Prove for every nontrivial connected graph G and every non-cut-vertex v of G that $\text{rad}(G - v) \geq \text{rad}(G) - 1$.

    (c) Let G be a nontrivial connected graph with $\text{rad}(G) = r$. Among all connected induced subgraphs of G having radius r, let H be one of minimum order. Prove that $\text{rad}(H - v) = r - 1$ for every non-cut-vertex v of H.

Section 3.2. Introduction to Trees

19. Let G be a connected graph of order 3 or more. Prove that if $e = uv$ is a bridge of G, then at least one of u and v is a cut-vertex of G.

20. Prove that every connected graph all of whose vertices have even degrees contains no bridges.

21. (a) Give an example of a tree of order 8 containing six vertices of degree 1 and two vertices of degree 4.
    (b) Find all trees T where 75% of the vertices of T have degree 1 and the remaining 25% of the vertices have another degree (a fixed degree).

22. Determine the average degree of a tree T of order n in terms of n.

23. Draw all forests of order 6.

24. Prove that a graph G is a forest if and only if every induced subgraph of G contains a vertex of degree at most 1.

25. Characterize those graphs with the property that every connected subgraph is an induced subgraph.

26. A graph G of order 8 has the degree sequence $s : 3, 3, 3, 1, 1, 1, 1, 1$. Prove or disprove: G is a tree.

27. Let G be a connected graph, and let e_1 and e_2 be two edges of G. Prove that $G - e_1 - e_2$ has three components if and only if both e_1 and e_2 are bridges in G.

28. Prove that a 3-regular graph G has a cut-vertex if and only if G has a bridge.

29. A tree is called **central** if its center is K_1 and **bicentral** if its center is K_2. Show that every tree is either central or bicentral.

30. Let T be a tree order 3 or more, and let T' be the tree obtained from T by deleting its end-vertices.
    (a) Show that $\text{diam}(T) = \text{diam}(T') + 2$, $\text{rad}(T) = \text{rad}(T') + 1$ and $\text{Cen}(T) = \text{Cen}(T')$.
    (b) Show that a tree T is central or bicentral (see Exercise 29) according to whether $\text{diam}(T) = 2\,\text{rad}(T)$ or $\text{diam}(T) = 2\,\text{rad}(T) - 1$.

31. Let G be a connected graph of order n and size m such that $V(G)=\{v_1, v_2, \ldots, v_n\}$, where v_i belongs to $b(v_i)$ blocks $(1 \leq i \leq n)$.
    (a) Show that $\sum_{i=1}^{n} b(v_i) \leq 2m$.
    (b) Show that $\sum_{i=1}^{n} b(v_i) = 2m$ if and only if G is a tree.

32. Determine all trees T such that $\overline{T}$ is also a tree.

33. Prove that if T is a tree of order at least 3, then T contains a cut-vertex v such that every vertex adjacent to v, with at most one exception, is an end-vertex.

34. Let T be a tree of order n with degree sequence $d_1, d_2, \ldots, d_n$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that $d_i \leq \left\lceil \frac{n-1}{i} \right\rceil$ for each integer i with $1 \leq i \leq n$.

35. Let G be a connected graph that is not a tree containing two distinct vertices u and v such that $G - u$ and $G - v$ are both trees. Show that $\deg u = \deg v$.

36. Show that there exists no tree T containing two distinct edges e_1 and e_2 such that the two components of $T - e_1$ are isomorphic *and* the two components of $T - e_2$ are isomorphic.

37. Let T be a tree of order n. Prove that T is isomorphic to a subgraph of $\overline{C}_{n+2}$.

38. Prove that if T is a tree of order $n \geq 2$ that is not a star, then T is isomorphic to a subgraph of $\overline{T}$.

39. Find all those graphs G of order $n \geq 4$ such that the subgraph induced by every three vertices of G is a tree, or show that no such graph exists.

40. Show that every tree with maximum degree k has at least k leaves.

41. Let $P = (u_0, u_1, \ldots, u_t)$ be a longest path in a tree T. Show, for every vertex u in T, that $e(u) = \max\{d(u, u_0), d(u, u_t)\}$.

42. Let T be a tree with $\text{diam}(T) \geq 3$. Use Exercise 41 to prove the following.

    (a) Every central vertex in T lies on a longest path in T.

    (b) For every integer k with $\text{rad}(T) < k \leq \text{diam}(T)$, the tree T contains at least two vertices with eccentricity k.

Section 3.3. Spanning Trees

43. Determine the Prüfer codes of the trees in Figure 3.11.

44. (a) Which trees have constant Prüfer codes?

    (b) Which trees have Prüfer codes each term of which is one of two numbers?

    (c) Which trees have Prüfer codes with distinct terms?

45. Determine the labeled tree having Prüfer code $(4, 5, 7, 2, 1, 1, 6, 6, 7)$.

46. In the Prüfer code of a certain tree T, seven numbers appear twice each, one number appears three times and no other number appears in the code. How many leaves does T contain?

47. The Prüfer code of a tree T of order n with vertex set $\{1, 2, \ldots, n\}$ is $(8, 6, 8, 8, 1, 6, 2, 2, 9)$. What is the degree sequence $d_1, d_2, \ldots, d_n$ of T where $d_1 \geq d_2 \geq \cdots \geq d_n$?

48. (a) Show that if G is a connected graph of order $n \geq 4$ having cycle rank 1, then for every two distinct vertices u and v of G, the graph G contains at most two $u - v$ paths.

    (b) Is the statement in (a) true if G has cycle rank 2?

49. Show that Theorem 3.24 is sharp by giving an example of a graph G of order n with $\text{rad}(G) = n/2$ for every even integer $n \geq 2$.

50. Show for every even integer $n \geq 2$ and every positive integer r with $r \leq n/2$ that there exists a tree of order n having radius r.

51. Prove that a sequence $s_n : d_1, d_2, \ldots, d_n$ $(n \geq 3)$ of integers with $1 \leq d_i \leq n - 1$ for $1 \leq i \leq n$ is a degree sequence of a unicyclic graph of order n if and only if at most $n - 3$ terms of s_n are 1 and $\sum_{i=1}^{n} d_i = 2n$.

52. (a) Show that for every connected graph G there is a spanning tree T of G such that $\text{diam}(T) \leq 2\,\text{diam}(G)$.

    (b) Prove or disprove: For every positive integer k, there exists a connected graph G and a spanning tree T of G such that $\text{diam}(T) > k\,\text{diam}(G)$.

53. Give an example of a connected graph G and a vertex v of G for which there exist two distinct spanning trees that are distance-preserving from v.

54. (a) Let G be a connected graph. Show that if T is a spanning tree of G that is distance-preserving from some vertex of G, then

$$\text{diam}(G) \leq \text{diam}(T) \leq 2\,\text{diam}(G).$$

    (b) Show that for every positive integer a, there exists a connected graph G of diameter a and a vertex v of G such that for every integer b with $a < b \leq 2a$, there is a spanning tree T that is distance-preserving from v and $\text{diam}(T) = b$.

55. Give an example of a connected graph G that is not a tree and two vertices u and v of G such that a distance-preserving spanning tree from v is the same as a distance-preserving spanning tree from u.

56. Let G be the labeled graph in Figure 3.22.

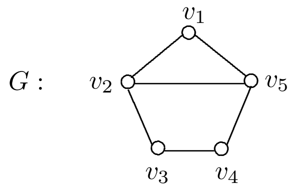


Figure 3.22: The graph G in Exercise 56

(a) Use the Matrix Tree Theorem to compute the number of distinct labeled spanning trees of G.

(b) Draw all the distinct labeled spanning trees of G.

57. (a) Let $G = K_4$ with $V(G) = \{v_1, v_2, v_3, v_4\}$. Draw all labeled spanning trees of G in which v_4 is an end-vertex.

    (b) Let v be a fixed vertex of $G = K_n$. Determine the number of labeled spanning trees of G in which v is an end-vertex.

58. Prove Theorem 3.23 as a corollary of Theorem 3.26.

59. Prove that an edge e of a connected graph is a bridge if and only if e belongs to every spanning tree of G.

60. Let F be a subgraph of a connected graph G. Prove that F is a subgraph of every spanning tree of G if and only if F contains no cycles.

61. What is the maximum number of spanning trees, no two of which have an edge in common, that a complete graph of order $n \geq 4$ can have?

62. (a) Let G be a nontrivial connected graph. Prove that if v is an end-vertex of some spanning tree of G, then v is not a cut-vertex of G.

    (b) Use (a) to give an alternate proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

    (c) Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

    (d) Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path.

63. Show that there is only one positive integer k such that no graph contains exactly k spanning trees.

64. Let T and T' be two distinct spanning trees of a connected graph G of order n. Show that there exists a sequence $T = T_0, T_1, \ldots, T_k = T'$ of spanning trees of G such that T_i and T_{i+1} have $n - 2$ edges in common for each i with $0 \leq i \leq k - 1$.

65. Show for every two integers r and t with $2 \leq r \leq t$ that there exists a connected graph G such that r is the minimum number of end-vertices in a spanning tree of G and t is the maximum number of end-vertices in a spanning tree of G.

66. For the graph G in Figure 3.23, determine

    (a) the adjacency matrix A of G and the degree matrix D of G,

    (b) a cofactor of the matrix $D - A$,

    (c) the matrix C described in the proof of Theorem 3.26,

    (d) the matrix C_3 described in the proof of Theorem 3.26,

    (e) the matrix $C_3 \cdot C_3^t$,

    (f) the major determinants of C_3 and C_3^t.

    Illustrate (3.3) in the case where $M = C_3$ and $M' = C_3^t$, and where $\det(M_0)$ and $\det(M'_0)$ are the corresponding major determinants of C_3 and C_3^t. Then show that the value of $\det(C_3 \cdot C_3^t)$ obtained is the expected value.

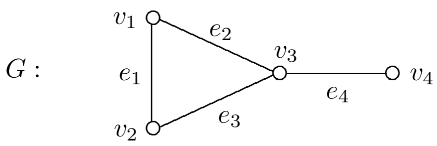


Figure 3.23: The graph G in Exercise 66

Section 3.4. The Minimum Spanning Tree Problem

67. Apply both Kruskal's and Prim's algorithms to find a minimum spanning tree in the weighted graph in Figure 3.24. In each case, show how this tree is constructed, as in Figures 3.20 and 3.21.

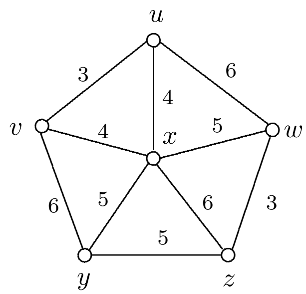


Figure 3.24: The weighted graph in Exercise 67

68. Apply both Kruskal's and Prim's algorithms to find a minimum spanning tree in the weighted graph in Figure 3.25. In each case, show how this tree is constructed, as in Figures 3.20 and 3.21.

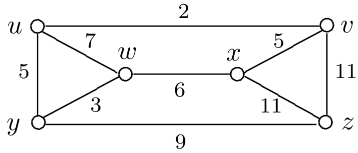


Figure 3.25: The weighted graph in Exercise 68

69. Let G be a connected weighted graph and T a minimum spanning tree of G. Show that T is a unique minimum spanning tree of G if and only if the weight of each edge e of G that is not in T exceeds the weight of every other edge on the cycle in $T + e$.

70. Let G be a connected weighted graph. Prove that every minimum spanning tree of G can be obtained by Kruskal's algorithm.

71. Let G be a connected weighted graph whose edges have distinct weights. Show that G has exactly one minimum spanning tree.

72. Modify Kruskal's algorithm to find a spanning tree of maximum weight (a maximum spanning tree) in a connected, weighted graph.

Chapter 4

Connectivity

We saw in Chapter 3 that each tree of order 3 or more contains at least one vertex whose removal results in a disconnected graph. In fact, every vertex in a tree that is not a leaf has this property. Furthermore, the removal of every edge in a tree results in a disconnected graph (with exactly two components). On the other hand, no vertex or edge in a nonseparable graph of order 3 or more has this property. Hence, in this sense, nonseparable graphs possess a greater degree of connectedness than trees. We now look at the two most common measures of connectedness of graphs. In the process of doing this, we will encounter some of the best known and most useful theorems dealing with the structure of graphs.

4.1 Connectivity and Edge-Connectivity

The first measure of graph connectedness that we discuss is expressed in terms of the removal of vertices from a graph.

The Connectivity of a Graph

A **vertex-cut** of a noncomplete graph G is a set S of vertices of G such that $G-S$ is disconnected. A vertex-cut of minimum cardinality in G is called a **minimum vertex-cut** of G and this cardinality is called the **vertex-connectivity** (or, more simply, the **connectivity**) of G and is denoted by $\kappa(G)$. (The symbol κ is the Greek letter *kappa*.)

Complete graphs do not contain vertex-cuts since the removal of any proper subset of vertices from a complete graph results in a smaller complete graph. The connectivity of the complete graph of order n is defined as $n-1$, that is, $\kappa(K_n) = n-1$. In general then, the **connectivity** $\kappa(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. Therefore, for every graph G of order n,

$$0 \le \kappa(G) \le n-1.$$

A graph G therefore has connectivity 0 if and only if either $G = K_1$ or G is disconnected; a graph G has connectivity 1 if and only if $G = K_2$ or G is a connected graph with cut-vertices; and a graph G has connectivity 2 or more if and only if G is a nonseparable graph of order 3 or more.

For a minimum vertex-cut S of a noncomplete connected graph G, let G_1, G_2, …, G_k ($k \geq 2$) be the components of $G - S$. Then the subgraphs $B_i = G[V(G_i) \cup S]$ are sometimes called the **branches** of G at S or the **S-branches** of G. For the minimum vertex-cut $S = \{u, v\}$ of the graph G of Figure 4.1, the three S-branches of G are also shown in that figure.

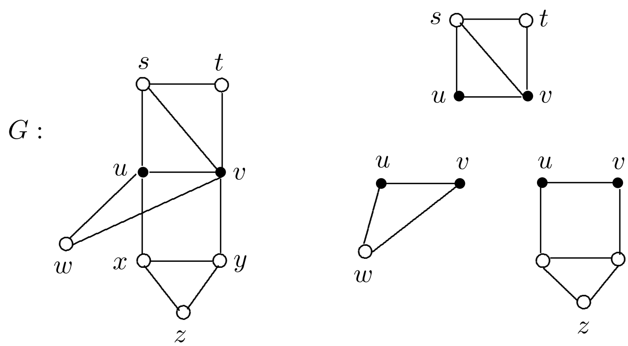


Figure 4.1: The branches of a graph at $S = \{u, v\}$

Often it is more useful to know that a given graph G cannot be disconnected by the removal of a certain number of vertices rather than to know the actual connectivity of G. A graph G is **k-connected**, $k \geq 1$, if $\kappa(G) \geq k$. That is, G is k-connected if the removal of fewer than k vertices from G results in neither a disconnected nor a trivial graph. In particular, to show that a graph G of order $n \geq k + 1$ is k-connected, it suffices to show that $G - S$ is connected for every set $S \subseteq V(G)$ with $|S| = k - 1$. The 1-connected graphs are then the nontrivial connected graphs, while the 2-connected graphs are the nonseparable graphs of order 3 or more.

As would be expected, the higher the degrees of the vertices of a graph, the more likely it is that the graph has large connectivity. There are several sufficient conditions of this type. The simplest result of this type, due to Gary Chartrand and Frank Harary [46], is stated next.

Theorem 4.1 *Let G be a graph of order n and let k be an integer with $1 \leq k \leq n - 1$. If*

$$\deg v \geq \left\lceil \frac{n + k - 2}{2} \right\rceil$$

for every vertex v of G, then G is k-connected.

Proof. Suppose that this theorem is false. Then there is a graph G satisfying the hypothesis of the theorem such that G is not k-connected. Certainly then,

G is not a complete graph. Hence, there exists a vertex-cut U of G such that $|U| = \ell \le k-1$. The graph $G-U$ is therefore disconnected and has order $n-\ell$.

Let G_1 be a component of $G-U$ of smallest order, say n_1. Thus $n_1 \le \lfloor (n-\ell)/2 \rfloor$. Let v be a vertex of G_1. Necessarily, v is adjacent in G only to vertices of U and to other vertices of G_1. Hence,

$$\begin{aligned} \deg v &\le \ell + (n_1 - 1) \le \ell + \lfloor (n-\ell)/2 \rfloor - 1 \\ &= \lfloor (n+\ell-2)/2 \rfloor \le \lfloor (n+k-3)/2 \rfloor, \end{aligned}$$

contrary to the hypothesis. ∎

The Edge-Connectivity of a Graph

How connected a graph G is can be measured not only in terms of the number of vertices that need to be deleted from G to arrive at a disconnected or trivial graph but in terms of the number of edges that must be deleted from G to arrive at such a graph.

An **edge-cut** of a graph G is a subset X of $E(G)$ such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a **minimum edge-cut** and this cardinality is the **edge-connectivity** of G, which is denoted by $\lambda(G)$. (The symbol λ is the Greek letter *lambda*.) The trivial graph K_1 does not contain an edge-cut but we define $\lambda(K_1) = 0$. Therefore, $\lambda(G)$ is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Since the set of edges incident with any vertex of a graph G of order n is an edge-cut of G, it follows that

$$0 \le \lambda(G) \le \delta(G) \le n-1. \tag{4.1}$$

A graph G is k**-edge-connected**, $k \ge 1$, if $\lambda(G) \ge k$. That is, G is k-edge-connected if the removal of fewer than k edges from G results in neither a disconnected graph nor a trivial graph. Thus, a 1-edge-connected graph is a nontrivial connected graph and a 2-edge-connected graph is a nontrivial connected bridgeless graph.

For the graph G of Figure 4.2, $\kappa(G) = 2$ and $\lambda(G) = 3$. Both $\{u, v_1\}$ and $\{u, v_2\}$ are minimum vertex-cuts, while $\{e_1, e_2, e_3\}$ is a minimum edge-cut.

The class of k-edge-connected graphs is characterized in the following simple but useful theorem.

Theorem 4.2 *A nontrivial graph G is k-edge-connected if and only if there exists no nonempty proper subset W of $V(G)$ such that the number of edges joining W and $V(G) - W$ is less than k.*

Proof. First assume that there exists no nonempty proper subset W of $V(G)$ such that the number of edges joining W and $V(G) - W$ is less than k but that G is not k-edge-connected. Since G is nontrivial, there exist ℓ edges, where

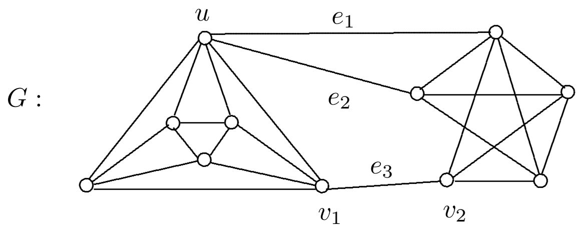


Figure 4.2: Connectivity and edge-connectivity

$0 < \ell < k$, such that their deletion from G results in a disconnected graph H. Let H_1 be a component of H. Since the number of edges joining $V(H_1)$ and $V(G) - V(H_1)$ is at most ℓ, where $\ell < k$, this is a contradiction.

Conversely, suppose that G is k-edge-connected. If there exists a nonempty proper subset W of $V(G)$ such that j edges ($j < k$) join W and $V(G) - W$, then the deletion of these j edges produces a disconnected graph, which is impossible. ■

That $\lambda(K_n) = \delta(K_n)$ for all complete graphs K_n is probably not surprising. This fact is verified in the next result.

Theorem 4.3 *For every positive integer n,*

$$\lambda(K_n) = n - 1.$$

Proof. Since the edge-connectivity of K_1 is defined to be 0, we may assume that $n \geq 2$. We observed in (4.1) that $\lambda(K_n) \leq n - 1$. Let X be a minimum edge-cut of K_n. Then $|X| = \lambda(K_n)$ and $K_n - X$ consists of two components, say G_1 and G_2. Suppose that G_1 has order k. Then G_2 has order $n - k$. Thus $|X| = k(n-k)$. Since $k \geq 1$ and $n-k \geq 1$, it follows that $(k-1)(n-k-1) \geq 0$ and so

$$(k-1)(n-k-1) = k(n-k) - (n-1) \geq 0,$$

which implies that

$$\lambda(K_n) = |X| = k(n-k) \geq n - 1.$$

Therefore, $\lambda(K_n) = n - 1$. ■

Whitney's Inequalities

That the edge-connectivity of a graph is bounded above by its minimum degree and bounded below by its connectivity was first observed by Hassler Whitney [255].

Theorem 4.4 *For every graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof. We already observed in (4.1) that $\lambda(G) \leq \delta(G)$ and so it remains to show that $\kappa(G) \leq \lambda(G)$. Let G be a graph of order n. If G is disconnected, then $\kappa(G) = \lambda(G) = 0$; while if G is complete, then $\kappa(G) = \lambda(G) = \delta(G) = n - 1$. Hence, we may assume that G is a connected graph that is not complete and so $\delta(G) \leq n - 2$. Let X be a minimum edge-cut of G. Then $|X| = \lambda(G) \leq n - 2$. Necessarily, $G - X$ consists of two components, say G_1 and G_2. Suppose that the order of G_1 is k. Then the order of G_2 is $n - k$, where $k \geq 1$ and $n - k \geq 1$. Also, every edge in X joins a vertex of G_1 and a vertex of G_2. We consider two cases.

Case 1. *Every vertex of G_1 is adjacent to every vertex of G_2.* Then $|X| = k(n - k)$. Hence, we can apply the argument used in the proof of Theorem 4.3. Since $k - 1 \geq 0$ and $n - k - 1 \geq 0$, it follows that

$$(k - 1)(n - k - 1) = k(n - k) - (n - 1) \geq 0$$

and so

$$\lambda(G) = |X| = k(n - k) \geq n - 1.$$

This, however, contradicts $\lambda(G) \leq n - 2$ and so Case 1 cannot occur.

Case 2. *There exist a vertex u in G_1 and a vertex v in G_2 such that $uv \notin E(G)$.* We now define a set U of vertices of G. Let $e \in X$. If e is incident with u, say $e = uv'$, then the vertex v' is placed in the set U. If e is not incident with u, say $e = u'v'$ where u' is in G_1, then the vertex u' is placed in U. Hence, for every edge $e \in X$, one of its two incident vertices belongs to U but $u, v \notin U$. Thus $|U| \leq |X|$ and U is a vertex-cut. Therefore,

$$\kappa(G) \leq |U| \leq |X| = \lambda(G),$$

as desired. ■

The connectivity of a graph G of a given order n and size m can only be so large. For example, if $m < n - 1$, then G is disconnected and so $\kappa(G) = 0$. On the other hand, if $m \geq n - 1$, then there is a sharp upper bound for $\kappa(G)$ in terms of the average degree of G, which we present next.

Theorem 4.5 *If G is a graph of order n and size $m \geq n - 1$, then*

$$\kappa(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor.$$

Proof. Since the sum of the degrees of the vertices of G is $2m$, the average degree of the vertices of G is $2m/n$ and so $\delta(G) \leq 2m/n$. Since $\delta(G)$ is an integer, $\delta(G) \leq \lfloor 2m/n \rfloor$. By Theorem 4.4, $\kappa(G) \leq \lfloor 2m/n \rfloor$. ■

The observation stated in Theorem 4.5 is due to Frank Harary. In the second book ever written on graph theory, *Théorie des Graphes et Ses Applications*, the author Claude Berge [22] wrote (translated into English):

What is the maximum connectivity of a graph with n vertices and m edges?

In 1962 Harary [119] answered Berge's question by showing that for every pair n, m of integers with $2 \leq n-1 \leq m \leq \binom{n}{2}$, there exists a graph G of order n and size m with $\kappa(G) = \left\lfloor \frac{2m}{n} \right\rfloor$. Of course, if $m = n-1 \geq 2$, every tree T of order n has the desired property since

$$\kappa(T) = \left\lfloor \frac{2m}{n} \right\rfloor = \left\lfloor \frac{2n-2}{n} \right\rfloor = 1.$$

We observed that $\kappa(G) = 2$ and $\lambda(G) = 3$ for the graph G of Figure 4.2. Since $\delta(G) = 4$, this graph shows that the two inequalities stated in Theorem 4.4 can be strict. The first of these inequalities cannot be strict for cubic graphs, however.

Theorem 4.6 *For every cubic graph G,*

$$\kappa(G) = \lambda(G).$$

Proof. For a cubic graph G, it follows that $\kappa(G) = \lambda(G) = 0$ if and only if G is disconnected. If $\kappa(G) = 3$, then $\lambda(G) = 3$ by Theorem 4.4. So two cases remain, namely $\kappa(G) = 1$ and $\kappa(G) = 2$. Let U be a minimum vertex-cut of G. Then $|U| = 1$ or $|U| = 2$ and so $G - U$ is disconnected. Let G_1 and G_2 be two components of $G - U$. Since G is cubic, for each $u \in U$, at least one of G_1 and G_2 contains exactly one neighbor of u.

Case 1. $\kappa(G) = |U| = 1$. Thus, U consists of a cut-vertex u of G. Since some component of $G - U$ contains exactly one neighbor w of u, the edge uw is a bridge of G and so $\lambda(G) = \kappa(G) = 1$.

Case 2. $\kappa(G) = |U| = 2$. Let $U = \{u, v\}$. Assume that each of u and v has exactly one neighbor, say u' and v', respectively, in the same component of $G - U$. (This is the case that holds if $uv \in E(G)$.) Then $X = \{uu', vv'\}$ is an edge-cut of G and $\lambda(G) = \kappa(G) = 2$. (See Figure 4.3(a) for the situation when u and v are not adjacent.)

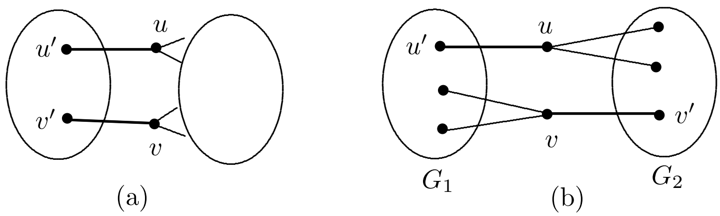


Figure 4.3: A step in the proof of Case 2

Hence, we may assume that u has one neighbor u' in G_1 and two neighbors in G_2, while v has two neighbors in G_1 and one neighbor v' in G_2 (see Figure 4.3(b)). Therefore, $uv \notin E(G)$ and $X = \{uu', vv'\}$ is an edge-cut of G; so $\lambda(G) = \kappa(G) = 2$. ■

According to Theorem 4.4, $\lambda(G) \leq \delta(G)$ for every graph G. The following theorem of Jan Plesník [188] gives a sufficient condition for equality to hold in this case.

Theorem 4.7 *If G is a connected graph of diameter at most 2, then $\lambda(G) = \delta(G)$.*

Proof. Since $\lambda(K_n) = \delta(K_n)$ for each positive integer n by Theorem 4.3, we may assume that $\text{diam}(G) = 2$. Let X be an edge-cut with $|X| = \lambda(G)$, and let H_1 and H_2 be the two components of $G - X$. Now it cannot occur that both H_1 and H_2 have vertices v_1 and v_2, respectively, that are not incident with any edge of X, for then $d(v_1, v_2) \geq 3$.

We may assume then that every vertex of H_1, say, is incident with an edge of X. Thus

$$n_1 = |V(H_1)| \leq |X| = \lambda(G) \leq \delta(G). \tag{4.2}$$

Since each vertex u in H_1 is incident with at most $n_1 - 1$ edges in H_1, it follows that u is incident with at least $\delta(G) - n_1 + 1$ edges in X. Consequently,

$$\lambda(G) = |X| \geq n_1(\delta(G) - n_1 + 1). \tag{4.3}$$

Since $n_1 \geq 1$, it follows that multiplying the inequality $\delta(G) \geq n_1$ in (4.2) by $n_1 - 1$ gives $\delta(G)(n_1 - 1) \geq n_1(n_1 - 1)$ and so $\lambda(G) \geq n_1(\delta(G) - n_1 + 1) \geq \delta(G)$. ■

For each integer $k \geq 3$, let G_k be the graph obtained from two copies F_1 and F_2 of K_k by joining one vertex of F_1 to a vertex of F_2. Then $\text{diam}(G_k) = 3$, $\delta(G_k) = k - 1 \geq 2$ and $\lambda(G_k) = 1$. Since $\{G_k\}_{k \geq 3}$ is an infinite class of connected graphs of diameter 3 with $\lambda(G_k) \neq \delta(G_k)$, Theorem 4.7 is best possible in that it cannot be extended to graphs whose diameter exceeds 2.

If G is a graph of diameter 2 and X is a set of $\lambda(G) = \delta(G)$ edges such that $G - X$ is disconnected, then either G contains a vertex incident with every edge in X or some component of $G - X$ is a particular complete subgraph of G.

Theorem 4.8 *Let G be a graph of diameter 2. If X is a set of $\lambda(G)$ edges whose removal disconnects G, then at least one of the components of $G - X$ is isomorphic to K_1 or to $K_{\delta(G)}$.*

Proof. Let X be a minimum edge-cut of G. Since $\text{diam}(G) = 2$, it follows by Theorem 4.7 that $|X| = \lambda(G) = \delta(G) = \delta$. Furthermore, $G - X$ has exactly two components G_1 and G_2, where $V(G_1) = V_1$ and $V(G_2) = V_2$. Since $\text{diam}(G) = 2$, at most one of V_1 and V_2 contains a vertex that is not incident with any edges in X; for otherwise, there are vertices $x \in V_1$ and $y \in V_2$ such that x and y are not incident with any edges in X, implying that $d(x, y) \geq 3$, which is impossible. Assume, without loss of generality, that every vertex in V_1 is incident with some edge in X; that is, every vertex in V_1 is adjacent to some vertex of V_2 in G.

Let $v \in V_1$ such that v is incident with a maximum number k of edges in X, say v is incident with the edges $e_1, e_2, \cdots, e_k$ in X. Since there are $\delta - k$ edges in $X - \{e_1, e_2, \cdots, e_k\}$ and every vertex in V_1 is incident with some edge in X, there are at most $\delta - k$ vertices in $V_1 - \{v\}$ and so $|V_1| \leq \delta - k + 1$. On the other hand, v must also be adjacent to $\deg v - k \geq \delta - k$ vertices in V_1 and so there are at least $\delta - k$ vertices in $V_1 - \{v\}$. Therefore, $|V_1| \geq \delta - k + 1$ and so $|V_1| = \delta - k + 1$. Let u be any other vertex of V_1, where u is incident, say, with ℓ edges, where then $\ell \leq k$. Thus u is incident with $\deg u - \ell$ vertices in $V_1 - \{u\}$ and so

$$\delta - k + 1 = |V_1| \geq \deg u - \ell + 1 \geq \delta - k + 1,$$

which implies that $\deg u = \delta$ and $\ell = k$. Hence

$$(1)\ G_1 = K_{\delta-k+1} \text{ and } (2)\ |X| = k(\delta - k + 1) = \delta.$$

Rewriting the equation in (2), we obtain $(k - \delta)(k - 1) = 0$ and so $\delta = k$ or $\delta = 1$. Thus, $G_1 = K_1$ or $G_1 = K_\delta$. ∎

If G is a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices u and v, then $\text{diam}(G) = 2$. Therefore, the following result of Linda Lesniak [154] is a consequence of Theorem 4.7.

Corollary 4.9 *If G is a graph of order n such that* $\deg u + \deg v \geq n - 1$ *for each pair u, v of nonadjacent vertices of G, then $\lambda(G) = \delta(G)$.*

4.2 Theorems of Menger and Whitney

A nontrivial graph G is connected (or, equivalently, 1-connected) if every two distinct vertices of G are connected by at least one path. This fact has been generalized in several ways, many of which involve, either directly or indirectly, a theorem [169] due to Karl Menger (1902–1985), who was born in Vienna, Austria. Menger developed a talent for mathematics and physics at an early age and entered the University of Vienna in 1920 to study physics. The following year he attended a lecture by Hans Hahn on *Neueres über den Kurvenbegriff* (*What's new concerning the concept of a curve*) and Menger's interests turned toward mathematics. In the lecture it was mentioned that (at that time) there was no satisfactory definition of a curve, despite attempts to do so by a number of distinguished mathematicians, including Georg Cantor, Camille Jordan and Giuseppe Peano. Some mathematicians, including Felix Hausdorff and Ludwig Bieberbach, felt that it was unlikely that this problem would ever be solved. Despite being an undergraduate with limited mathematical background, Menger solved the problem and presented his solution to Hahn. This led Menger to work on curve and dimension theory. After completing his studies at Vienna, Menger left to broaden his mathematics in Amsterdam.

In 1927 Menger returned to the University of Vienna to accept the position of Chair of Geometry. It was during that year that he published the paper "Zur allgemeinen Kurventheorie" (which contained *Menger's Theorem*). Menger himself referred to this result as the "n-arc theorem" and proved it as a lemma for a theorem in curve theory.

In the spring of 1930, Karl Menger traveled to Budapest and met many Hungarian mathematicians, including Dénes König. Menger had read some of König's papers. During his visit, Menger learned that König was working on a book that would contain what was known about graph theory at that time. Menger was pleased to hear this and mentioned his theorem to König. However, König was not aware of Menger's work and, in fact, didn't believe that the theorem was true. Indeed, the very evening of their meeting, König set out to construct a counterexample. When the two met again the next day, König greeted Menger with "A sleepless night!". König then asked Menger to describe his proof, which he did. After that, König said that he would add a final section to his book on the theorem. As a result, König added a chapter to his 1936 book *Theorie der endlichen und unendliehen Graphen*, which would become the first book written on graph theory [148]. This was a major reason why Menger's theorem became so widely known among those interested in graph theory.

While Menger had visiting positions in the United States during 1930–31 (at Harvard University and the Rice Institute), he held professorships at the University of Notre Dame during 1937–46 and the newly founded Illinois Institute of Technology during 1946–71.

Menger's Theorem

Before stating and presenting a proof of Menger's theorem, some additional terminology is needed. For two nonadjacent vertices u and v in a graph G, a $u - v$ **separating set** is a set $S \subseteq V(G) - \{u, v\}$ such that u and v lie in different components of $G - S$. A $u - v$ separating set of minimum cardinality is called a **minimum** $u - v$ **separating set**.

For two distinct vertices u and v in a graph G, a collection of $u - v$ paths is **internally disjoint** if every two paths in the collection have only u and v in common. Menger's theorem states that the concepts of internally disjoint $u - v$ paths and $u - v$ separating sets are linked. In the graph G of Figure 4.4, there is a set $S = \{w_1, w_2, w_3\}$ of vertices of G that separate the vertices u and v. No set with fewer than three vertices separates u and v. According to Menger's theorem, there are three internally disjoint $u - v$ paths in G.

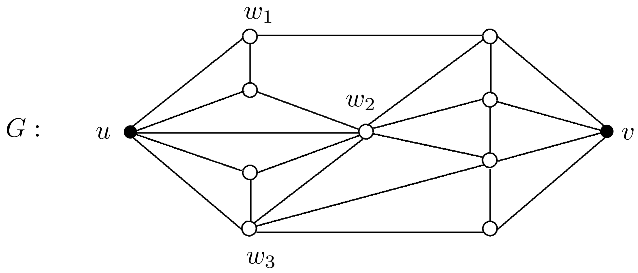


Figure 4.4: A graph illustrating Menger's theorem

Theorem 4.10 (**Menger's Theorem**) *Let u and v be nonadjacent vertices in a graph G. The minimum number of vertices in a $u-v$ separating set equals the maximum number of internally disjoint $u-v$ paths in G.*

Proof. We proceed by induction on the size of graphs. The theorem is certainly true for every empty graph. Assume that the theorem holds for all graphs of size less than m, where $m \geq 1$, and let G be a graph of size m. Moreover, let u and v be two nonadjacent vertices of G. If u and v belong to different components of G, then the result follows. So we may assume that u and v belong to the same component of G. Suppose that a minimum $u-v$ separating set consists of $k \geq 1$ vertices. Then G contains at most k internally disjoint $u-v$ paths. We show, in fact, that G contains k internally disjoint $u-v$ paths. Since this is obviously true if $k = 1$, we may assume that $k \geq 2$. We now consider three cases.

Case 1. *Some minimum $u-v$ separating set X in G contains a vertex x that is adjacent to both u and v.* Then $X - \{x\}$ is a minimum $u-v$ separating set in $G - x$ consisting of $k-1$ vertices. Since the size of $G - x$ is less than m, it follows by the induction hypothesis that $G - x$ contains $k-1$ internally disjoint $u-v$ paths. These paths together with the path $P = (u, x, v)$ produce k internally disjoint $u-v$ paths in G.

Case 2. *For every minimum $u-v$ separating set S in G, either every vertex in S is adjacent to u and not to v or every vertex in S is adjacent to v and not to u.* Necessarily then, $d(u,v) \geq 3$. Let $P = (u, x, y, \ldots, v)$ be a $u-v$ geodesic in G, where $e = xy$. Every minimum $u-v$ separating set in $G - e$ contains at least $k-1$ vertices. We show, in fact, that every minimum $u-v$ separating set in $G - e$ contains k vertices.

Assume, to the contrary, that there is some minimum $u-v$ separating set in $G-e$ with $k-1$ vertices, say $Z = \{z_1, z_2, \ldots, z_{k-1}\}$. Then $Z \cup \{x\}$ is a $u-v$ separating set in G and therefore a minimum $u-v$ separating set in G. Since x is adjacent to u (and not to v), it follows that every vertex z_i $(1 \leq i \leq k-1)$ is also adjacent to u and not adjacent to v.

Since $Z \cup \{y\}$ is also a minimum $u-v$ separating set in G and each vertex z_i $(1 \leq i \leq k-1)$ is adjacent to u but not to v, it follows that y is adjacent to u. This, however, contradicts the assumption that P is a $u-v$ geodesic. Thus, k is the minimum number of vertices in a $u-v$ separating set in $G-e$. Since the size of $G-e$ is less than m, it follows by the induction hypothesis that there are k internally disjoint $u-v$ paths in $G-e$ and in G as well.

Case 3. *There exists a minimum $u-v$ separating set W in G such that* (1) *no vertex of W is adjacent to both u and v and* (2) *W contains at least one vertex not adjacent to u and at least one vertex not adjacent to v.* Let $W = \{w_1, w_2, \ldots, w_k\}$. Let G_u be the subgraph of G consisting of, for each i with $1 \leq i \leq k$, all $u-w_i$ paths in G in which $w_i \in W$ is the only vertex of the path belonging to W. Let G'_u be the graph constructed from G_u by adding a new vertex v' and joining v' to each vertex w_i for $1 \leq i \leq k$. The graphs G_v and G'_v are defined similarly.

Since W contains a vertex that is not adjacent to u and a vertex that is not adjacent to v, the sizes of both G'_u and G'_v are less than m. So G'_u contains k internally disjoint $u-v'$ paths A_i $(1 \leq i \leq k)$, where A_i contains w_i. Also, G'_v contains k internally disjoint $u'-v$ paths B_i $(1 \leq i \leq k)$, where B_i contains w_i. Let A'_i be the $u-w_i$ subpath of A_i and let B'_i be the w_i-v subpath of B_i for $1 \leq i \leq k$. The k paths constructed from A'_i and B'_i for each i $(1 \leq i \leq k)$ are internally disjoint $u-v$ paths in G. ■

Whitney's Theorem

We mentioned that Karl Menger visited Dénes König in Budapest, Hungary early in 1930, which led to his theorem being included in König's book. Later in 1930, Menger went to the United States and spent the period from September of 1930 to February of 1931 as a visiting lecturer at Harvard University. It was at one of his seminar talks that he presented his theorem. During that period, Hassler Whitney (1907–1989) was doing research for his doctoral thesis in graph theory at Harvard. For a short period after receiving his Ph.D. in 1932 from Harvard, Whitney worked on graph theory, making important contributions, but thereafter turned to topology when the area was just being called *topology.* Whitney was a faculty member at Harvard until 1952, when he went to the Institute for Advanced Study in Princeton. Whitney was at Harvard during the period that the United States was involved in World War II. He had great interest and ability in applied mathematics. Because of this, he was brought in as a consultant to the Applied Mathematics Group at Columbia University. That part of the Applied Mathematics Panel was primarily the responsibility of Whitney. He developed mathematical principles to discover best techniques for aerial gunnery and was involved in making improvements to weapons systems.

Although research was a large part of Whitney's professional life, he contributed to mathematics in many ways. During 1944–1949 he was editor of the *American Journal of Mathematics* and during 1949–1954 he was editor of *Mathematical Reviews.* During 1953–1956, he chaired the National Science Foundation mathematical panel. On the personal side, Whitney was an avid mountain climber. In fact, the Whitney-Gilman Ridge on Cannon Cliff in Franconia, New Hampshire, was named for his cousin and him, who were the first to climb it (on August 3, 1939).

If G is a k-connected graph ($k \geq 1$) and v is a vertex of G, then $G - v$ is $(k-1)$-connected. In fact, if $e = uv$ is an edge of G, then $G - e$ is also $(k-1)$-connected (see Exercise 13). With the aid of Menger's theorem, a useful characterization of k-connected graphs, due to Whitney [255], was established. Since nonseparable graphs of order 3 or more are 2-connected, this theorem by Whitney is a generalization of Corollary 3.4.

Theorem 4.11 (**Whitney's Theorem**) *A nontrivial graph G is k-connected for some integer $k \geq 2$ if and only if for each pair u, v of distinct vertices of G, there are at least k internally disjoint $u - v$ paths in G.*

Proof. First, suppose that G is a k-connected graph, where $k \geq 2$, and let u and v be two distinct vertices of G. Assume first that u and v are not adjacent. Let U be a minimum $u - v$ separating set. Then

$$k \leq \kappa(G) \leq |U|.$$

By Menger's theorem, G contains at least k internally disjoint $u - v$ paths.

Next, assume that u and v are adjacent, where $e = uv$. As observed earlier, $G - e$ is $(k-1)$-connected. Let W be a minimum $u - v$ separating set in $G - e$ and so

$$k - 1 \leq \kappa(G - e) \leq |W|.$$

By Menger's theorem, $G - e$ contains at least $k - 1$ internally disjoint $u - v$ paths, implying that G contains at least k internally disjoint $u - v$ paths.

For the converse, assume that G contains at least k internally disjoint $u - v$ paths for every pair u, v of distinct vertices of G. If G is complete, then $G = K_n$, where $n \geq k + 1$, and so $\kappa(G) = n - 1 \geq k$. Hence, G is k-connected. Thus, we may assume that G is not complete.

Let U be a minimum vertex-cut of G. Then $|U| = \kappa(G)$. Let x and y be vertices in distinct components of $G - U$. Thus, U is an $x - y$ separating set of G. Since there are at least k internally disjoint $x - y$ paths in G, it follows by Menger's theorem that

$$k \leq |U| = \kappa(G)$$

and so G is k-connected. ∎

The following three results are consequences of Theorem 4.11 (see Exercises 17–19).

Corollary 4.12 *Let G be a k-connected graph, $k \geq 1$, and let S be any set of k vertices of G. If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.*

Corollary 4.13 *If G is a k-connected graph, $k \geq 2$, and $u, v_1, v_2, \ldots, v_t$ are $t+1$ distinct vertices of G, where $2 \leq t \leq k$, then G contains a $u - v_i$ path for each i $(1 \leq i \leq t)$, every two paths of which have only u in common.*

Corollary 4.14 *A graph G of order $n \geq 2k$ is k-connected if and only if for every two disjoint sets V_1 and V_2 of k distinct vertices each, there exist k pairwise disjoint paths connecting V_1 and V_2.*

By Theorem 3.3, every two vertices in a 2-connected graph lie on a common cycle of the graph. Gabriel Dirac [72] generalized this to k-connected graphs.

Theorem 4.15 *If G is a k-connected graph, $k \geq 2$, then every k vertices of G lie on a common cycle of G.*

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of k vertices of G. Among all cycles in G, let C be one containing a maximum number ℓ of vertices of S. Then $\ell \leq k$. If $\ell = k$, then the result follows, so we may assume that $\ell < k$. Since G is k-connected, G is 2-connected and so by Theorem 3.3, $\ell \geq 2$. We may further assume that $v_1, v_2, \ldots, v_\ell$ lie on C. Let u be a vertex of S that does not lie on C. We consider two cases.

Case 1. *The cycle C contains exactly ℓ vertices, say $C = (v_1, v_2, \ldots, v_\ell, v_1)$.* By Corollary 4.13, G contains a $u - v_i$ path P_i for each i with $1 \leq i \leq \ell$ such that every two of the paths $P_1, P_2, \ldots, P_\ell$ have only u in common. Replacing the edge v_1v_2 on C by P_1 and P_2 produces a cycle containing at least $\ell + 1$ vertices of S. This is a contradiction.

Case 2. *The cycle C contains at least $\ell + 1$ vertices.* Let v_0 be a vertex on C that does not belong to S. Since $2 < \ell + 1 \leq k$, it follows by Corollary 4.13 that G contains a $u - v_i$ path P_i for each i with $0 \leq i \leq \ell$ such that every two of the paths $P_0, P_1, \ldots, P_\ell$ have only u in common. For each i $(0 \leq i \leq \ell)$, let u_i be the first vertex of P_i that belongs to C and let P_i' be the $u - u_i$ subpath of P_i. Suppose that the vertices u_i $(0 \leq i \leq \ell)$ are encountered in the order $u_0, u_1, \ldots, u_\ell$ as we proceed about C in some direction. For some i with $0 \leq i \leq \ell$ and $u_{\ell+1} = u_0$, there is a $u_i - u_{i+1}$ path P on C, none of whose internal vertices belong to S. Replacing P on C by P_i' and P_{i+1}' produces a cycle containing at least $\ell + 1$ vertices of S. Again, this is a contradiction. ■

The converse of Theorem 4.15 is not true, however. For example, for $n \geq k \geq 3$, every k vertices of $G = C_n$ lie on a common cycle of G but G is not k-connected. There is, however, a theorem related to both Theorem 3.3 and Theorem 4.15. The following is due to Don Lick [157].

Theorem 4.16 *Let k and n be integers with $n \geq k+1 \geq 3$. A graph G of order n is k-connected if and only if for each set S of k vertices of G and for each 2-element subset T of S, there is a cycle of G that contains both vertices of T but no vertices of $S - T$.*

Proof. First, suppose that $T = \{u, v\}$. By Theorem 4.11, there exist k internally disjoint $u - v$ paths in G. Since the $k - 2$ vertices of $S - T$ belong to at most $k - 2$ of these paths, there are two of these paths that contain no vertices of $S - T$. These two paths produce a cycle containing the two vertices of T and no vertices of $S - T$.

For the converse, let G be a graph having the property that for each set S of k distinct vertices of G and each 2-element subset T of S, there is a cycle containing both vertices of T but containing no vertices of $S - T$. Assume, to the contrary, that G is not k-connected. Then G contains a vertex-cut $W = \{w_1, w_2, \cdots, w_{k-1}\}$. Since $n \geq k + 1$, the graph $G - W$ contains at least two vertices. Let u and v be two vertices belonging to different components of $G - W$. Now, let $S = \{w_1, w_2, \cdots, w_{k-2}, u, v\}$ and $T = \{u, v\}$. By assumption, there is a cycle C containing the vertices of T and no vertices of $S - T$. Since every cycle containing u and v must contain at least two vertices of $S - T$, this is a contradiction. ■

There are analogues to Menger's theorem (Theorem 4.10) and to Whitney's theorem (Theorem 4.11) in terms of edge-cuts. For two distinct vertices u and v in a graph G, an edge-cut X of G is a $u - v$ **separating set** if u and v lie in different components of $G - X$. We begin with the edge analogue of Theorem 4.10.

Theorem 4.17 *For distinct vertices u and v in a graph G, the minimum cardinality of a $u - v$ separating set $X \subseteq E(G)$ equals the maximum number of pairwise edge-disjoint $u - v$ paths in G.*

Proof. It is convenient to actually prove a stronger result here by allowing G to be a multigraph.

If u and v are vertices in different components of a multigraph G, then the theorem is immediate. Hence we may assume that the multigraphs under consideration are connected. If the minimum number of edges that separate u and v is 1, then G contains a bridge e so that u and v lie in different components of $G - e$. Thus, every $u - v$ path in G contains e, so the maximum number of pairwise edge-disjoint $u - v$ paths in G is also 1.

Suppose that the statement is false. Then there is a smallest integer $k \geq 2$ such that there exist multigraphs containing two vertices u and v for which the minimum number of edges that separate u and v is k but there do not exist k pairwise edge-disjoint $u - v$ paths. Among all such multigraphs, let G be one of minimum size.

If every $u - v$ path of G has length 1 or 2, then since the minimum number of edges of G that separate u and v is k, it follows that there are k pairwise edge-disjoint $u - v$ paths in G, which produces a contradiction. Thus, G contains a $u - v$ path P of length 3 or more.

Let e_1 be an edge of P that is incident with neither u nor v. Consider the multigraph $G - e_1$. Since the size of $G - e_1$ is less than the size of G, it is impossible for $G - e_1$ to contain k edges that separate u and v. Thus, e_1 must belong to every set of k edges that separate u and v. Let $S = \{e_1, e_2, \ldots, e_k\}$ be one such set.

We now subdivide each edge of S, that is, each edge $e_i = u_i v_i$ of S is replaced by a new vertex w_i and the two new edges $u_i w_i$ and $w_i v_i$ for $1 \leq i \leq k$. The k vertices w_i ($1 \leq i \leq k$) are now identified, producing a new vertex w and a new multigraph H. (See Figure 4.5 for a possible situation.) Observe that w is a cut-vertex in H and every $u - v$ path of H contains w.

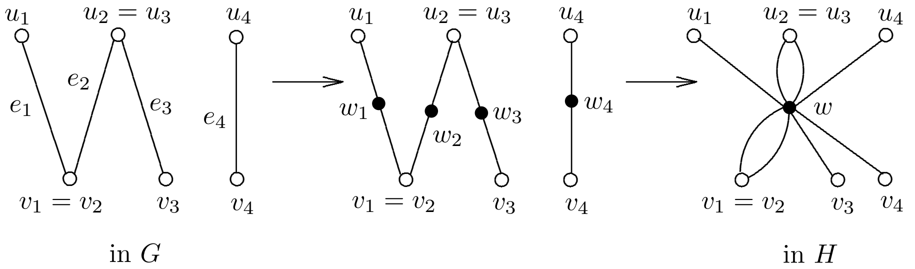


Figure 4.5: A step in the proof of Theorem 4.17

Denote by H_u the submultigraph of H consisting of all $u - w$ paths of H and denote by H_v the submultigraph consisting of all $v - w$ paths of H. The minimum number of edges separating u and w in H_u is k and the minimum number of edges separating v and w in H_v is k. Since each of H_u and H_v has smaller size than G, it follows that H_u contains k pairwise edge-disjoint $u - w$ paths and that H_v contains k pairwise edge-disjoint $w - v$ paths.

For $i = 1, 2, \ldots, k$, we can pair off a $u - w$ path in H_u and a $w - v$ path in H_v to produce a $u - v$ path in H containing the two edges $u_i w$ and $w v_i$. This produces k pairwise edge-disjoint $u - v$ paths in H. The process of subdividing the edges $u_i v_i$ of G and identifying the vertices w_i to obtain w can now be reversed to produce k pairwise edge-disjoint $u - v$ paths in G. This, however, produces a contradiction.

Since the theorem has been proved for multigraphs, it is valid for graphs. ■

With the aid of Theorem 4.17, it is now possible to present an edge analogue of Theorem 4.11 (see Exercise 23).

Theorem 4.18 *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G.*

Exercises for Chapter 4

Section 4.1. Connectivity and Edge-Connectivity

1. Determine the connectivity and edge-connectivity of each complete k-partite graph.

2. Let $v_1, v_2, \ldots, v_k$ be k distinct vertices of a k-connected graph G. Let H be the graph formed from G by adding a new vertex w of degree k that is adjacent to each of $v_1, v_2, \ldots, v_k$. Show that $\kappa(H) = k$.

3. (a) Prove that if G is a k-connected graph, then $G \vee K_1$ is $(k+1)$-connected.

   (b) Prove that if G is a k-edge-connected graph, then $G \vee K_1$ is $(k+1)$-edge-connected.

4. Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$ where $d_1 \geq d_2 \geq \cdots \geq d_n$. Determine $\lambda(G \vee K_1)$.

5. Show for every k-connected graph G and every tree T of order $k+1$ that there exists a subgraph of G isomorphic to T.

6. (a) Let G be a noncomplete graph of order n and connectivity k such that $\deg v \geq (n+2k-2)/3$ for every vertex v of G. Show that if S is a minimum vertex-cut of G, then $G-S$ has exactly two components.

   (b) Let G be a noncomplete graph of order n and connectivity k such that $\deg v \geq (n+kt-t)/(t+1)$ for some integer $t \geq 2$. Show that if S is a vertex-cut of cardinality $\kappa(G)$, then $G-S$ has at most t components.

7. For a graph G of order $n \geq 2$, define the **k-connectivity** $\kappa_k(G)$ of G $(2 \leq k \leq n)$ as the minimum number of vertices whose removal from G results in a graph with at least k components or a graph of order less than k. (Therefore, $\kappa_2(G) = \kappa(G)$.) A graph G is defined to be **(ℓ, k)-connected** if $\kappa_k(G) \geq \ell$. Let G be a graph of order n containing a set of at least k pairwise nonadjacent vertices. Show that if

$$\deg_G v \geq \left\lceil \frac{n+(k-1)(\ell-2)}{k} \right\rceil$$

   for every $v \in V(G)$, then G is (ℓ, k)-connected.

8. Verify that Theorem 4.1 is best possible by showing that for each positive integer k, there exists a graph G of order $n \geq k+1$ such that $\delta(G) = \left\lceil \frac{n+k-3}{2} \right\rceil$ and $\kappa(G) < k$.

9. Let a, b and c be positive integers with $a \leq b \leq c$. Prove that there exists a graph G with $\kappa(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.

10. The **connection number** $con(G)$ of a connected graph G of order $n \geq 2$ is the smallest integer k with $2 \leq k \leq n$ such that *every* induced subgraph of order k in G is connected. State and prove a theorem that gives a relationship between $\kappa(G)$ and $con(G)$ for a graph G of order n.

11. For an even integer $k \geq 2$, show that the minimum size of a k-connected graph of order n is $kn/2$.

12. Prove or disprove: Let G be a nontrivial graph. For every vertex v of G, $\kappa(G - v) = \kappa(G)$ or $\kappa(G - v) = \kappa(G) - 1$.

13. (a) Prove that if G is a k-connected graph and e is an edge of G, then $G - e$ is $(k-1)$-connected.

    (b) Prove that if G is a k-edge-connected graph and e is an edge of G, then $G - e$ is $(k-1)$-edge-connected.

14. (a) Show that if G is a 0-regular graph, then $\kappa(G) = \lambda(G)$.

    (b) Show that if G is a 1-regular graph, then $\kappa(G) = \lambda(G)$.

    (c) Show that if G is a 2-regular graph, then $\kappa(G) = \lambda(G)$.

    (d) By (a) – (c) and Theorem 4.6, if G is r-regular, where $0 \leq r \leq 3$, then $\kappa(G) = \lambda(G)$. Find the minimum positive integer r for which there exists an r-regular graph G such that $\kappa(G) \neq \lambda(G)$.

    (e) Find the minimum positive integer r for which there exists an r-regular graph G such that $\lambda(G) \geq \kappa(G) + 2$.

15. For a graph G, define $\overline{\kappa}(G) = \max\{\kappa(H)\}$ and $\overline{\lambda}(G) = \max\{\lambda(H)\}$, where each maximum is taken over all subgraphs H of G. How are $\overline{\kappa}(G)$ and $\overline{\lambda}(G)$ related to $\kappa(G)$ and $\lambda(G)$, respectively, and to each other?

16. Let G_1 and G_2 be two k-connected graphs, where $k \geq 2$, and let $\mathcal{G}$ be the set of all graphs obtained by adding k edges between G_1 and G_2. Determine $\max\{\kappa(G) : G \in \mathcal{G}\}$.

Section 4.2. Theorems of Menger and Whitney

17. Prove Corollary 4.12: *Let G be a k-connected graph, $k \geq 1$, and let S be any set of k vertices of G. If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.*

18. Prove Corollary 4.13: *If G is a k-connected graph, $k \geq 2$, and $u, v_1, v_2, \ldots, v_t$ are $t+1$ distinct vertices of G, where $2 \leq t \leq k$, then G contains a $u - v_i$ path for each i $(1 \leq i \leq t)$, every two paths of which have only u in common.*

19. Prove Corollary 4.14: *A graph G of order $n \geq 2k$ is k-connected if and only if for every two disjoint sets V_1 and V_2 of k distinct vertices each, there exist k pairwise disjoint paths connecting V_1 and V_2.*

20. Let G be a k-connected graph and let v be a vertex of G. For a positive integer t, define G_t to be the graph obtained from G by adding t new vertices $u_1, u_2, \ldots, u_t$ and all edges of the form $u_i w$, where $1 \le i \le t$ and for which $vw \in E(G)$. Show that G_t is k-connected.

21. Show that if G is a k-connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ such that each path P_i is a $u - v$ path for some $u \in S_1$ and some $v \in S_2$ for $i = 1, 2, \ldots, k$ and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.

22. Let G be a k-connected graph, $k \ge 3$, and let $v, v_1, v_2, \ldots, v_{k-1}$ be k vertices of G. Show that G has a cycle containing all of $v_1, v_2, \ldots, v_{k-1}$ but not v and $k-1$ internally disjoint $v - u_i$ paths P_i $(1 \le i \le k-1)$ such that for each i, the vertex u_i is the only vertex of P_i on C.

23. Prove Theorem 4.18: *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G.*

24. Prove or disprove: If G is a k-edge-connected graph and $v, v_1, v_2, \ldots, v_k$ are $k+1$ vertices of G, then for $i = 1, 2, \ldots, k$, there exist $v - v_i$ paths P_i such that each path P_i contains exactly one vertex of $\{v_1, v_2, \ldots, v_k\}$, namely v_i, and for $i \ne j$, P_i and P_j are edge-disjoint.

25. Prove or disprove: If G is a k-edge-connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist k edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that for each i, P_i is a $u - v$ path for some $u \in S_1$ and some $v \in S_2$ for $i = 1, 2, \ldots, k$ and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.

26. Show that $\kappa(Q_n) = \lambda(Q_n) = n$ for all positive integer n.

27. Assume that G is a *graph* in the proof of Theorem 4.17. Does the proof go through? If not, where does it fail?

28. Let G be a graph of order n with $\kappa(G) \ge 1$. Prove that

$$n \ge \kappa(G)[\text{diam}(G) - 1] + 2.$$

29. A **chorded cycle** is a cycle C (of length at least 4) together with an edge that joins two nonconsecutive vertices of C. Prove that every 3-connected graph contains a chorded cycle but that this need not be the case for a 2-connected graph.

30. (a) Show that for every two vertices u and v of a 3-connected graph G, there exist two internally disjoint $u - v$ paths of different lengths in G.

    (b) Show that the result in (a) is not true in general if G is 2-connected.

31. (a) Prove that if G is a connected graph of order n and k is an integer with $1 \leq k \leq n-3$ such that the maximum number of internally disjoint $u-v$ paths in G is k for every pair u, v of nonadjacent vertices of G, then G contains a vertex-cut with exactly $k+1$ vertices.

    (b) Let G be a k-connected graph of diameter k, where $k \geq 2$. Prove that G contains $k+1$ distinct vertices $v, v_1, v_2, \ldots, v_k$ and k internally disjoint $v - v_i$ paths P_i $(1 \leq i \leq k)$ such that P_i has length i.

32. By Theorem 2.3, if G is a nontrivial graph of order n such that $\deg u + \deg v \geq n-1$ for every two nonadjacent vertices u and v of G, then G is connected, that is, G is 1-connected. Also, by Exercise 12 in Chapter 3, if G is a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n$ for every two nonadjacent vertices u and v of G, then G is nonseparable, that is, G is 2-connected. Consequently, for $k = 1, 2$ and $k < n$, if G is a graph of order n such that $\deg u + \deg v \geq n + k - 2$ for every two nonadjacent vertices u and v of G, then G is k-connected. Prove that this is true for every positive integer k.

Chapter 5

Eulerian Graphs

Many of the early concepts and theorems of graph theory came about quite indirectly, often from recreational mathematics, through puzzles, or games or problems that, as were seen later, could be phrased in terms of graphs. The very first of these was a problem called the *Königsberg Bridge Problem*, which was not only solved by one of the most famous mathematicians of all time but whose solution is considered the origin of graph theory and would lead to an important class of graphs.

5.1 The Königsberg Bridge Problem

Early in the 18th century, the East Prussian city of Königsberg (now called Kaliningrad and located in Russia) occupied both banks of the River Pregel and the island of Kneiphof, lying in the river at a point where the river branches into two parts. There were seven bridges that spanned various sections of the river. (See Figure 5.1.)

A popular problem, called the **Königsberg Bridge Problem**, asks whether there is a route that crosses each of these bridges exactly once. Although such a route was long thought to be impossible, the first mathematical verification of this was presented by the famed mathematician Leonhard Euler (1707–1783) at the Petersburg Academy on 26 August 1735. Euler's proof was contained in a paper [86] that would turn out to be the beginning of graph theory. This paper appeared in the 1736 volume of the proceedings of the Petersburg Academy. Euler's paper, written in Latin, started as follows (translated into English):

> *In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the geometry of position. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them.*

> *It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position – especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.*

Euler denoted the island Kneiphof in Königsberg by the letter A and the other three land regions by B, C and D. The seven bridges that crossed the River Pregel were denoted by a, b, c, d, e, f and g (see Figure 5.1).

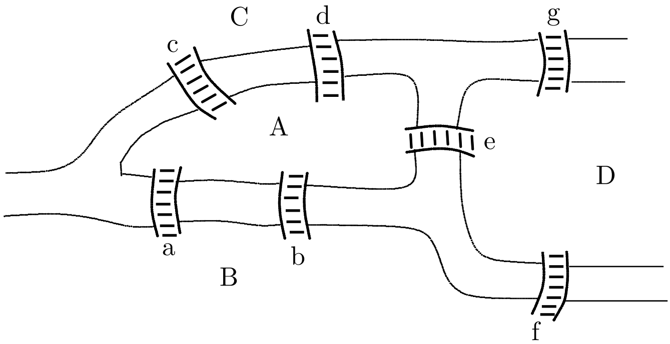


Figure 5.1: The bridges of Königsberg

In his paper, Euler describes what must occur if there was a route that crossed each of the seven bridges exactly once. Such a route could be represented as a sequence of letters, each term of which is one of the letters A, B, C and D. A particular term in this sequence would indicate that the route had reached that land region and the term immediately following it would indicate the land region to which the route had progressed after crossing a bridge. Since there are seven bridges, the sequence must consist of eight terms.

Because there are five bridges leading into (or out of) land region A (the island Kneiphof), each occurrence of A must indicate that either the route began at A, ended at A or had progressed to and then exited from A. Thus, A must appear three times in the sequence. In a similar manner, each of B, C and D must appear twice in the sequence. However, this implies that such a sequence must contain nine terms, which is impossible. Therefore, there is no route in Königsberg that crosses each bridge exactly once.

As Euler mentioned in his paper, he also formulated a more general problem. In order to describe and present a solution to this general problem, we turn to the modern-day approach in which both the Königsberg Bridge Problem and its generalization are described in terms of graphs.

5.2 Eulerian Circuits and Trails

A circuit C in a nontrivial connected graph G that contains every edge of G (necessarily exactly once) is an **Eulerian circuit**, while an open trail that contains every edge of G is an **Eulerian trail**. (Some refer to an Eulerian circuit as an **Euler tour**.) The graph G_1 of Figure 5.2 contains an Eulerian trail while G_2 contains an Eulerian circuit.

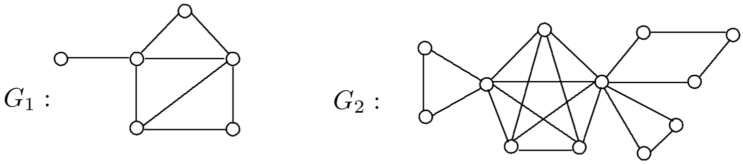


Figure 5.2: Graphs with Eulerian trails and Eulerian circuits

These terms are defined in exactly the same way if G is a nontrivial connected multigraph. In fact, the map of Königsberg in Figure 5.1 can be represented by the multigraph shown in Figure 5.3. Then the Königsberg Bridge Problem can be reformulated as follows:

> Does the multigraph shown in Figure 5.3 contain either an Eulerian circuit or an Eulerian trail?

As Euler showed (although not using this terminology, of course), the answer to this question is *no*. In fact, the term *graph* did not appear in Euler's article as this term, with this meaning, had yet to appear in the literature.

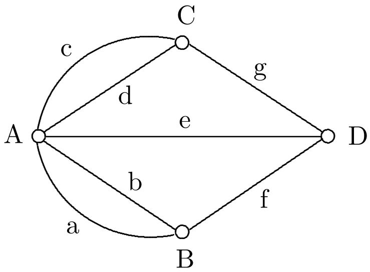


Figure 5.3: The multigraph of Königsberg

Euler's Theorem

A connected graph containing an Eulerian circuit is an **Eulerian graph**. Thus, the graph G_2 of Figure 5.2 is Eulerian. Simple but useful characterizations of both Eulerian graphs and graphs with Eulerian trails exist; in fact, in each case the characterization is attributed to Euler [86].

Theorem 5.1 *A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

Proof. Assume first that G is an Eulerian graph. Then G contains an Eulerian circuit C. Let v be a vertex of G. Suppose first that v is not the initial vertex of C (and thus not the terminal vertex of C either). Since each occurrence of v in C indicates that v is both entered and exited on C and produces a contribution of 2 to its degree, the degree of v is even. Next, suppose that v is the initial and terminal vertex of C. As the initial vertex of C, this represents a contribution of 1 to the degree of v. There is also a contribution of 1 to the degree of v because v is the terminal vertex of C as well. Any other occurrence of v on C again represents a contribution of 2 to the degree of v. Here too, v is even.

We now turn to the converse. Let G be a nontrivial connected graph in which every vertex is even. Let u be a vertex of G. First, we show that G contains a $u - u$ circuit. Construct a trail T beginning at u that contains a maximum number of edges of G. We claim that T is, in fact, a circuit; for suppose that T is a $u - v$ trail, where $v \neq u$. Then there is an odd number of edges incident with v and belonging to T. Since the degree of v in G is even, there is at least one edge incident with v that does not belong to T. Suppose that vw is such an edge. However then, T followed by w produces a trail T' with initial vertex u containing more edges than T, which is impossible. Thus, T is a circuit with initial and terminal vertex u. We now denote T by C.

If C is an Eulerian circuit of G, then the proof is complete. Hence, we may assume that C does not contain all edges of G. Since G is connected, there is a vertex x on C that is incident with an edge that does not belong to C. Let $H = G - E(C)$. Since every vertex on C is incident with an even number of edges on C, it follows that every vertex of H is even. Let H' be the component of H containing x. Consequently, every vertex of H' has positive even degree. By the same argument as before, H' contains a circuit C' with initial and terminal vertex x. By inserting C' at some occurrence of x in C, a $u-u$ circuit C'' in G is produced having more edges than C. This is a contradiction. ∎

Trails in Graphs

A characterization of connected graphs containing an Eulerian trail is also due to Euler.

Theorem 5.2 *A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.*

Proof. If G contains an Eulerian trail $u - v$ trail T, then we construct a new connected graph H from G by adding a new vertex w of degree 2 and joining it to u and v. Then T together with the two edges uw and wv form an Eulerian circuit in H. By Theorem 5.1, every vertex of H is even and so only u and v have odd degrees in $G = H - w$.

For the converse, let G be a connected graph containing exactly two vertices u and v of odd degree. We show that G contains an Eulerian trail T, where T is either a $u - v$ trail or a $v - u$ trail. Add a new vertex w of degree 2 to G and join it to u and v, calling the resulting graph H. Therefore, H is a connected graph all of whose vertices are even. By Theorem 5.1, H is an Eulerian graph containing an Eulerian circuit C. The circuit C necessarily contains uw and wv as consecutive edges. Deleting w from C results in an Eulerian trail of G that begins at u or v and terminates at the other. ■

In the next-to-last paragraph of Euler's paper, Euler wrote (again an English translation):

> *So whatever arrangement may be proposed, one can easily determine whether or not a journey can be made, crossing each bridge once, by the following rules:*
>
> *If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.*
>
> *If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these areas.*
>
> *If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished from any starting point.*
>
> *With these rules, the given problem can also be solved.*

Euler ended his paper by writing:

> *When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across*

the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

Consequently, in Euler's paper, he actually only verified that every vertex being even is a necessary condition for a connected graph to be Eulerian and that exactly two vertices being odd is a necessary condition for a connected graph to contain an Eulerian trail. Euler did not show that these are sufficient conditions. The first proof that these are also sufficient conditions would not be published for another 137 years, in an 1873 paper authored by Carl Hierholzer [128]. Hierholzer received his Ph.D. in 1870 and died in 1871. Thus, his paper was published two years after his death. He had told colleagues of what he had done but died before he could write a paper containing this work. His colleagues wrote the paper on his behalf and had it published for him.

Theorems 5.1 and 5.2 hold for multigraphs as well as graphs. By these theorems then, the multigraph of Figure 5.3 contains neither an Eulerian trail nor an Eulerian circuit.

As we have now seen, if G is a nontrivial connected graph with no odd vertices, then G contains an Eulerian circuit. If G contains exactly two odd vertices, then G contains an Eulerian trail. If G contains more than two odd vertices, then G contains neither an Eulerian circuit nor an Eulerian trail. However, the following result by Gary Chartrand, Albert D. Polimeni and M. James Stewart [50] shows that every connected graph with odd vertices must contain a certain number of trails of a certain type and of certain lengths.

The **distance between two subgraphs** F **and** H in a connected graph G is $\min\{d(u,v) : u \in V(F), v \in V(H)\}$.

Theorem 5.3 *If G is a connected graph containing $2k$ odd vertices ($k \geq 1$), then G contains k pairwise edge-disjoint open trails connecting odd vertices and such that every edge of G lies on one of these trails and at most one of these trails has odd length.*

Proof. Let $y_1, y_2, \ldots, y_k$ and $z_1, z_2, \ldots, z_k$ be the odd vertices of G. We construct a new graph H from G by adding k new vertices $x_1, x_2, \ldots, x_k$ to G and joining x_i to y_i and z_i for $i = 1, 2, \ldots, k$. Thus, H is Eulerian and therefore contains an Eulerian circuit C. Since $y_i x_i$ and $x_i z_i$ are consecutive on C for $i = 1, 2, \ldots, k$, deleting the k vertices x_i ($1 \leq i \leq k$) from H results in k pairwise edge-disjoint trails in G connecting odd vertices such that every edge of G lies on one of these trails.

It remains to show that there are k such trails, at most one of which has odd length. Assume, to the contrary, that in any collection of k pairwise edge-disjoint trails of G connecting odd vertices and such that every edge of G lies on one of these trails, there are at least two trails of odd length. Among all such collections of k trails, consider those collections containing a minimum number of trails of odd length; and, among those, consider one, say $\{T_1, T_2, \ldots, T_k\}$, where the distance between some pair of trails of odd length is minimum. If two trails T_a and T_b of odd length have a vertex in common, then they may be replaced by two trails T_a^* and T_b^* of even length connecting odd vertices such that

$$E(T_a) \cup E(T_b) = E(T_a^*) \cup E(T_b^*).$$

Otherwise, let T_r and T_s be two trails of odd length where the distance between T_r and T_s attains this minimum distance. Suppose that P is a path of minimum length connecting a vertex w_r in T_r and a vertex w_s in T_s, and let $w_r x$ be the edge of P incident with w_r. Then $w_r x$ belongs to a trail T_p among $T_1, T_2, \ldots, T_k$. Necessarily, T_p has even length. However, T_r and T_p have the vertex w_r in common. So, T_r and T_p may be replaced by trails T_r' and T_p' connecting odd vertices such that T_r' has even length, T_p' has odd length,

$$E(T_r) \cup E(T_p) = E(T_r') \cup E(T_p')$$

and $w_r x$ belongs to T_p'. Since the distance between T_p' and T_s is less than the distance between T_r and T_s, this is a contradiction. ■

Veblen's Theorem

Among other characterizations of Eulerian graphs are two dealing with cycles in graphs. One of these is expressed in terms of the edge-disjoint cycles a graph may possess and is due to Oswald Veblen [244], known primarily for his work in topology. This theorem will be encountered again in Chapter 13.

Theorem 5.4 (**Veblen's Theorem**) *A nontrivial connected graph G is Eulerian if and only if G contains pairwise edge-disjoint cycles such that every edge of G lies on one of these cycles.*

Proof. First, suppose that G is Eulerian. We proceed by induction on the size m of G. If $m = 3$, then $G = K_3$ has the desired property. Assume then, that every Eulerian graph of size less than m, where $m \geq 4$, contains pairwise edge-disjoint cycles such that every edge of the graph lies on one of these cycles. Let G be an Eulerian graph of size m. Since G is Eulerian, every vertex of G has even degree. Thus, G is not a tree and so G contains at least one cycle C. If $G = C_m$, then the proof is complete. Otherwise, there are edges of G not in C. Removing the edges of C from G produces a graph G' in which every vertex is even. Thus each nontrivial component of G' is Eulerian and has fewer than m edges. By the induction hypothesis, each nontrivial component of G' contains pairwise edge-disjoint cycles such that every edge of this component lies on one of these cycles. Now, all of these cycles of the components of G', together with C, give the desired result.

For the converse, suppose that G contains pairwise edge-disjoint cycles such that every edge of G lies on one of these cycles. Then every vertex G is even and so G is Eulerian by Theorem 5.1. ■

For another characterization of Eulerian graphs involving cycles, the necessity is due to Shunichi Toida [236] and the sufficiency to Terry A. McKee [168].

Theorem 5.5 *A nontrivial connected graph G is Eulerian if and only if every edge of G lies on an odd number of cycles in G.*

Proof. First, let G be an Eulerian graph and let $e = uv$ be an edge of G. Then $G - e$ is connected. Consider the set of all $u - v$ trails in $G - e$ in which v appears exactly once, namely as the terminal vertex. There is an odd number of edges possible for the initial edge of such a trail. Once the initial edge has been chosen and the trail has then proceeded to the next vertex, say w, then again there is an odd number of choices for the edges that are incident with w but different from uw. We continue this process until we arrive at vertex v. At each vertex different from v in such a trail, there is an odd number of edges available for a continuation of the trail. Hence, there is an odd number of these trails.

Suppose that T_1 is a $u - v$ trail that is not a $u - v$ path and T_1 contains v only once. Then some vertex v_1 ($\neq v$) occurs at least twice on T_1, implying that T_1 contains a $v_1 - v_1$ circuit, say $C = (v_1, v_2, \ldots, v_k, v_1)$. Hence, there exists a $u - v$ trail T_2 identical to T_1 except that C is replaced by the "reverse" circuit $C' = (v_1, v_k, v_{k-1}, \ldots, v_2, v_1)$. This implies that the $u - v$ trails that are not $u - v$ paths occur in pairs. Therefore, there is an even number of such $u - v$ trails that are not $u - v$ paths and, consequently, there is an odd number of $u - v$ paths in $G - e$. This, in turn, implies that there is an odd number of cycles containing e.

For the converse, suppose that G is a nontrivial connected graph that is not Eulerian. We show that some edge of G lies on an even number of cycles in G. Since G is not Eulerian, G contains a vertex v of odd degree. For each edge e incident with v, denote by $c(e)$ the number of cycles of G containing e. Since any such cycle contains two edges incident with v, it follows that the sum $\sum c(e)$, taken over all edges e incident with v, equals twice the number of cycles containing v. Because there is an odd number of terms in this sum, some term $c(e)$ is even. ■

Exercises for Chapter 5

Section 5.1. The Königsberg Bridge Problem

1. In present-day Königsberg (Kaliningrad), there are two additional bridges, one between regions B and C and one between regions B and D. Is it now possible to devise a route over all bridges of Königsberg without recrossing any of them?

2. Euler's approach to solve the Königsberg Bridge Problem was to observe that if there was a route that crossed each of the seven bridges exactly once, then this route could be represented by a sequence of eight letters, where each term represents a land region to which the route had progressed and each of the seven pairs of consecutive letters in the sequence represents a bridge that the route had crossed. Another observation that Euler could have made was that there are at least two land regions that were neither the beginning nor the end of such a route. Show that this observation could be used to solve the Königsberg Bridge Problem.

3. Suppose that the Königsberg Bridge Problem had asked instead whether it was possible to take a route about Königsberg that crossed each bridge exactly twice. What would have been the answer in this case?

4. Suppose that there is a boat in Königsberg that moves along the River Pregel and travels under its bridges. In such a boat ride, a boat travels under each bridge at most once. What is the maximum number of bridges under which a boat can travel?

Section 5.2. Eulerian Circuits and Trails

5. Let F and H be two disjoint connected non-Eulerian regular graphs and let $G = (F + H) \vee K_1$; that is, G is obtained from F and H by adding a new vertex v and joining v to each vertex in F and H. Prove that G is Eulerian.

6. Let G be a connected graph of order $n \geq 4$ that has neither an Eulerian circuit nor an Eulerian trail. A graph H is constructed by adding a new vertex v to G and joining v to every odd vertex of G. Prove or disprove: H is Eulerian.

7. Find a necessary and sufficient condition for the Cartesian product $G \,\square\, H$ of two nontrivial connected graphs G and H to be Eulerian.

8. Prove that if a graph of order $n \geq 6$ has an Eulerian $u-v$ trail such that $\deg u - \deg v \geq n - 2$, then n must be even.

9. Suppose that G is an r-regular graph of order n such that both G and its complement $\overline{G}$ are connected. Is it possible that neither G nor $\overline{G}$ is Eulerian?

10. Show that if T is a tree containing at least one vertex of degree 2, then $\overline{T}$ is not Eulerian.

11. Prove that an Eulerian graph G has even size if and only if G has an even number of vertices v for which $\deg v \equiv 2 \pmod 4$.

12. Prove or disprove: Every Eulerian bipartite graph has even size.

13. Let G be a connected graph with exactly two odd vertices u and v, where $\deg u \geq 3$ and $\deg v \geq 3$. Prove or disprove:

    (a) There exist two edge-disjoint $u - v$ trails in G.

    (b) If G is 2-edge-connected, then there exist two edge-disjoint $u - v$ trails in G such that every edge of G lies on one of these trails.

14. Let G be a connected graph of order n and size $m > n$ containing exactly two odd vertices u and v such that $d(u, v) < m - n$. Show that G contains a $u - v$ path P and two or more cycles such that every edge of G belongs either to P or exactly one of the cycles.

15. Let G be a connected graph of order n and size m with $\delta(G) \geq 3$ such that every vertex of G has odd degree.

    (a) Find a sharp upper bound (in terms of m and n) for the size of an Eulerian subgraph of G.

    (b) Give an example of such a graph G where the maximum size of an Eulerian subgraph of G is less than the upper bound in (a).

16. (a) Prove that every Eulerian graph of odd order has three vertices of the same degree.

    (b) Prove that for each odd integer $n \geq 3$, there exists exactly one Eulerian graph of order n containing exactly three vertices of the same degree and at most two vertices of any other degree.

17. (a) To how many cycles does each edge of $G = K_4$ and $H = K_5$ belong?

    (b) According to Theorem 5.5, what does (a) say about the graphs G and H?

Chapter 6

Hamiltonian Graphs

In the previous chapter, we were introduced to Eulerian graphs, which are those graphs G possessing a circuit containing every edge of G. In this chapter, we turn our attention to those graphs G possessing a cycle containing every vertex of G.

6.1 Hamilton's Icosian Game

William Rowan Hamilton (1805–1865) was gifted even as a child and his numerous interests and talents ranged from languages (having mastered many by age 10) to mathematics and physics. In 1832 he predicted that a ray of light passing through a biaxial crystal would be refracted into the shape of a cone. When this was experimentally confirmed, it was considered a major discovery and led to his being knighted in 1835, thereby becoming *Sir* William Rowan Hamilton. Even today, Hamilton is regarded as one of the leading mathematicians and physicists of the 19th century.

Although Hamilton's accomplishments were many, one of his best known in mathematics was his creation of a new algebraic system called *quaternions*, an extension of the complex numbers. On 16 October 1843, while walking with his wife along the Royal Canal in Dublin, Hamilton suddenly discovered a collection of 4-dimensional numbers $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where a, b, c and d are real numbers, that formed a structure known as a division algebra. Furthermore,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Hamilton carved these equations into the stone of the Brougham Bridge. In the quaternions, $\mathbf{ij} = \mathbf{k}$ and $\mathbf{ji} = -\mathbf{k}$; so the quaternions are not commutative.

In 1856 Hamilton developed another example of a non-commutative algebraic system in a game he called the *Icosian Game*, initially exhibited by Hamilton at a meeting of the British Association in Dublin. The Icosian Game (the prefix *icos* is from the Greek for *twenty*) consisted of a board on which were placed twenty holes and some lines between certain pairs of holes. The diagram for this game is shown in Figure 6.1, where the holes are designated by the twenty consonants of the English alphabet.

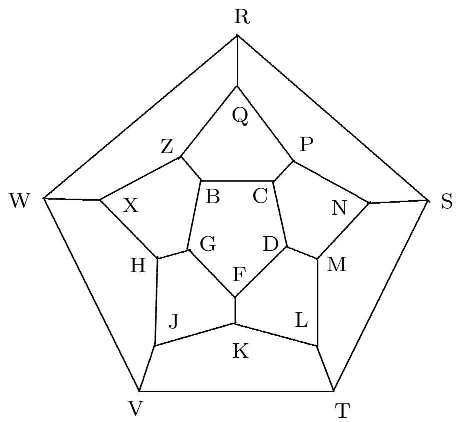


Figure 6.1: Hamilton's Icosian Game

Hamilton later sold the rights of his game for 25 pounds to John Jaques & Son, a game manufacturer especially well known as a dealer in chess sets. The preface to the instruction pamphlet for the Icosian Game, prepared by Hamilton for marketing the game in 1859, read as follows:

> *In this new Game (invented by Sir WILLIAM ROWAN HAMILTON, LL.D., & c., of Dublin, and by him named Icosian from a Greek word signifying 'twenty') a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed along the lines of the figure, and also to fulfill certain other conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in proposing as well as in resolving problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be cyclical, that is, so that No.* 20 *may be adjacent to No.* 1*; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No.* 6 *in hole H, as above, it is also allowed (by the supposed conditions) to put No.* 7 *in X instead of J, and then to conclude with the succession, W R S T V J K L*

M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.

Another (traveler) version of Hamilton's Icosian Game was labeled as

NEW PUZZLE
TRAVELLER'S DODECAHEDRON
or
A VOYAGE ROUND THE WORLD

In this game, the twenty vertices of the dodecahedron, labeled with the twenty consonants, stood for twenty cities of the world:

B. Brussels	H. Hanover	N. Naples	T. Toholsk
C. Canton	J. Jeddo	P. Paris	V. Vienna
D. Delhi	K. Kashmere	Q. Quebec	W. Washington
F. Frankfort	L. London	R. Rome	X. Xenia
G. Geneva	M. Moscow	S. Stockholm	Z. Zanzibar

The idea was thus to construct a round trip around the world where each of the 20 cities would be visited on the trip exactly once.

Of course, the diagram of Hamilton's Icosian game shown in Figure 6.1 can be immediately interpreted as a graph (see Figure 6.2), where the lines in the diagram are the edges of the graph and the holes are its vertices. Indeed, the graph of Figure 6.2 can be considered as the graph of the geometric solid called the **dodecahedron** (where the prefix *dodec* is from the Greek for *twelve*, pertaining to the twelve faces of the solid). This subject will be discussed in more detail in Chapter 10.

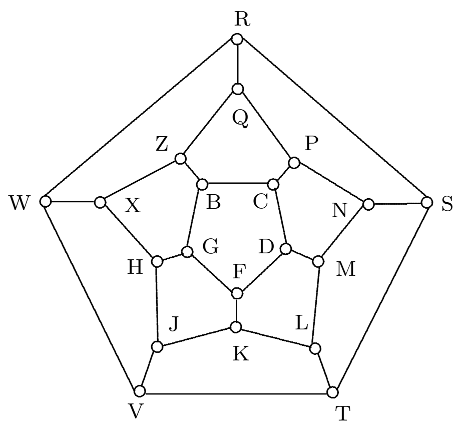
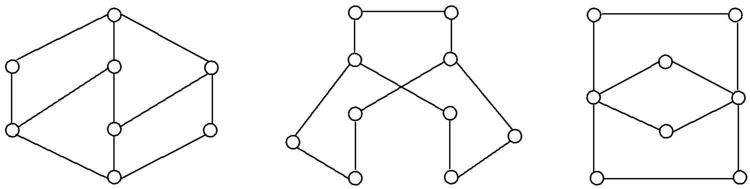


Figure 6.2: The graph of the dodecahedron

6.2 Sufficient Conditions for Hamiltonicity

The problems proposed by Hamilton in his Icosian Game gave rise to concepts in graph theory, which eventually became a popular subject of study by mathematicians. Let G be a graph. A path in G that contains every vertex of G is called a **Hamiltonian path** of G, while a cycle in G that contains every vertex of G is called a **Hamiltonian cycle** of G. A graph that contains a Hamiltonian cycle is itself called **Hamiltonian**. Certainly, the order of every Hamiltonian graph is at least 3 and every Hamiltonian graph contains a Hamiltonian path. On the other hand, a graph with a Hamiltonian path need not be Hamiltonian. The graph G_1 of Figure 6.3 is Hamiltonian and therefore contains both a Hamiltonian cycle and a Hamiltonian path. The graph G_2 contains a Hamiltonian path but is not Hamiltonian; while G_3 contains neither a Hamiltonian cycle nor a Hamiltonian path.



G_1 G_2 G_3

Figure 6.3: Hamiltonian paths and cycles in graphs

In 1855 (the year before Hamilton developed his Icosian Game) the Reverend Thomas Penyngton Kirkman (1806–1895) studied such questions as whether it is possible to visit all corners (vertices) of a polyhedron exactly once by moving along edges of the polyhedron and returning to the starting vertex. He observed that this could be done for some polyhedra but not all. While Kirkman had studied *Hamiltonian cycles* on general polyhedra and had preceded Hamilton's work on the dodecahedron by several months, it is Hamilton's name that became associated with spanning cycles in graphs, not Kirkman's.

Ore's Theorem

Since the concepts of a circuit that contains every edge of a graph and a cycle that contains every vertex seem so very similar and since there is a simple and useful characterization of graphs that are Eulerian, one might very well anticipate the existence of such a characterization of graphs that are Hamiltonian. However, no such theorem has ever been discovered. On the other hand, it is much more likely that a graph is Hamiltonian if the degrees of its vertices are large. It wasn't until 1952 that a general theorem by Gabriel Andrew Dirac appeared, giving a sufficient condition for a graph to be Hamiltonian. However, in 1960 an even more general theorem [179], due to Oystein Ore (1899–1968), would be discovered and lead to a host of other sufficient conditions. For a graph G, we write $\sigma_2(G)$ to denote the minimum degree sum of two nonadjacent vertices of G.

Theorem 6.1 (**Ore's Theorem**) *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is Hamiltonian.*

Proof. Suppose that the statement is false. Then for some integer $n \geq 3$, there exists a graph H of order n such that $\sigma_2(H) \geq n$ but yet H is not Hamiltonian. Add as many edges as possible between pairs of nonadjacent vertices of H so that the resulting graph G is still not Hamiltonian. Hence, G is a maximal non-Hamiltonian graph (that is, G is not Hamiltonian but $G + uv$ is Hamiltonian for every two nonadjacent vertices u and v of G). Certainly, G is not a complete graph. Furthermore, $\sigma_2(G) \geq n$.

If the edge xy were to be added between two nonadjacent vertices x and y of G, then necessarily $G+xy$ is Hamiltonian and so $G+xy$ contains a Hamiltonian cycle C. Since C must contain the edge xy, the graph G contains a Hamiltonian $x - y$ path $(x = v_1, v_2, \ldots, v_n = y)$. If $xv_i \in E(G)$, where $2 \leq i \leq n-1$, then $yv_{i-1} \notin E(G)$; for otherwise,

$$C' = (x = v_1, v_2, \ldots, v_{i-1}, y = v_n, v_{n-1}, v_{n-2}, \ldots, v_i, x)$$

is a Hamiltonian cycle of G, which is impossible. Hence, for each vertex of G adjacent to x, there is a vertex of $V(G) - \{y\}$ not adjacent to y. However then,

$$\deg_G y \leq (n-1) - \deg_G x,$$

that is, $\deg_G x + \deg_G y \leq n - 1$, contradicting the fact that $\sigma_2(G) \geq n$. ■

Dirac's Theorem

The aforementioned 1952 paper of Dirac [70] contained the following sufficient condition for a graph to be Hamiltonian, which is then a consequence of Theorem 6.1.

Corollary 6.2 (**Dirac's Theorem**) *If G is a graph of order $n \geq 3$ such that*

$$\deg v \geq n/2$$

for each vertex v of G, then G is Hamiltonian.

We have now seen fundamental results by Dirac on both connectivity (Theorem 4.15) and on Hamiltonian graphs. We will encounter him often again later in the book. Gabriel Andrew Dirac (1925–1984) was born in Budapest, Hungary. He was a stepson of Paul Adrien Maurice Dirac and a nephew of Eugene Paul Wigner, both recipients of the Nobel Prize in physics. Dirac became interested in graph theory early in his mathematical career, at a time when the respectability of graph theory occurred primarily within Hungary. However, Dirac was attracted to graph theory and became one of those mathematicians whose contributions to graph theory brought it to a place of prominence. In fact, the mathematician Carsten Thomassen wrote:

> *He (Dirac) was more interested in the intrinsic beauty of graph theory than its applications.*

With the aid of Theorem 6.1, a sufficient condition for a graph to have a Hamiltonian path can also be given.

Corollary 6.3 *Let G be a graph of order $n \geq 2$. If $\sigma_2(G) \geq n-1$, then G contains a Hamiltonian path.*

Proof. Let $H = G \vee K_1$ be the join of G and K_1, where w is the vertex of H that does not belong to G. Then $\sigma_2(H) \geq n+1$. Since the order of H is $n+1$, it follows by Theorem 6.1 that H is Hamiltonian. Let C be a Hamiltonian cycle of H. Deleting w from C produces a Hamiltonian path in G. ■

The Closure of a Graph

J. Adrian Bondy and Vašek Chvátal [34] observed that the proof of Ore's theorem (Theorem 6.1) neither uses nor needs the full strength of the requirement that the degree sum of each pair of nonadjacent vertices is at least the order of the graph being considered. Initially, Bondy and Chvátal made the following observation.

Theorem 6.4 *Let u and v be nonadjacent vertices in a graph G of order n such that $\deg u + \deg v \geq n$. Then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.*

Proof. Certainly, if G is Hamiltonian, then $G + uv$ is Hamiltonian. For the converse, suppose that $G + uv$ is Hamiltonian but G is not. Then every Hamiltonian cycle in $G + uv$ contains the edge uv, implying that G contains a Hamiltonian $u - v$ path. We can now proceed exactly as in the proof of Theorem 6.1 to produce a contradiction. ■

The preceding result inspired a definition. Let G be a graph of order n. The **closure** $CL(G)$ of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n (in the resulting graph at each stage) until no such pair remains. A graph G and its closure are shown in Figure 6.4.

First, we show that the closure is a well-defined operation on graphs, that is, the same graph is obtained regardless of the order in which edges are added.

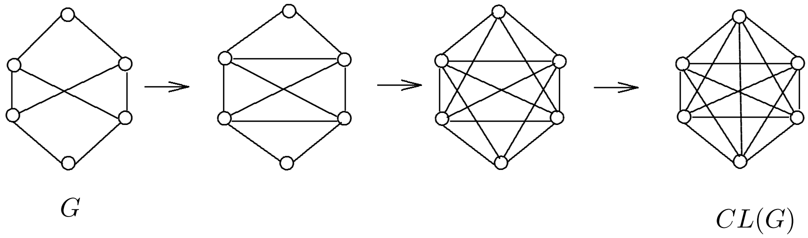


Figure 6.4: Constructing the closure of a graph

Theorem 6.5 *Let G be a graph of order n. If G_1 and G_2 are graphs obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains, then $G_1 = G_2$.*

Proof. Suppose that G_1 is obtained by adding the edges $e_1, e_2, \ldots, e_r$ to G in the given order and G_2 is obtained from G by adding the edges $f_1, f_2, \ldots, f_s$ in the given order. Assume, to the contrary, that $G_1 \neq G_2$. Then $E(G_1) \neq E(G_2)$. Thus, we may assume that there is a first edge $e_i = xy$ in the sequence $e_1, e_2, \ldots, e_r$ that does not belong to G_2. If $i = 1$, then let $G = H$; otherwise, let $H = G + \{e_1, e_2, \ldots, e_{i-1}\}$. Then H is a subgraph of G_2 and x and y are nonadjacent vertices of H. Since $\deg_H x + \deg_H y \geq n$, it follows that $\deg_{G_2} x + \deg_{G_2} y \geq n$, which produces a contradiction. ∎

Repeated application of Theorem 6.4 gives us the following result.

Theorem 6.6 *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Since each complete graph with at least three vertices is Hamiltonian, we obtain the following sufficient condition for a graph to be Hamiltonian due to Bondy and Chvátal [34].

Theorem 6.7 *Let G be a graph with at least three vertices. If $CL(G)$ is complete, then G is Hamiltonian.*

If a graph G satisfies the conditions of Theorem 6.1, then $CL(G)$ is complete and so, by Theorem 6.7, G is Hamiltonian. Thus, Ore's theorem is an immediate corollary of Theorem 6.7 (although chronologically it preceded the theorem of Bondy and Chvátal by several years). In fact, many sufficient conditions for a graph to be Hamiltonian based on the degrees of the vertices of a graph can be deduced from Theorem 6.7. The following result of Chvátal [54] is an example of one of the strongest of these.

Theorem 6.8 *Let G be a graph of order $n \geq 3$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $k < n/2$ for which $d_k \leq k$ and $d_{n-k} \leq n - k - 1$, then G is Hamiltonian.*

Proof. Let $H = CL(G)$. We show that H is complete which, by Theorem 6.7, implies that G is Hamiltonian. Assume, to the contrary, that H is not complete. Let u and w be nonadjacent vertices of H for which $\deg_H u + \deg_H w$ is as large as possible. Since u and w are nonadjacent vertices of H, it follows that $\deg_H u + \deg_H w \leq n-1$. Assume, without loss of generality, that $\deg_H u \leq \deg_H w$. Thus, if $k = \deg_H u$, we have that $k \leq (n-1)/2 < n/2$ and $\deg_H w \leq n-1-k$. Let W denote the vertices other than w that are not adjacent to w in H. Then $|W| = n-1-\deg_H w \geq k$. Also, by the choice of u and w, every vertex $v \in W$ satisfies

$$\deg_G v \leq \deg_H v \leq \deg_H u = k.$$

Thus, G has at least k vertices of degree at most k and so $d_k \leq k$. Similarly, let U denote the vertices other than u that are not adjacent to u in H. Then $|U| = n-1-\deg_H u = n-k-1$. Every vertex $v \in U$ satisfies

$$\deg_G v \leq \deg_H v \leq \deg_H w \leq n-1-k,$$

implying that $d_{n-k-1} \leq n-k-1$. However,

$$\deg_G u \leq \deg_H u \leq \deg_H w \leq n-1-k,$$

so $d_{n-k} \leq n-k-1$. This, however, contradicts the hypothesis of the theorem. Thus, $H = CL(G)$ is complete. ■

All of the sufficient conditions presented thus far for a graph G to be Hamiltonian involve the degrees of the vertices of G. In fact, if G has order n, then each of these conditions requires some of the vertices of G to have degree at least $n/2$. In the case of regular graphs, however, this situation can be improved. Bill Jackson [134] showed that every 2-connected r-regular graph of order at most $3r$ is Hamiltonian. As we are about to see, the Petersen graph shows that $3r$ cannot be replaced by $3r+1$. We mentioned earlier that the girth (the length of a smallest cycle) of the Petersen graph P is 5 and its circumference (the length of a longest cycle) is 9. In fact, $P - v$ contains a 9-cycle for every vertex v of P. We now verify that the Petersen graph has circumference 9 by showing that it is not Hamiltonian.

Theorem 6.9 *The Petersen graph is not Hamiltonian.*

Proof. Assume, to the contrary, that the Petersen graph P is Hamiltonian. Then P has a Hamiltonian cycle

$$C = (v_1, v_2, \ldots, v_{10}, v_1).$$

Since P is cubic, v_1 is adjacent to exactly one of the vertices $v_3, v_4, \ldots, v_9$. However, since P contains neither a 3-cycle nor a 4-cycle, v_1 is adjacent to exactly one of v_5, v_6 and v_7. Because of the symmetry of v_5 and v_7, we may assume that v_1 is adjacent either to v_5 or to v_6.

Case 1. *v_1 is adjacent to v_5 in P.* Then v_{10} is adjacent to exactly one of v_4, v_5 and v_6, which results in a 4-cycle, a 3-cycle or a 4-cycle, respectively, each of which is impossible.

Case 2. *v_1 is adjacent to v_6 in P.* Again, v_{10} is adjacent to exactly one of v_4, v_5 and v_6. Since P does not contain a 3-cycle or a 4-cycle, v_{10} must be adjacent to v_4. However, then this returns us to Case 1, where v_1 and v_5 are replaced by v_{10} and v_4, respectively. ■

The next sufficient condition for a graph G to be Hamiltonian does not involve the degrees of the vertices of G but, rather, involves the cardinality of sets of pairwise nonadjacent vertices of a graph and its connectivity. A set U of vertices in a graph G is **independent** if no two vertices in U are adjacent. (Some refer to an independent set of vertices as a **stable set**.) The maximum number of vertices in an independent set of vertices of G is called the **vertex independence number** or, more simply, the **independence number** of G and is denoted by $\alpha(G)$. For example, $\alpha(K_{r,s}) = \max\{r, s\}$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(K_n) = 1$. Vašek Chvátal and Paul Erdős [58] showed that if G is a graph of order at least 3 whose connectivity is at least as large as its independence number, then G must be Hamiltonian.

Theorem 6.10 *Let G be a graph of order at least 3. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Proof. If $\alpha(G) = 1$, then G is complete and therefore Hamiltonian. Hence, we may assume that $\alpha(G) = k \geq 2$. Since $\kappa(G) \geq 2$, it follows that G is 2-connected and so G contains a cycle by Theorem 4.15. Let C be a longest cycle in G. By Theorem 4.15, C contains at least k vertices. We show that C is a Hamiltonian cycle. Assume, to the contrary, that C is not a Hamiltonian cycle. Then there is some vertex w of G that does not lie on C. Since G is k-connected, it follows with the aid of Corollary 4.13 that G contains k paths $P_1, P_2, \cdots, P_k$ such that P_i is a $w - v_i$ path where v_i is the only vertex of P_i on C and such that the paths are pairwise-disjoint except for w.

In some cyclic ordering of the vertices of C, let u_i be the vertex that follows v_i on C for each i $(1 \leq i \leq k)$. No vertex u_i is adjacent to w, for otherwise, replacing the edge $v_i u_i$ by P_i and wu_i produces a cycle whose length exceeds that of C. Let $S = \{w, u_1, u_2, \ldots, u_k\}$. Since $|S| = k + 1 > \alpha(G)$ and $wu_i \notin E(G)$ for each i $(1 \leq i \leq k)$, there are distinct integers r and s such that $1 \leq r, s \leq k$ and $u_r u_s \in E(G)$. Replacing the edges $u_r v_r$ and $u_s v_s$ by the edge $u_r u_s$ and the paths P_r and P_s produces a cycle that is longer than C. This is a contradiction. ■

For the non-Hamiltonian Petersen graph P, $\kappa(P) = 3$ and $\alpha(P) = 4$; so, as anticipated by Theorem 6.10, $\kappa(P) < \alpha(P)$. On the other hand, $\kappa(P) \geq \alpha(P) - 1$ and, consequently, P has a Hamiltonian path (see Exercise 13).

6.3 Toughness of Graphs

While we have described several sufficient conditions for a graph to be Hamiltonian, there are also useful necessary conditions. Certainly, every Hamiltonian graph is connected. Since every pair u, v of distinct vertices of a Hamiltonian graph G lies on a Hamiltonian cycle of G, it follows that G contains at least two internally disjoint $u-v$ paths. By Theorem 4.11, every Hamiltonian graph is 2-connected. Consequently, if S is a subset of $V(G)$ with $|S| = 1$, then $G-S$ consists of a single component, that is $k(G-S) \leq |S|$, where, recall, that $k(G-S)$ is the number of components in $G-S$. The next result generalizes this observation.

A Necessary Condition for Hamiltonian Graphs

Theorem 6.11 *If G is a Hamiltonian graph, then*

$$k(G-S) \leq |S|$$

for every nonempty proper subset S of $V(G)$.

Proof. Let S be a nonempty proper subset of $V(G)$. If $G-S$ is connected, then certainly, $k(G-S) \leq |S|$. Hence, we may assume that $k(G-S) = k \geq 2$ and that $G_1, G_2, \ldots, G_k$ are the components of $G-S$. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle of G. Without loss of generality, we may assume that $v_1 \in V(G_1)$. For $1 \leq j \leq k$, let v_{i_j} be the last vertex of C that belongs to G_j. Necessarily then, $v_{i_j+1} \in S$ for $1 \leq j \leq k$ and so $|S| \geq k = k(G-S)$. ■

Because Theorem 6.11 presents a necessary condition for a graph to be Hamiltonian, it is most useful in its contrapositive formulation:

> *If there exists a nonempty proper subset S of the vertex set $V(G)$ of a graph G such that $k(G-S) > |S|$, then G is not Hamiltonian.*

For example, the graph G of Figure 6.5 is not Hamiltonian, for if we let $S = \{w, x\}$, then $k(G-S) = 3$ and so $k(G-S) > |S|$.

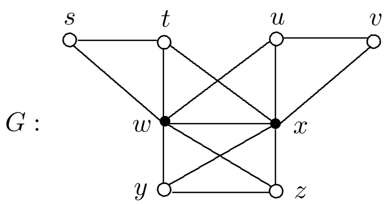


Figure 6.5: A non-Hamiltonian graph

t-Tough Graphs

We saw from Theorem 6.11 that for every Hamiltonian graph G,

$$k(G - S) \leq |S|$$

for every nonempty proper set S of vertices of G. So, if G is a Hamiltonian graph, then

$$\frac{|S|}{k(G - S)} \geq 1$$

for every nonempty proper set S of vertices of G.

For a nonnegative real number t, a noncomplete graph G is called **t-tough** if

$$\frac{|S|}{k(G - S)} \geq t$$

for every vertex-cut S of G. If G is a t-tough graph and s is a nonnegative real number such that $s < t$, then G is also s-tough.

Consequently, every Hamiltonian graph is 1-tough. The converse is not true, however. As the next theorem states, the non-Hamiltonian Petersen graph is 1-tough.

Theorem 6.12 *The Petersen graph is* 1*-tough.*

Proof. The Petersen graph P shown in Figure 6.6 is 3-connected; in fact, $\kappa(P) = 3$. We show that $|S|/k(P - S) \geq 1$ for every vertex-cut S of P. Indeed, we show for every nonempty set S of vertices of P with $|S| \geq 3$ that $|S|/k(P - S) \geq 1$. This is obvious if $|S| \geq 5$, so it remains only to show that this is the case for $|S| = 3$ and $|S| = 4$.

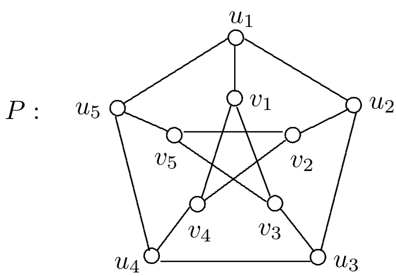


Figure 6.6: The Petersen graph

Let $C = (u_1, u_2, \ldots, u_5, u_1)$ be the exterior cycle of P and $C'=(v_1, v_3, v_5, v_2, v_4, v_1)$ the interior cycle. Assume, without loss of generality, that at least as many vertices of S lie on C as on C'. We consider three cases.

Case 1. *No vertex of S lies on C'.* Then $P - S$ is connected and $|S|/k(P - S) > 1$.

Case 2. *One vertex of S lies on C'.* Then $P - S$ contains a component of order at least 5 and so $|S|/k(P - S) \geq 1$.

Case 3. *Two vertices of S lie on C'.* Then $|S| = 4$ and $P - S$ contains at most four components. Thus, $|S|/k(P - S) \geq 1$. ∎

The maximum real number t for which a noncomplete graph G is t-tough is called the **toughness** of G and is denoted by $t(G)$. Certainly, the toughness of a noncomplete graph is a rational number. Also, $t(G) = 0$ if and only if G is disconnected. Indeed, it follows that if G is a noncomplete graph, then

$$t(G) = \min \left\{ \frac{|S|}{k(G - S)} \right\}, \tag{6.1}$$

where the minimum is taken over all vertex-cuts S of G.

For the graph G of Figure 6.7, $S_1 = \{u, v, w\}$, $S_2 = \{w\}$ and $S_3 = \{u, v\}$ are three (of many) vertex-cuts. Observe that

$$\frac{|S_1|}{k(G - S_1)} = \frac{3}{8}, \quad \frac{|S_2|}{k(G - S_2)} = \frac{1}{3} \quad \text{and} \quad \frac{|S_3|}{k(G - S_3)} = \frac{2}{7}.$$

Since there is no vertex-cut S of G with $|S|/k(G - S) < \frac{2}{7}$, it follows that $t(G) = \frac{2}{7}$.

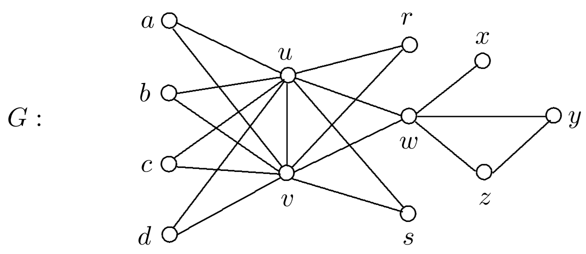


Figure 6.7: A graph with toughness $\frac{2}{7}$

The toughness of a graph G can be considered as a measure of how tightly the subgraphs of G are held together. Thus, the smaller the toughness, the more vulnerable the graph is. A 1-tough graph, for example, has the property that breaking the graph into k components (if this is possible) requires the removal of at least k vertices; while breaking a 2-tough graph into k components requires the removal of at least $2k$ vertices.

A parameter that plays an important role in the study of toughness is the independence number. The independence number is related to toughness in the sense that among all the vertex-cuts S of a noncomplete graph G, the maximum value of $k(G - S)$ is the independence number $\alpha(G)$ of G; so for every vertex-cut S of G, we have that $\kappa(G) \leq |S|$ and $k(G - S) \leq \alpha(G)$. This leads to bounds for the toughness of a graph obtained by Chvátal [55].

Theorem 6.13 *For every noncomplete graph G,*

$$\frac{\kappa(G)}{\alpha(G)} \le t(G) \le \frac{\kappa(G)}{2}.$$

Proof. According to (6.1),

$$t(G) = \min\left\{\frac{|S|}{k(G-S)}\right\} \ge \frac{\kappa(G)}{\alpha(G)}.$$

Let S' be a vertex-cut with $|S'| = \kappa(G)$. Thus, $k(G-S') \ge 2$; so

$$t(G) = \min\left\{\frac{|S|}{k(G-S)}\right\} \le \frac{|S'|}{k(G-S')} \le \frac{\kappa(G)}{2},$$

as desired. ■

For a given graph F, a graph G is **F-free** if G contains no induced subgraph isomorphic to F. A K_3-free graph is called a **triangle-free** graph. In this context, a graph of particular interest is $K_{1,3}$, which, as mentioned earlier, is referred to as a claw. A $K_{1,3}$-free graph is then referred to as a **claw-free graph**. The following result by Manton Matthews and David Sumner [163] provides a class of graphs for which the upper bound in Theorem 6.13 is attained.

Theorem 6.14 *If G is a noncomplete claw-free graph, then $t(G) = \frac{1}{2}\kappa(G)$.*

Proof. If G is disconnected, then $t(G) = \kappa(G) = 0$ and the result follows. So we assume that $\kappa(G) = r \ge 1$. Let S be a vertex-cut such that $t(G) = |S|/k(G-S)$. Suppose that $k(G-S) = k$ and that $G_1, G_2, \ldots, G_k$ are the components of $G - S$.

Let $u_i \in V(G_i)$ and $u_j \in V(G_j)$, where $i \ne j$. Since G is r-connected, it follows by Theorem 4.11 that G contains at least r internally disjoint $u_i - u_j$ paths. Each of these paths contains a vertex of S. Consequently, there are at least r edges joining the vertices of S and the vertices of G_i for each i $(1 \le i \le k)$ such that no two of these edges are incident with the same vertex of S.

Hence, there is a set X containing at least kr edges between S and $G - S$ such that any two edges incident with a vertex of S are incident with vertices in distinct components of $G - S$. However, since G is claw-free, no vertex of S is joined to vertices in three components of $G - S$. Therefore,

$$kr \le |X| \le 2|S| = 2kt(G);$$

so $kr \le 2kt(G)$. Thus, $t(G) \ge \frac{r}{2} = \frac{1}{2}\kappa(G)$. By Theorem 6.13, $t(G) = \frac{1}{2}\kappa(G)$. ■

In defining the toughness of a graph we were in some sense fine-tuning the idea of connectivity. For example, if a graph G is 2-connected, then the removal of one vertex from G does not result in a disconnected graph. The removal of two vertices, however, may not only result in a disconnected graph but in fact may result in a graph with many components. If, however, we know that G is 1-tough, then not only is G a 2-connected graph but the removal of any two vertices of G can result in a graph with at most two components.

The t_0-Tough Conjecture

Chvátal [55] introduced the concept of toughness because he believed that it was strongly related to the existence of Hamiltonian cycles in graphs. In fact, he proposed the following conjecture.

The t_0-Tough Conjecture *There exists a constant t_0 such that every t_0-tough graph is Hamiltonian.*

In other words, the t_0-Tough Conjecture speculates the existence of a constant t_0 such that if G is a graph with $t(G) \geq t_0$, then G is Hamiltonian.

For many years, it was believed, in fact, that every 2-tough graph is Hamiltonian. In 2000, however, Douglas Bauer, Hajo Broersma and Henk Jan Veldman [14] showed that the graph G of Figure 6.8 (the **Bauer-Broersma-Veldman graph**), which is the join of F and K_2, is 2-tough but not Hamiltonian. Hence, if the t_0-Tough Conjecture is true, then $t_0 > 2$. However, Bauer, Broersma and Veldman [14] showed even more. They described a sequence $\{G_k\}$ of non-Hamiltonian graphs for which $\lim_{k\to\infty} t(G_k) = 9/4$. Thus, $t_0 \geq 9/4$ if the t_0-Tough Conjecture is true.

NP-Completeness

As we have now seen, while the problem of determining whether a graph is Eulerian is quite easy to solve, the problem of determining whether a graph is Hamiltonian is, in general, difficult to solve. In fact, this Hamiltonian problem belongs to a much studied class of mathematical problems, both within and outside of graph theory.

Most problems in graph theory apply to all members of some family of graphs (such as all connected graphs or all graphs of order n the degrees of whose vertices are at least $n/2$). When referring to an instance of a problem we mean that the problem is being applied to a particular member of the family. For example, an instance of the Minimum Spanning Tree Problem is the problem of finding a minimum spanning tree in a specific weighted connected graph.

The class of all problems solvable by polynomial-time algorithms is denoted by **P**. For example, the Minimum Spanning Tree Problem belongs to **P**. Determining the problems belonging to **P** is a question of great interest to many. There is a related class of problems denoted by **NP**, which stands for *nondeterministic polynomial-time.*

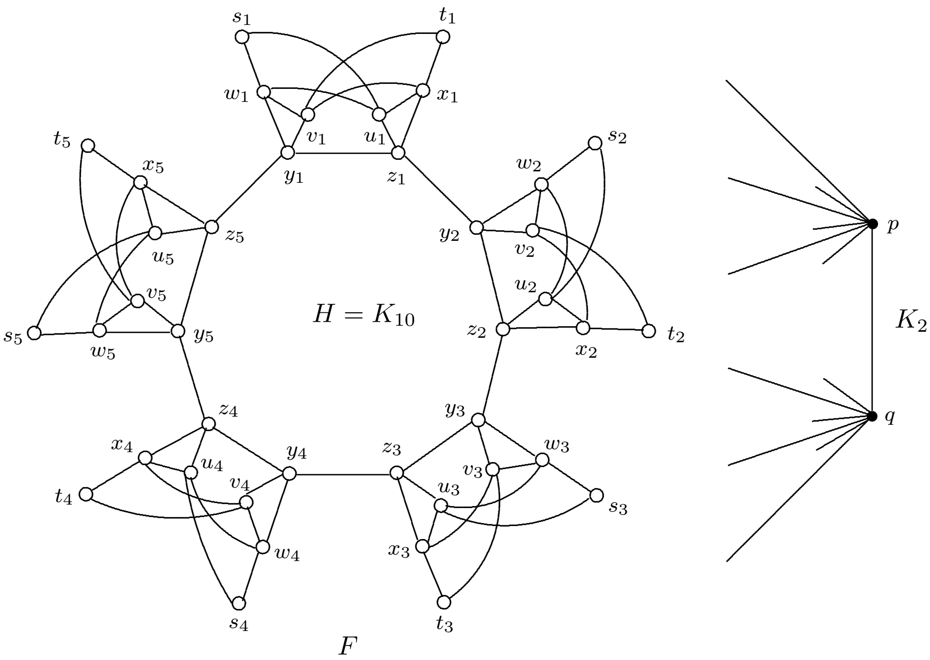


Figure 6.8: The Bauer-Broersma-Veldman graph:
A 2-tough graph that is not Hamiltonian

A **decision problem** is a question that can be answered *yes* or *no*. A decision problem belongs to the class **P** if there is a polynomial-time algorithm that solves every instance of the problem in polynomial time. There are decision problems that are difficult to solve but once a solution is revealed, it is easy to verify that this is in fact a solution. For example, it is difficult in general to determine whether a given graph G is Hamiltonian but it is easy to verify that a cyclic sequence of vertices of G constitutes a Hamiltonian cycle of G. The class of such difficult-to-solve easy-to-verify problems is denoted by **NP**.

The problems in **NP** and the problems in **P** have one property in common, namely: Given a solution to a problem in either class, the solution can be verified in polynomial-time. Thus, $\mathbf{P} \subseteq \mathbf{NP}$. One of the best known problems in mathematics asks whether every problem in **NP** is also in **P**. This is called the **P = NP Problem** and is considered by many as the most important problem in theoretical computer science. Its importance and fame have only been magnified because of a million dollar prize offered by the Clay Mathematics Institute for its solution.

A problem in the set **NP** is called **NP-complete** if a polynomial-time algorithm for a solution to the problem would result in polynomial-time solutions for all problems in **NP**. The **NP**-complete problems are among the most difficult in the set **NP** and can be reduced from and to all other **NP**-complete problems in polynomial time. The concept of **NP**-completeness was initiated in 1971 by Stephen Cook [61] who gave an example of the first **NP**-complete problem. The following year Richard M. Karp [139] described some twenty diverse **NP**-complete problems. There are now known to be thousands of **NP**-complete problems. For example, the problem of determining whether a graph is Hamiltonian is **NP**-complete.

6.4 Highly Hamiltonian Graphs

As we have already remarked, obtaining an applicable characterization of Hamiltonian graphs remains an unsolved problem in graph theory. In view of the lack of success in developing such a characterization, it is not surprising that special classes of Hamiltonian graphs have been singled out for investigation. We now discuss several types of such *highly Hamiltonian* graphs.

Hamiltonian-Connected Graphs

A graph G is **Hamiltonian-connected** if for every pair u, v of vertices of G, there is a Hamiltonian $u - v$ path in G. Necessarily, every Hamiltonian-connected graph of order 3 or more is Hamiltonian but the converse is not true. The cubic graph $G_1 = C_3 \square K_2$ of Figure 6.9 is Hamiltonian-connected, while the 3-cube $G_2 = C_4 \square K_2 = Q_3$ is not Hamiltonian-connected. The graph G_2 contains no Hamiltonian $u - v$ path, for example. (See Exercise 36.)

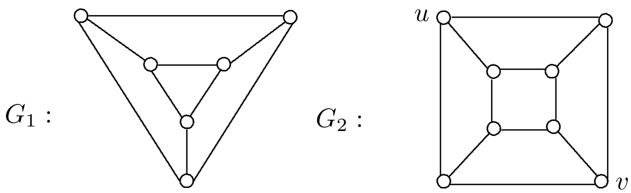


Figure 6.9: Hamiltonian-connected and non-Hamiltonian-connected graphs

There is a sufficient condition for a graph to be Hamiltonian-connected that is similar in statement to the sufficient condition for a graph to contain a Hamiltonian cycle presented in Theorem 6.1. The following theorem is also due to Oystein Ore [181] and provides a sufficient condition for a graph to be Hamiltonian-connected.

Theorem 6.15 *Let G be a graph of order n. If $\sigma_2(G) \geq n + 1$, then G is Hamiltonian-connected.*

Proof. Let u and v be two vertices of G and let H be the graph of order $n+1$ obtained from G by adding a new vertex w that is joined to u and v. We now construct the closure $F = CL(H)$ of H. Since $\deg_H x + \deg_H y \geq n+1$ for every two nonadjacent vertices x and y of H that belong to G, it follows that $F[V(G)] = K_n$. Furthermore, if $x \in V(G) - \{u, v\}$, then $\deg_F x + \deg_F w \geq (n-1) + 2 = n+1$ and so $xw \in E(F)$. Therefore, $F = CL(H) = K_{n+1}$, which implies by Theorem 6.7 that H is Hamiltonian. Since $\deg_H w = 2$, every Hamiltonian cycle C of H must contain the edges uw and vw. Removing w from C produces a Hamiltonian $u - v$ path in G. ■

There is now an immediate corollary, similar in statement to the sufficient condition given in Corollary 6.2 for a graph to be Hamiltonian.

Corollary 6.16 *Let G be a graph of order n such that*

$$\deg v \geq (n+1)/2$$

for every vertex v of G, then G is Hamiltonian-connected.

Hamiltonian Extension Numbers

As indicated by Hamilton's remarks, the graph of the dodecahedron in Figure 6.2 is Hamiltonian. Indeed, Hamilton's statement implies that this graph has a much stronger property. As was mentioned in the instruction pamphlet to his Icosian Game, Hamilton also observed that every path of order 5 in the graph G of the dodecahedron lies on a Hamiltonian cycle of G. (He didn't use that terminology of course.) Although Hamilton didn't mention that this is also true of paths of smaller order but not true of all paths of order 6 (see Exercise 1), this is nevertheless the case. This leads to another class of highly Hamiltonian graphs.

The **Hamiltonian extension number** of a Hamiltonian graph G, denoted by $\text{he}(G)$, is the maximum positive integer k for which every path in G having order k or less lies on a Hamiltonian cycle of G. Hence, $\text{he}(G) = 5$ for the graph G of the dodecahedron.

By Dirac's theorem (Corollary 6.2), if G is a graph of even order $n \geq 4$ with $\delta(G) \geq n/2$, then G contains a Hamiltonian cycle. Obviously, every vertex of G lies on a Hamiltonian cycle of G and so $\text{he}(G) \geq 1$. For each even integer $n \geq 4$, the graph $H = K_2 \vee \left(2K_{(n-2)/2}\right)$ has order n and $\delta(H) = n/2$ such that not every edge of H lies on a Hamiltonian cycle of H. For example, the Hamiltonian graph H of order $n = 6$ in Figure 6.10 has $\delta(H) = n/2 = 3$ but no Hamiltonian cycle of H contains the edge uv (that is, no Hamiltonian cycle of H contains the path (u, v) of order 2). Thus, $\text{he}(H) = 1$. Because every edge of a Hamiltonian-connected graph G lies on a Hamiltonian cycle of G, it follows by Corollary 6.16 that if G is a graph of order n such that $\delta(G) \geq (n+1)/2$, then $\text{he}(G) \geq 2$. This is a special case of a more general result.

Corollary 6.17 *Let k and n be integers such that $n \geq 3$ and $1 \leq k < n$. If G is a graph of order $n \geq 3$ and $\delta(G) \geq (n+k-1)/2$, then* $\text{he}(G) \geq k$.

H :

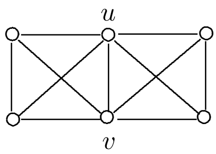


Figure 6.10: A Hamiltonian graph in which no Hamiltonian cycle contains uv

Proof. Since the result is true for $k = 1$ and $k = 2$, we may assume that $k \geq 3$ and so $n \geq 4$. Let P be a path of order ℓ in G, where $3 \leq \ell \leq k$, say $P = (u = u_1, u_2, \ldots, u_\ell = v)$. Let $S = \{u_2, u_3, \ldots, u_{\ell-1}\}$ and let $H = G - S$. Then H has order $n' = n - \ell + 2$ and

$$\delta(H) \geq \delta(G) - \ell + 2 \geq \frac{n + \ell - 1}{2} - \ell + 2 = \frac{n - \ell + 3}{2} = \frac{n' + 1}{2}.$$

By Corollary 6.16, H is Hamiltonian-connected and so H contains a Hamiltonian $u - v$ path P'. Thus, P' and P produce a Hamiltonian cycle in G containing P. Therefore, $\text{he}(G) \geq k$. ■

With the aid of Corollary 6.16, a lower bound for $\text{he}(G)$ in terms of $\delta(G)$ can be obtained in [45].

Theorem 6.18 *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then*

$$\text{he}(G) \geq 2\delta(G) - n + 1.$$

Proof. If $\delta(G) = n/2$, then $2\delta(G) - n + 1 = 1$ and the theorem follows trivially. Thus, we may assume that $\delta(G) > n/2$. Hence, $\delta(G) \geq (n+1)/2$. By Corollary 6.16, G is Hamiltonian-connected, which implies that every edge of G lies on a Hamiltonian cycle of G. Furthermore, since $\delta(G) \leq n - 1$, it follows that $2\delta(G) - n + 1 \leq n - 1$. Since G is Hamiltonian, G contains paths of order $2\delta(G) - n + 1$ or less. Because the result is immediate if $\delta(G) = n - 1$, we may assume that $n/2 < \delta(G) \leq n - 2$. Consequently, $n \geq 5$ and $2\delta(G) - n + 1 \geq 2$.

Let P be a $u - v$ path of order ℓ, where $3 \leq \ell \leq 2\delta(G) - n + 1$ and let H be the subgraph of G induced by $(V(G) - V(P)) \cup \{u, v\}$. Thus, the order of H is

$$n' = n - \ell + 2 \geq n - (2\delta(G) - n + 1) + 2 = 2(n - \delta(G)) + 1 \geq 5.$$

Moreover,

$$\begin{aligned} \delta(H) &\geq \delta(G) - \ell + 2 \geq \delta(G) - (2\delta(G) - n + 1) + 2 \\ &= n - \delta(G) + 1 = \frac{n' + 1}{2}. \end{aligned}$$

By Corollary 6.16, H is Hamiltonian-connected. Therefore, H contains a Hamiltonian $u - v$ path P', which, together with P, produce a Hamiltonian cycle of G containing P. ■

As we have observed, it is a consequence of Dirac's theorem that if G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{1}{2}n$, then every vertex (path of order 1) lies on a Hamiltonian cycle of G. By replacing $\frac{1}{2}$ by a larger rational number, an even stronger result can be obtained.

Corollary 6.19 *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq rn$ for some rational number $r \in \left[\frac{1}{2}, 1\right)$, then* $\text{he}(G) \geq (2r-1)n+1$.

Proof. By Theorem 6.18, $\text{he}(G) \geq 2\delta(G)-n+1 \geq 2rn-n+1 = (2r-1)n+1$. ■

We now show that the lower bound presented in Corollary 6.19 is sharp. We know that this is true for $r = \frac{1}{2}$, so assume that $\frac{1}{2} < r < 1$. Then $r = a/b$ for some positive integers a and b, where $\frac{1}{2} < \frac{a}{b} < 1$ and so $a < b < 2a$. Let $G = K_{a,a,\ldots,a,a(b-a)}$ be the complete $(a+1)$-partite graph with partite sets V_i $(1 \leq i \leq a+1)$ where $|V_i| = a$ for $1 \leq i \leq a$ and $|V_{a+1}| = a(b-a)$. This graph G has order $n = ab$ and $\delta(G) = a^2 = \frac{a}{b}(ab) = rn$. By Corollary 6.19, every path of order $(2r-1)n+1 = 2a^2 - ab + 1$ lies on a Hamiltonian cycle of G. Let P be a path of order $a(2a-b)+2 = (2r-1)n+2$ such that $V(P) \subseteq \bigcup_{i=1}^{a} V_i$. Since

$$\left|\bigcup_{i=1}^{a} V_i\right| - |V(P)| = a^2 - [a(2a-b)+2] = a(b-a) - 2$$

and $|V_{a+1}| = a(b-a)$, there is no Hamiltonian path on the remaining vertices of G and so there is no Hamiltonian cycle of G containing this path P. Consequently, $\text{he}(G) = (2r-1)n+1$ and the bound for $\text{he}(G)$ in Corollary 6.19 is sharp.

Corollaries 6.17 and 6.19 are equivalent (see Exercise 33).

Panconnected and Pancyclic Graphs

We now consider a property that a graph may possess that is even stronger than being Hamiltonian-connected. A connected graph G of order n is said to be **panconnected** if for each pair u, v of distinct vertices of G, there exists a $u-v$ path of length ℓ for each integer ℓ satisfying $d(u,v) \leq \ell \leq n-1$. If a graph is panconnected, then it is Hamiltonian-connected. The next example indicates that these concepts are not equivalent.

For $k \geq 3$, let G_k be that graph such that $V(G_k) = \{v_1, v_2, \ldots, v_{2k}\}$ and

$$\begin{aligned} E(G_k) &= \{v_i v_{i+1} : i = 1, 2, \ldots, 2k\} \cup \{v_i v_{i+3} : i = 2, 4, \ldots, 2k-4\} \cup \\ &\quad \{v_1 v_3, v_{2k-2} v_{2k}\}, \end{aligned}$$

where all subscripts are expressed modulo $2k$. Although for each pair u, v of distinct vertices and for each integer ℓ satisfying $k \leq \ell \leq 2k-1$, the graph G_k contains a $u-v$ path of length ℓ, there is no $v_1 - v_{2k}$ path of length ℓ if $1 < \ell < k$. Since $d(v_1, v_{2k}) = 1$, it follows that G_k is not panconnected.

James Williamson [257] obtained a sufficient condition for a graph G to be panconnected in terms of the minimum degree of G.

Theorem 6.20 *If G is a graph of order $n \geq 4$ such that*

$$\deg v \geq (n+2)/2$$

for every vertex v of G, then G is panconnected.

Proof. If $n = 4$, then $G = K_4$ and the statement is true.

Suppose that the theorem is not true. Thus, there exists a graph G of order $n \geq 5$ with $\delta(G) \geq (n+2)/2$ such that G is not panconnected; that is, there are vertices u and v of G and an integer ℓ with $d(u,v) < \ell < n-1$ such that there is no $u-v$ path of length ℓ. Let $G^* = G - \{u, v\}$. Then G^* has order $n^* = n - 2 \geq 3$ and $\delta(G^*) \geq (n+2)/2 - 2 = n^*/2$. Therefore, by Corollary 6.2, the graph G^* contains a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{n^*}, v_1)$.

If $uv_i \in E(G)$, $1 \leq i \leq n^*$, then $vv_{i+\ell-2} \notin E(G)$, where the subscripts are expressed modulo n^*; for otherwise,

$$(u, v_i, v_{i+1}, \ldots, v_{i+\ell-2}, v)$$

is a $u-v$ path of length ℓ in G. Thus, for each vertex of C that is adjacent to u in G, there is a vertex of C that is not adjacent to v in G. Since $\deg_G u \geq (n+2)/2$, we conclude that u is adjacent to at least $n/2$ vertices of C; so,

$$\deg_G v \leq 1 + n^* - \frac{n}{2} = \frac{n}{2} - 1.$$

This, however, produces a contradiction. ■

The result presented in Theorem 6.20 cannot be improved in general. Let $n = 2k+1 \geq 7$, and consider the graph $K_{k,k+1}$ with partite sets V_1 and V_2, where $|V_1| = k$ and $|V_2| = k+1$. The graph G is obtained from $K_{k,k+1}$ by constructing a path P_{k-1} on $k-1$ vertices of V_2. Join the remaining two vertices x and y of V_2 by an edge (see Figure 6.11 for $k = 4$). Then $\deg v \geq (n+1)/2$ for every vertex v but G is not panconnected since G contains no $x-y$ path of length 3.

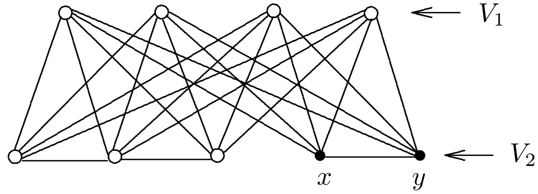


Figure 6.11: A graph that is not panconnected

Perhaps surprisingly, not only does $\sigma_2(G) \geq n+2$ fail to imply that G is panconnected but there is no constant c for which $\sigma_2(G) \geq n+c$ implies that G

is panconnected. For example, for an integer $c \geq 2$, let G be the graph of order $n = 2c + 4$, where $V(G) = \{u, w\} \cup U \cup W$ with $|U| = |W| = c + 1$ such that $G[U \cup W] = K_{2c+2}$ and where u is adjacent to every vertex of U, w is adjacent to every vertex of W and $uw \in E(G)$. (See Figure 6.12 for this graph G when $c = 2$.) Then $\sigma_2(G) = 3c + 4 = n + c$. However, since $d(u, w) = 1$ and there is no $u - w$ path of length 2 in G, it follows that G is not panconnected.

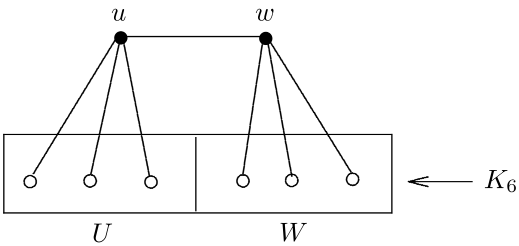


Figure 6.12: A graph that is not panconnected

We have, however, seen a number of results concerning a graph G of order n and size m such that if $\deg u + \deg v$ is at least some expression $f(n)$ involving n for all pairs u, v of nonadjacent vertices of G, then G has a certain property. In particular, by Theorem 6.1, if $\deg u + \deg v \geq n$, then G is Hamiltonian. Bondy [33] showed that graphs satisfying this condition have yet another *highly Hamiltonian* property.

A graph G of order $n \geq 3$ is called **pancyclic** if G contains a cycle of every possible length, that is, G contains a cycle of length ℓ for each ℓ with $3 \leq \ell \leq n$. The following theorem of Bondy [33] states that every graph G of order $n \geq 3$ for which $\sigma_2(G) \geq n$ is not only Hamiltonian but is pancyclic with one exception when n is even.

Theorem 6.21 *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then either G is pancyclic or n is even and $G = K_{\frac{n}{2}, \frac{n}{2}}$.*

6.5 Powers of Graphs and Line Graphs

A number of graph operations have been defined and studied that have led to results dealing with Hamiltonian and Eulerian properties. One of the simplest operations is that of the subdivision graph of a graph. The **subdivision graph** $S(G)$ of a graph G is that graph obtained from G by replacing each edge $e = uv$ of G by a new vertex w_e and the two new edges uw_e and vw_e. The subdivision graph of K_4 is shown in Figure 6.13.

If G is a graph of order n and size m, then the order of $S(G)$ is $n + m$ and its size is $2m$. Furthermore, $S(G)$ is a bipartite graph with partite sets $V(G)$ and $V(S(G)) - V(G)$.

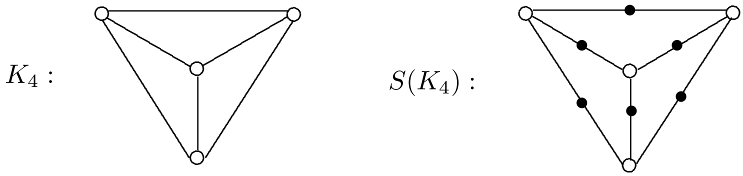


Figure 6.13: The subdivision graph of a graph

The Square and Cube of a Graph

Associated with each connected graph of order n and diameter d is a class of graphs defined in terms of distance. For each positive integer k, the **kth power** G^k of a graph G is that graph with $V(G^k) = V(G)$ and $uv \in E(G^k)$ if and only if $1 \le d_G(u,v) \le k$. Thus, $G^1 = G$ and $G^k = K_n$ if $k \ge d$. The graph G^2 is also called the **square** of G, while G^3 is called the **cube** of G. A graph G with its square and cube are shown in Figure 6.14. While the graph G^3 of Figure 6.14 is Hamiltonian, G^2 is not.

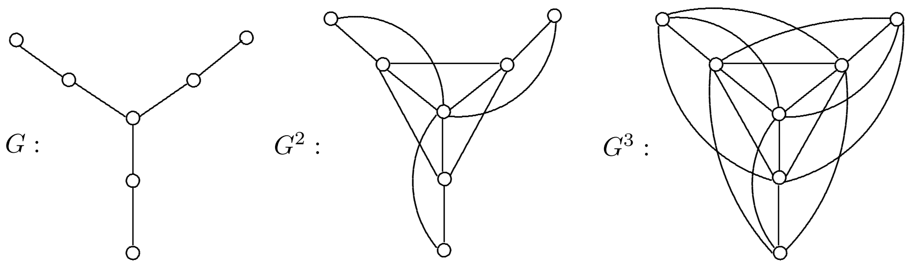


Figure 6.14: A graph and its square and cube

Since the kth power G^k $(k \ge 2)$ of a connected graph G contains G as a subgraph (as a proper subgraph if G is not complete), it follows that G^k is Hamiltonian if G is Hamiltonian. Even if G is not Hamiltonian, G^k is Hamiltonian for a sufficiently large integer k. It is therefore natural to ask for the minimum positive integer k for which G^k is Hamiltonian. Certainly, for connected graphs in general, $k = 2$ will not suffice since the graph G^2 of Figure 6.14 is not Hamiltonian. We noted that G^3 *is* Hamiltonian, however. In fact, the cube of every connected graph of order at least 3 is Hamiltonian. Indeed, a stronger result exists, discovered by Milan Sekanina [218] and later, but independently, by Jerome Karaganis [138].

Theorem 6.22 *If G is a connected graph, then G^3 is Hamiltonian-connected.*

Proof. If H is a spanning tree of G and H^3 is Hamiltonian-connected, then G^3 is Hamiltonian-connected. Hence, it suffices to prove that the cube of every tree is Hamiltonian-connected. To show this, we proceed by induction on n, the order of the tree. For small values of n, the result is obvious.

Assume for all trees H of order less than n that H^3 is Hamiltonian-connected and let T be a tree of order n. Let u and v be any two vertices of T. We consider two cases.

Case 1. *u and v are adjacent in T.* Let $e = uv$ and consider the forest $T - e$. This forest has two components, one tree T_u containing u and the other tree T_v containing v. By hypothesis, T_u^3 and T_v^3 are Hamiltonian-connected. Let u_1 be any vertex of T_u adjacent to u, and let v_1 be any vertex of T_v adjacent to v. If T_u or T_v is trivial, we define $u_1 = u$ or $v_1 = v$, respectively. Note that u_1 and v_1 are adjacent in T^3 since $d_T(u_1, v_1) \leq 3$. Let P_u be a Hamiltonian $u - u_1$ path (which may be trivial) of T_u^3 and let P_v be a Hamiltonian $v_1 - v$ path of T_v^3. The path formed by beginning with P_u followed by the edge u_1v_1 and then the path P_v is a Hamiltonian $u - v$ path of T^3.

Case 2. *u and v are not adjacent in T.* Since T is a tree, there exists a unique path between every two of its vertices. Let P be the unique $u - v$ path of T and let $f = uw$ be the edge of P incident with u. The graph $T - f$ consists of two trees, one tree T_u containing u and the other tree T_w containing w. By hypothesis, there exists a Hamiltonian $w - v$ path P_w in T_w^3. Let u_1 be a vertex of T_u adjacent to u, or let $u_1 = u$ if T_u is trivial, and let P_u be a Hamiltonian $u - u_1$ path in T_u^3. Because $d_T(u_1, w) \leq 2$, the edge u_1w is present in T^3. Hence, the path formed by starting with P_u followed by u_1w and then P_w is a Hamiltonian $u - v$ path of T^3. ■

It is, of course, an immediate consequence of Theorem 6.22 that for every connected graph G of order at least 3, its cube G^3 is Hamiltonian. Although it is not true that the square of every connected graph of order at least 3 is Hamiltonian, it was conjectured independently by Crispin Nash-Williams and Michael D. Plummer that for 2-connected graphs, this is the case. In 1974, Herbert Fleischner [92] verified this conjecture. More recent proofs of this theorem have been given by Stanislav Řiha [199] in 1991 and Agelos Georgakopoulos [103] in 2009.

Theorem 6.23 *If G is a 2-connected graph, then G^2 is Hamiltonian.*

A variety of results strengthening (but employing) Fleischner's work have been obtained. For example, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [47] showed that the square of a 2-connected graph is Hamiltonian-connected.

Theorem 6.24 *If G is a 2-connected graph, then G^2 is Hamiltonian-connected.*

Proof. Since G is 2-connected, G has order at least 3. Let u and v be any two vertices of G. Let $G_1, G_2, \ldots, G_5$ be five distinct copies of G and let u_i and v_i $(i = 1, 2, \ldots, 5)$ be the vertices in G_i corresponding to u and v in G. Form a new graph F by adding to the union $G_1+G_2+\cdots+G_5$ two new vertices w_1 and w_2 and ten new edges w_1u_i and w_2v_i $(i=1,2,\ldots,5)$. The graph F is shown in Figure 6.15. Clearly, neither w_1 nor w_2 is a cut-vertex of F. Furthermore, since each graph G_i is 2-connected and contains two vertices adjacent to vertices in $V(F)-V(G_i)$, no vertex of G_i is a cut-vertex of F. Hence, F is 2-connected.

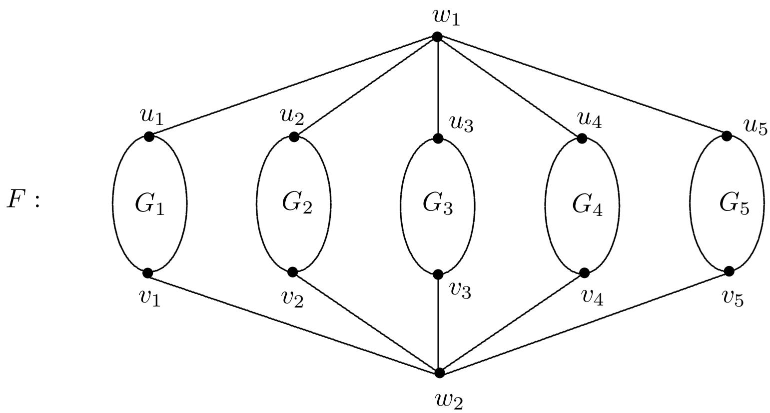


Figure 6.15: The graph F in the proof of Theorem 6.24

By Theorem 6.23, F^2 has a Hamiltonian cycle C, which, of course, contains w_1 and w_2. At least one of the graphs G_i, say G_1, contains no vertex adjacent to either w_1 or w_2 on C. Since u_1 and v_1 are the only vertices of G_1 adjacent on C to vertices not in G_1, it follows that C has a u_1-v_1 path containing all vertices of G_1. Thus, G_1^2 has a Hamiltonian u_1-v_1 path, which implies that G^2 contains a Hamiltonian $u-v$ path. ■

The Line Graph of a Graph

The most familiar graph operation of a graph is that of the line graph. The **line graph** $L(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. A graph and its line graph are shown in Figure 6.16, where the vertex u_i $(1\le i\le 6)$ of $L(G)$ corresponds to the edge e_i of G.

It is relatively easy to determine the number of vertices and the number of edges in the line graph $L(G)$ of a graph G in terms of easily computed quantities in G. Indeed, if G is a graph of order n and size m with degree sequence $d_1,d_2,\ldots,d_n$ and its line graph $L(G)$ has order n' and size m', then $n'=m$ and

$$m'=\sum_{i=1}^{n}\binom{d_i}{2}$$

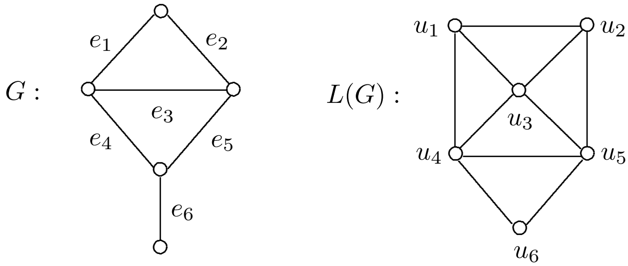


Figure 6.16: A graph and its line graph

since each edge of $L(G)$ corresponds to a pair of adjacent edges of G. For each nontrivial connected graph G, its line graph $L(G)$ is also connected (see Exercise 50). For the set $\mathcal{G}$ of all connected graphs and the set $\mathcal{G}'$ of all nonempty connected graphs, we may think of L as a function, namely $L : \mathcal{G}' \to \mathcal{G}$. Hassler Whitney [255] showed that this function is very nearly injective.

Theorem 6.25 *Let G_1 and G_2 be nontrivial connected graphs. If $L(G_1) \cong L(G_2)$, then $G_1 \cong G_2$ unless one of G_1 and G_2 is K_3 and the other is $K_{1,3}$.*

The function $L : \mathcal{G}' \to \mathcal{G}$ is not close to being surjective, however. The graph $K_{1,3}$ is one of many graphs that is not isomorphic to the line graph of any graph. To see this, suppose that there is a graph H such that $L(H) = K_{1,3}$. Then H is a graph of size 4 containing an edge that is adjacent to the other three edges, no two of which are adjacent to each other. Such a graph H does not exist and so $K_{1,3}$ is not the line graph of any graph. Indeed, Lowell W. Beineke [17] obtained the following result.

Theorem 6.26 *A graph G is isomorphic to the line graph of some graph if and only if none of the nine graphs of Figure* 6.17 *is isomorphic to an induced subgraph of G.*

We turn to the problem of determining characteristics possessed by a graph that yield certain Hamiltonian properties of its line graph. Frank Harary and Crispin Nash-Williams [122] characterized those graphs having a Hamiltonian line graph. A circuit C in a graph G is called a **dominating circuit** if every edge of G either belongs to C or is adjacent to an edge of C. Equivalently, a circuit C in a graph G is a dominating circuit if every edge of G is incident with a vertex of C. Although we did not present proofs of Theorems 6.25 and 6.26, we do give a proof of the following result, which characterizes those graphs G for which $L(G)$ is Hamiltonian.

Theorem 6.27 *Let G be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if $G = K_{1,\ell}$ for some $\ell \geq 3$ or G contains a dominating circuit.*

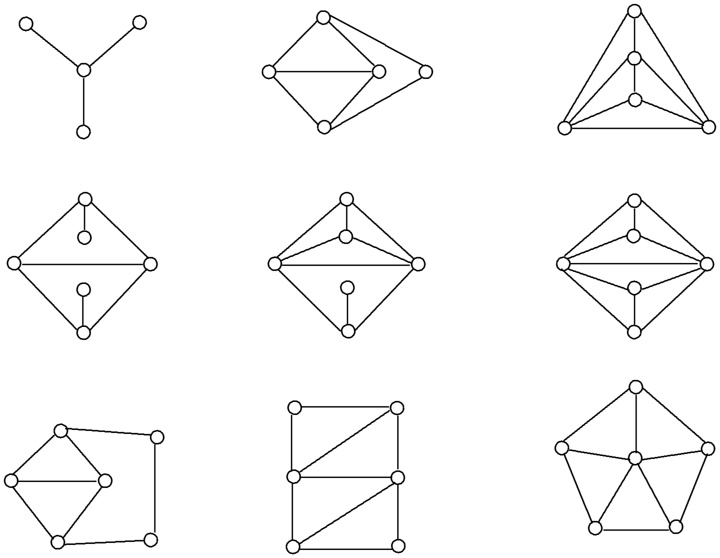


Figure 6.17: The induced subgraphs not contained in any line graph

Proof. If $G = K_{1,\ell}$ for some $\ell \geq 3$, then $L(G)$ is Hamiltonian since $L(G) = K_\ell$. Suppose, then, that G contains a dominating circuit

$$C = (v_1, v_2, \ldots, v_t, v_1).$$

It suffices to show that there exists an ordering $s : e_1, e_2, \ldots, e_m$ of the m edges of G such that e_i and e_{i+1} are adjacent edges of G, for $1 \leq i \leq m-1$, as are e_1 and e_m, since such an ordering s corresponds to a Hamiltonian cycle of $L(G)$. Begin the ordering s by selecting, in any order, all edges of G incident with v_1 that are not edges of C, followed by the edge v_1v_2. At each successive vertex v_i, $2 \leq i \leq t-1$, select, in any order, all edges of G incident with v_i that are neither edges of C nor previously selected edges, followed by the edge v_iv_{i+1}. This process terminates with the edge $v_{t-1}v_t$. The ordering s is completed by adding the edge v_tv_1. Since C is a dominating circuit of G, every edge of G appears exactly once in s. Furthermore, consecutive edges of s as well as the first and last edges of s are adjacent in G.

Conversely, suppose that G is not a star but $L(G)$ is Hamiltonian. We show that G contains a dominating circuit. Since $L(G)$ is Hamiltonian, there is an ordering $s : e_1, e_2, \ldots, e_m$ of the m edges of G such that e_i and e_{i+1} are adjacent edges of G for $1 \leq i \leq m-1$, as are e_1 and e_m. For $1 \leq i \leq m-1$, let v_i be the vertex of G incident with both e_i and e_{i+1}. (Note that $1 \leq k \neq q \leq m-1$ does not necessarily imply that $v_k \neq v_q$.) Since G is not a star, there is a smallest integer j_1 exceeding 1 such that $v_{j_1} \neq v_1$. The vertex v_{j_1-1} is incident with e_{j_1}, the vertex v_{j_1} is incident with e_{j_1} and $v_{j_1-1} = v_1$. Thus, $e_{j_1} = v_1v_{j_1}$. Next, let j_2 (if it exists) be the smallest integer exceeding j_1 such that $v_{j_2} \neq v_{j_1}$.

The vertex v_{j_2-1} is incident with e_{j_2}, the vertex v_{j_2} is incident with e_{j_2} and $v_{j_2-1} = v_{j_1}$. Thus, $e_{j_2} = v_{j_1}v_{j_2}$. Continuing in this fashion, we finally arrive at a vertex v_{j_t} such that $e_{j_t} = v_{j_t-1}v_{j_t}$, where $v_{j_t} = v_{m-1}$. Since every edge of G appears exactly once in s and since $1 < j_1 < j_2 < \cdots < j_t = m-1$, this construction yields a trail

$$T = (v_1, e_{j_1}, v_{j_1}, e_{j_2}, v_{j_2}, \ldots, v_{j_{t-1}}, e_{j_t}, v_{j_t} = v_{m-1})$$

in G with the properties that (i) every edge of G is incident with a vertex of T and (ii) neither e_1 nor e_m is an edge of T.

Let w be the vertex of G incident with both e_1 and e_m. We consider four possible cases.

Case 1. $w = v_1 = v_{m-1}$. Then T itself is a dominating circuit of G.

Case 2. $w = v_1$ *and* $w \neq v_{m-1}$. Since e_m is incident with both w and v_{m-1}, it follows that $e_m = v_{m-1}w = v_{m-1}v_1$. Thus, $C = (T, e_m, v_1)$ is a dominating circuit of G.

Case 3. $w = v_{m-1}$ *and* $w \neq v_1$. Since e_1 is incident with both w and v_1, we have that $e_1 = wv_1 = v_{m-1}v_1$. Thus, $C = (T, e_1, v_1)$ is a dominating circuit of G.

Case 4. $w \neq v_{m-1}$ *and* $w \neq v_1$. Since e_m is incident with both w and v_{m-1}, it follows that $e_m = v_{m-1}w$. Because e_1 is incident with both w and v_1, we have that $e_1 = wv_1$. Thus, $v_1 \neq v_{m-1}$ and $C = (T, e_m, w, e_1, v_1)$ is a dominating circuit of G. ■

As a consequence of Theorem 6.27, if G is either Eulerian or Hamiltonian, then $L(G)$ is Hamiltonian. In fact, successively taking the line graph of a connected graph has some interesting consequences.

For a nonempty graph G, we write $L^0(G)$ to denote G and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the **iterated line graph** $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. The following result is due to Gary Chartrand and Curtiss E. Wall [51].

Theorem 6.28 *If G is a connected graph such that* $\deg v \geq 3$ *for every vertex v of G, then $L^2(G)$ is Hamiltonian.*

Proof. Let v be a vertex of G, where $\deg v = r \geq 3$. Then in $L(G)$, the edges incident with v give rise to a subgraph H_v, where $H_v \cong K_r$. Let C_v be a Hamiltonian cycle in H_v. Let H be the spanning subgraph of $L(G)$ defined by

$$V(H) = V(L(G)) \text{ and } E(H) = \textstyle\bigcup_{v \in V(G)} E(C_v).$$

Then H is connected and every vertex of H is even. Consequently, H is Eulerian and so H is a dominating circuit of $L(G)$. By Theorem 6.27, $L^2(G)$ is Hamiltonian. ■

If $G = P_n$, where $n \geq 2$, then $L(G) = P_{n-1}$. Thus, $L^{n-1}(G) = P_1$ and $L^k(G)$ is not defined for $k \geq n$. If $G = K_{1,3}$, then $L(G) = C_3$. If $G = C_n$ for some $n \geq 3$, then $L(G) = C_n$ and $L^k(C_n) = C_n$ for every nonnegative integer k. Suppose then that G is a connected graph that is not a path, $K_{1,3}$ or a cycle. Then $\Delta(G) \geq 3$. Suppose that G contains a vertex u of degree 1. Then G contains a $u - v$ path $P = (u = u_0, u_1, \ldots, u_\ell = v)$ of minimum length $\ell \geq 1$ where $\deg v \geq 3$. This gives rise to an $x_0 - x$ path P' of length ℓ in $L(G)$, where x_0 corresponds to u_0u_1, x corresponds to $u_{\ell-1}u_\ell$, $\deg x_0 = 1$ if $\ell \geq 2$, and $\deg x \geq 3$. By successively taking the line graph of G, $L(G)$, $L^2(G)$ and so on, we eventually arrive at a graph $L^k(G)$ for some k where there are no end-vertices. If a connected graph H contains no end-vertices but does contain vertices of degree 2, then G contains a $y - z$ path (or $y - z$ cycle if $y = z$) $Q = (y = y_0, y_1, \ldots, y_t = z)$ where $t \geq 2$, $\deg y_i = 2$ for $1 \leq i \leq t-1$, $\deg y \geq 3$ and $\deg z \geq 3$. This gives rise to a $w - x$ path $Q' = (w = w_1, w_2, \ldots, w_t = x)$ of length $t-1$ in $L(G)$ where w_1 corresponds to the edge y_0y_1 in G, w_t corresponds to the edge $y_{t-1}y_t$, $\deg w_i = 2$ for $2 \leq i \leq t-1$, $\deg w_1 \geq 3$ and $\deg w_t \geq 3$. By successively taking the line graph of H, $L(H)$, $L^2(H)$ and so on, we arrive at a graph $L^r(H)$ in which all vertices have degree at least 3. This is illustrated for the graph G of Figure 6.18. It then follows that there is a sufficiently large integer s such that the degree of every vertex of $L^s(G)$ is at least 3. By the discussion above and Theorem 6.28, we then have the following.

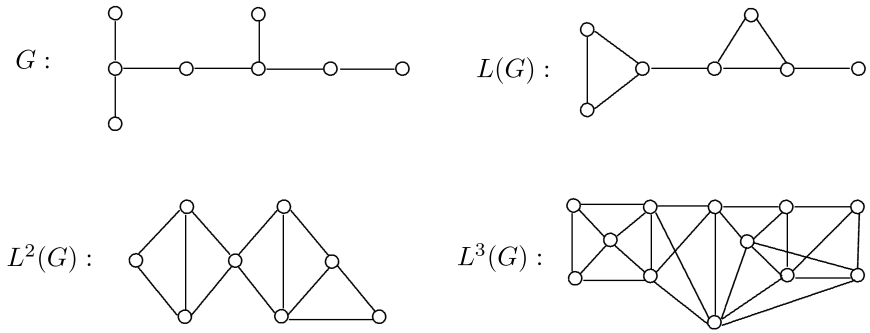


Figure 6.18: A graph and iterated line graphs

Theorem 6.29 *If G is a connected graph that is not a path, then there exists an integer k_0 such that $L^k(G)$ is Hamiltonian for every integer $k \geq k_0$.*

The Total Graph of a Graph

A graph operation related to the line graph is the total graph. The **total graph** $T(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the elements of the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if the corresponding elements in G are two adjacent vertices, two adjacent edges or an incident vertex and edge. A graph H and its total graph $T(H)$ are shown in Figure 6.19.

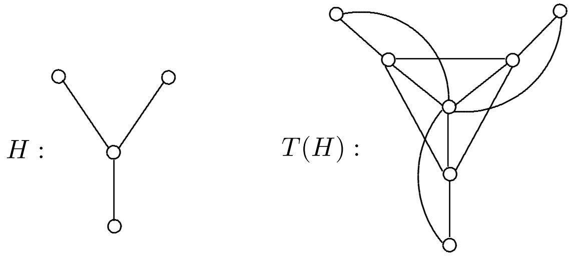


Figure 6.19: A graph and its total graph

One might observe that $T(H) \cong G^2$ for the graph H of Figure 6.19 and the graph G of Figure 6.14. We saw that G^2 is not Hamiltonian. Thus, for the graph H of Figure 6.19, $T(H)$ is not Hamiltonian. This, in fact, is not surprising for if F is any graph, then $T(F) \cong [S(F)]^2$, that is, the total graph of F is the square of the subdivision graph of F. Since the graph G of Figure 6.14 is isomorphic to $S(K_{1,3})$ and the graph H of Figure 6.19 is isomorphic to $K_{1,3}$, it follows that $T(H) \cong G^2$, where G is the graph of Figure 6.14.

While the total graph of a nontrivial connected graph G need not be Hamiltonian, the same cannot be said for $T(T(G))$.

Theorem 6.30 *If G is a nontrivial connected graph, then $T(T(G))$ is Hamiltonian.*

Proof. First, we consider $T(G)$. Since $S(G)$ is a nontrivial connected graph, $[S(G)]^2$ is 2-connected and $T(G) \cong [S(G)]^2$, it follows that $H = T(G)$ is 2-connected. Hence, $S(H)$ is 2-connected. By Theorem 6.23, $[S(H)]^2$ is Hamiltonian. Thus, $T(H) \cong [S(H)]^2$ and so $T(T(G))$ is Hamiltonian. ■

Exercises for Chapter 6

Section 6.2. Sufficient Conditions for Hamiltonian Graphs

1. (a) Show that the graph of the dodecahedron is Hamiltonian.
   (b) Hamilton observed that every path of order 5 lies on a Hamiltonian cycle of the graph of the dodecahedron. Show that there is a path of order 6 in this graph that does not have this property.

2. Show that if G is a graph containing a vertex that is adjacent to at least three vertices of degree 2, then G is not Hamiltonian.

3. (a) Show that if G is a bipartite graph of odd order, then G is not Hamiltonian.
   (b) Show that the Herschel graph of Figure 6.20 is not Hamiltonian.

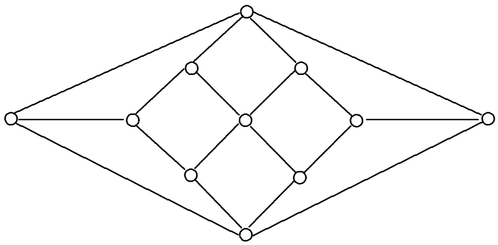


Figure 6.20: The Herschel graph

4. (a) Prove that if G and H are Hamiltonian graphs, then $G \,\square\, H$ is Hamiltonian.
   (b) Prove that the n-cube Q_n, $n \geq 2$, is Hamiltonian.

5. Show that the bound presented in Theorem 6.1 is sharp, that is, show that for infinitely many integers $n \geq 3$ there are non-Hamiltonian graphs G of order n such that $\sigma_2(G) = n - 1$.

6. A Hamiltonian graph G of order n is **k-ordered Hamiltonian** for an integer k with $1 \leq k \leq n$ if for every ordered set $S = \{v_1, v_2, \ldots, v_k\}$ of k vertices of G, there is a Hamiltonian cycle of G encountering these k vertices of S in the order listed.

   (a) Let G be a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for every vertex v of G. Show that G is 3-ordered Hamiltonian.
   (b) Let G be a graph of order $n \geq 4$ such that $\deg v \geq n/2$ for every vertex v of G. Show that G need not be 4-ordered Hamiltonian.
   (c) Show that if G is a 4-ordered Hamiltonian graph, then G is 3-connected.

7. Let G be a graph of order $n \geq 3$ having the property that for each vertex v of G, there is a Hamiltonian path with initial vertex v. Show that G is 2-connected but not necessarily Hamiltonian.

8. (a) Let G be a graph of order $n \geq 2$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$. Show that if there is no integer $k < (n+1)/2$ for which $d_k \leq k-1$ and $d_{n+1-k} \leq n-k-1$, then G has a Hamiltonian path.
   (b) Show that every self-complementary graph has a Hamiltonian path.

9. Let G be a bipartite graph with partite sets U and W such that $|U| = |W| = k \geq 2$. Prove that if $\deg v > k/2$ for every vertex v of G, then G is Hamiltonian.

10. (a) Prove that if G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian.
    (b) Prove that if G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 1$, then G has a Hamiltonian path.

11. Show that the bound presented in Theorem 6.10 is sharp, that is, show that for infinitely many integers $n \geq 3$ there are non-Hamiltonian graphs G of order n such that $\kappa(G) \geq \alpha(G) - 1$.

12. (a) Show that a connected graph G of order $n = 2k+1$ having independence number $k+1$ is not Hamiltonian.
    (b) Give an example of a Hamiltonian graph H of order $n = 2k$ for some $k \geq 2$, where k vertices have degree 2, no two vertices of which are adjacent, while the remaining vertices have degree 3 or more.

13. Show that if G is a graph of order at least 2 for which $\kappa(G) \geq \alpha(G) - 1$, then G has a Hamiltonian path.

14. Prove that if T is a tree of order at least 4 that is not a star, then $\overline{T}$ contains a Hamiltonian path.

Section 6.3. Toughness of Graphs

15. (a) Prove that $K_{r,2r,3r}$ is Hamiltonian for every positive integer r.
    (b) Prove that $K_{r,2r,3r+1}$ is Hamiltonian for no positive integer r.
    (c) Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph of order at least 3, where $n_1 \leq n_2 \leq \cdots \leq n_k$. Find a necessary and sufficient condition for the graph G to be Hamiltonian.

16. Show that every 1-tough graph is 2-connected.

17. (a) Give an example of a graph G containing a Hamiltonian path for which $k(G-S) > |S|$ for some nonempty proper subset S of $V(G)$.

(b) State and prove a result analogous to Theorem 6.11 that gives a necessary condition for a graph to contain a Hamiltonian path.

18. Show that the graph G of Figure 6.21 is 1-tough but not Hamiltonian.

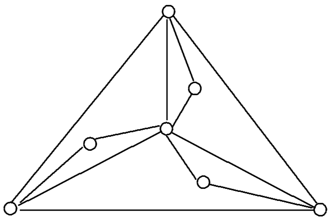


Figure 6.21: A 1-tough non-Hamiltonian graph

19. (a) Prove that if G is a graph of order 101 and $\delta(G) = 51$, then every vertex of G lies on a cycle of length 27.

(b) State and prove a generalization of (a).

20. Determine the toughness of the regular complete 3-partite graph $K_{r,r,r}$ $(r \geq 2)$.

21. Show that if H is a spanning subgraph of a noncomplete graph G, then $t(H) \leq t(G)$.

22. (a) Show that if G is a noncomplete graph of order n, then $t(G) \leq (n - \alpha(G))/\alpha(G)$.

(b) Show that the order of every noncomplete connected graph G is at least $\alpha(G)(1 + t(G))$.

23. Show for positive integers r and s with $r + s \geq 3$ that

$$t(K_{r,s}) = \min\{r, s\}/\max\{r, s\}.$$

24. Prove or disprove: There is a constant k_0 such that every k_0-connected graph is Hamiltonian.

25. (a) Determine the toughness of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ where $n_1 \leq n_2 \leq \cdots \leq n_k$.

(b) Show that for every nonnegative rational number r, there exists a graph G with $t(G) = r$.

26. Determine a formula for the toughness of a tree.

27. Show that the Bauer-Broersma-Veldman graph of Figure 6.8 is not Hamiltonian.

28. A well-known conjecture of Matthews and Sumner [163] states that every 4-connected claw-free graph is Hamiltonian. Restate this conjecture in terms of toughness for noncomplete graphs.

Section 6.4. Highly Hamiltonian Graphs

29. Show that if G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.

30. Prove that every Hamiltonian-connected graph of order 4 or more is 3-connected.

31. Give a proof by contradiction of Theorem 6.15:

    Let G be a graph of order $n \geq 4$. If $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected.

    by first observing that $G - v$ is Hamiltonian for every vertex v of G.

32. Let n and k be positive integers such that $n \geq k+2$. Prove that if G is a graph of order n such that if $\sigma_2(G) \geq n+k-1$, then $\text{he}(G) \geq k$.

33. Show that Corollary 6.17:

    Let k and n be integers such that $n \geq 3$ and $1 \leq k < n$. If G is a graph of order $n \geq 3$ and $\delta(G) \geq (n+k-1)/2$, then $\text{he}(G) \geq k$.

    and Corollary 6.19:

    If G is a graph of order $n \geq 3$ such that $\delta(G) \geq rn$ for some rational number $r \in \left[\frac{1}{2}, 1\right)$, then $\text{he}(G) \geq (2r-1)n + 1$.

    are equivalent by showing that

    (a) Corollary 6.17 implies Corollary 6.19 and

    (b) Corollary 6.19 implies Corollary 6.17.

34. (a) Give an example of Hamiltonian graph G of order $n \geq 4$ for which $\text{he}(G) = n-3$.

    (b) Prove that if G is a graph of order $n \geq 3$ such that $\text{he}(G) \geq n-2$, then $\text{he}(G) = n$.

35. Give an example of a graph G that is pancyclic but not panconnected.

36. Prove that no bipartite graph of order 3 or more is Hamiltonian-connected, panconnected or pancyclic.

37. Let $f(n)$ denote the function of n. Determine a sharp lower bound $f(n)$ such that if G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq f(n)$, then G is pancyclic.

38. A graph G is called **hypo-Hamiltonian** if G is non-Hamiltonian but $G - v$ is Hamiltonian for every vertex v of G. Show that the Petersen graph is hypo-Hamiltonian.

Section 6.5. Powers of Graphs and Line Graphs

39. Determine all those connected graphs G for which $S(G)$ is Eulerian.

40. Determine all those connected graphs G for which $S(G)$ is Hamiltonian.

41. Show that if G is a connected graph of order n and size m, then $\alpha(S(G)) = m$ unless G belongs to a familiar class of graphs.

42. Show that if G is a connected graph of order $n \geq 2$ and k is an integer with $1 \leq k \leq n-1$, then G^k is k-connected.

43. Show that if G is a connected graph of diameter ℓ and $1 \leq k \leq \ell$, then $\text{diam}(G^k) = \lceil \ell/k \rceil$.

44. A graph H is called a **square root** of a connected graph G if $H^2 = G$.

    (a) Give an example of a connected graph with two non-isomorphic square roots.

    (b) Give an example of a connected graph with a unique square root.

45. Show that the graph G^2 of Figure 6.14 is not Hamiltonian.

46. Prove that if v is any vertex of a connected graph G of order at least 4, then $G^3 - v$ is Hamiltonian.

47. Prove that if G is a self-complementary graph of order at least 5, then G^2 is Hamiltonian-connected.

48. Let G be a connected graph G. Prove that if $k = 2^j$ for some positive integer j, then $t(G^k) \geq \frac{k}{2}\ \kappa(G)$.

49. According to Theorem 6.26, every line graph is claw-free. Is the converse true?

50. Prove that the line graph of every nontrivial connected graph is connected.

51. Determine a formula for the number of triangles in the line graph $L(G)$ in terms of quantities in G.

52. Prove that $L(G)$ is Eulerian if G is Eulerian.

53. (a) Find a necessary and sufficient condition for a graph G to have the property that $G \cong L(G)$.

    (b) Find a necessary and sufficient condition for a graph G to have the property that $L(G) \cong L^2(G)$.

54. For each of the following, prove or disprove.

    (a) If G is Hamiltonian, then G^2 is Hamiltonian-connected.

    (b) If G is connected and $L(G)$ is Eulerian, then G is Eulerian.

    (c) If G is Hamiltonian, then $L(G)$ is Hamiltonian-connected.

    (d) If G has a dominating circuit, then $L(G)$ has a dominating circuit.

55. Prove that if G is a connected graph and $L^3(G)$ is Eulerian, then $L^2(G)$ is Eulerian.

56. Give an example of a connected graph G such that $\deg v \geq 3$ for every vertex v of G but $L(G)$ is not Hamiltonian.

57. (a) Show that if G is a k-edge-connected graph, $k \geq 2$, then $L(G)$ is k-connected.

    (b) Show that if G is a k-edge-connected graph, $k \geq 2$, then $L(G)$ is $(2k-2)$-edge-connected.

58. Show that there exists a graph G that is not isomorphic to the total graph of any graph.

59. (a) Prove that if G is a nontrivial connected graph, then $T(G^2)$ is Hamiltonian.

    (b) Prove that if G is a nontrivial connected graph, then $(T(G))^2$ is Hamiltonian.

Chapter 7

Digraphs

There are occasions when the symmetric nature of graphs does not provide a desirable structure to represent a situation we may encounter. This leads us to the concept of directed graphs (digraphs).

7.1 Introduction to Digraphs

A **directed graph** or **digraph** D is a finite nonempty set of objects called **vertices** together with a (possibly empty) set of ordered pairs of distinct vertices of D called **arcs** or **directed edges**. For vertices u and v in D, an arc (u, v) is sometimes denoted by writing $u \to v$ (or $v \leftarrow u$). As with graphs, the vertex set of D is denoted by $V(D)$ or simply V and the arc set (or directed edge set) of D is denoted by $E(D)$ or E. A digraph D with vertex set $V = \{u, v, w, x\}$ and arc set $E = \{(u, v), (v, u), (u, w), (w, v), (w, x)\}$ is shown in Figure 7.1. When a digraph is described by means of a diagram, the "direction" of each arc is indicated by an arrowhead. Observe that in a digraph, it is possible for two arcs to join the same pair of vertices if the arcs are directed oppositely.

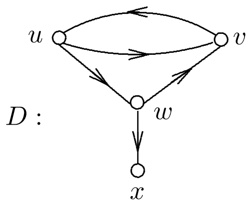


Figure 7.1: A digraph

Much of the terminology used for digraphs is quite similar to that used for graphs. The cardinality of the vertex set of a digraph D is called the **order** of D and is ordinarily denoted by n, while the cardinality of its arc set is the **size** of D and is ordinarily denoted by m. If $a = (u, v)$ is an arc of a digraph D, then u is said to be **adjacent to** v and v is **adjacent from** u. For a vertex v in a digraph D, the **outdegree** $\operatorname{od} v$ of v is the number of vertices of D to which v is adjacent, while the **indegree** $\operatorname{id} v$ of v is the number of vertices of D from which v is adjacent. The **out-neighborhood** $N^+(v)$ of a vertex v in a digraph D is the set of vertices adjacent from v, while the **in-neighborhood** $N^-(v)$ of v is the set of vertices adjacent to v. Thus, $\operatorname{od} v = |N^+(v)|$ and $\operatorname{id} v = |N^-(v)|$. The **degree** $\deg v$ of a vertex v is defined by

$$\deg v = \operatorname{od} v + \operatorname{id} v.$$

For the vertex v in the digraph of Figure 7.2, $\operatorname{od} v = 3$, $\operatorname{id} v = 2$ and $\deg v = 5$.

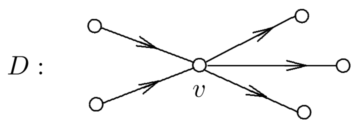


Figure 7.2: The outdegree, indegree and degree of a vertex

The First Theorem of Digraph Theory

The directed graph version of Theorem 1.4 is stated next.

Theorem 7.1 (**The First Theorem of Digraph Theory**) *If D is a digraph of size m, then*

$$\sum_{v \in V(G)} \operatorname{od} v = \sum_{v \in V(G)} \operatorname{id} v = m.$$

Proof. When the outdegrees of the vertices are summed, each arc is counted once. Similarly, when the indegrees of the vertices are summed, each arc is counted just once. ∎

A digraph D_1 is **isomorphic** to a digraph D_2, written $D_1 \cong D_2$, if there exists a bijective function $\phi : V(D_1) \to V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(\phi(u), \phi(v)) \in E(D_2)$. The function ϕ is called an **isomorphism** from D_1 to D_2.

There is only one digraph of order 1, namely the **trivial digraph**. Also, there is only one digraph of order 2 and size m for each m with $0 \leq m \leq 2$. There are four digraphs of order 3 and size 3, all of which are shown in Figure 7.3.

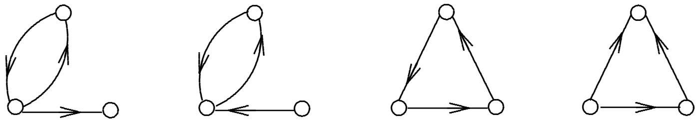


Figure 7.3: The digraphs of order 3 and size 3

A digraph D_1 is a **subdigraph** of a digraph D if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$. We use $D_1 \subseteq D$ to indicate that D_1 is a subdigraph of D. A subdigraph D_1 of D is a **spanning subdigraph** of D if $V(D_1) = V(D)$. Vertex-deleted, arc-deleted, induced and arc-induced subdigraphs are defined in the expected manner. These last two concepts are illustrated for the digraph D of Figure 7.4, where

$$V(D) = \{v_1, v_2, v_3, v_4\},\ U = \{v_1, v_2, v_3\} \text{ and } X = \{(v_1, v_2), (v_2, v_4)\}.$$

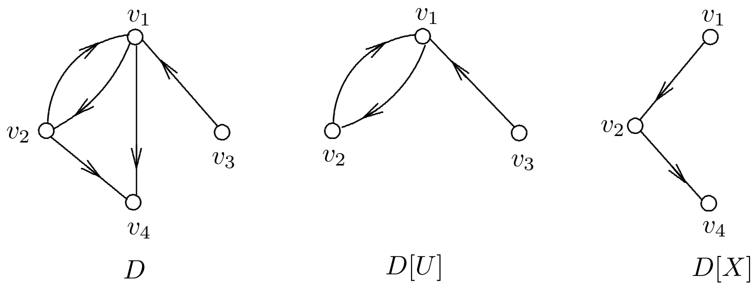


Figure 7.4: Induced and arc-induced subdigraphs

We now consider certain types of digraphs that occur periodically. A digraph is **symmetric** if whenever (u, v) is an arc of D, then (v, u) is an arc of D as well. There is a natural one-to-one correspondence between symmetric digraphs and graphs. The **complete symmetric digraph** K_n^* of order n has both arcs (u, v) and (v, u) for every two distinct vertices u and v. A digraph is called an **oriented graph** if whenever (u, v) is an arc of D, then (v, u) is *not* an arc of D. Thus, an oriented graph D can be obtained from a *graph* G by assigning a direction to (or by "orienting") each edge of G, thereby transforming each edge of a graph G into an arc and transforming G itself into an oriented graph. The digraph D is also called an **orientation** of G. Figure 7.5 shows three digraphs D_1, D_2 and D_3. While D_1 is a symmetric digraph and D_2 is an oriented graph, the digraph D_3 is neither. The **underlying graph** of a digraph D is that graph obtained by replacing each arc (u, v) or symmetric pair (u, v), (v, u) of arcs by the edge uv. The underlying graph of each digraph in Figure 7.5 is the graph G.

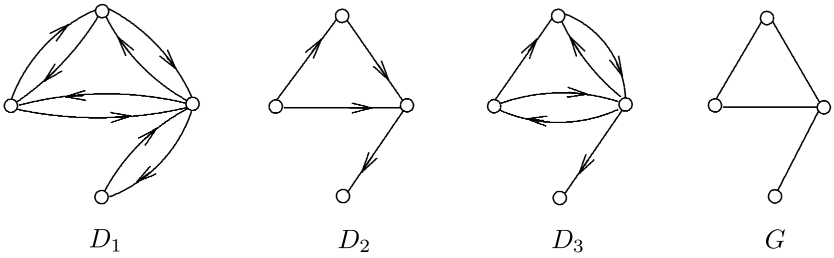


Figure 7.5: Digraphs with the same underlying graph

An orientation of a complete graph is called a **tournament** and will be studied in some detail in Sections 7.4–7.6. A digraph D is **regular of degree** r or r**-regular** if $\operatorname{od} v = \operatorname{id} v = r$ for every vertex v of D. A 1-regular digraph D_1 and a 2-regular digraph D_2 are shown in Figure 7.6. The digraph D_2 is a tournament.

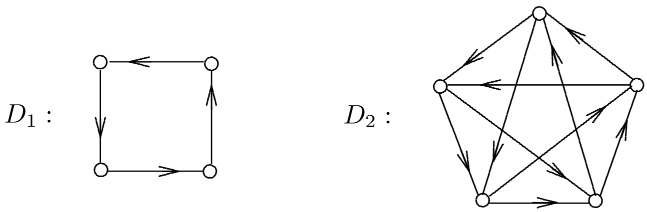


Figure 7.6: Regular digraphs

The terms walk, open and closed walk, trail, path, circuit and cycle for graphs have natural counterparts in digraph theory as well, the important difference being that the directions of the arcs must be followed in each of these walks. In particular, when referring to digraphs, the terms **directed path**, **directed cycle** and **directed circuit** are synonymous with the terms **path**, **cycle** and **circuit**. More formally, for vertices u and v in a digraph D, a **directed** $u - v$ **walk** W (or simply a $u - v$ **walk**) in D is a finite sequence

$$W = (u = u_0, u_1, u_2, \ldots, u_k = v)$$

of vertices, beginning with u and ending with v such that (u_i, u_{i+1}) is an arc for $0 \leq i \leq k-1$. The number k of occurrences of arcs (including repetition) in the walk W is its **length**. Digraphs in which every vertex has positive outdegree must contain cycles (see Exercise 10).

Theorem 7.2 *If D is a digraph such that $\operatorname{od} v \geq k \geq 1$ for every vertex v of D, then D contains a cycle of length at least $k+1$.*

Connected Digraphs

A digraph D is **connected** (or **weakly connected**) if the underlying graph of D is connected. A digraph D is **strong** (or **strongly connected**) if for every pair u, v of vertices, D contains both a $u - v$ path and a $v - u$ path. While all digraphs of Figure 7.7 are connected, only D_1 is strong.

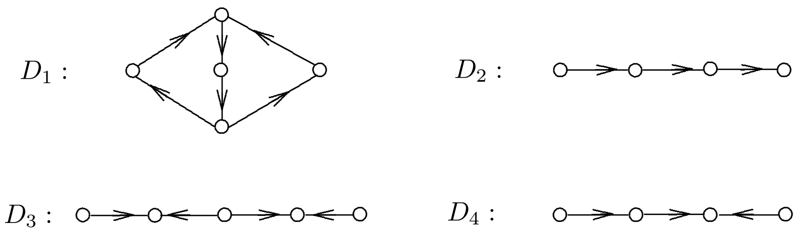


Figure 7.7: Connectedness properties of digraphs

Distance can be defined in digraphs as well. For vertices u and v in a digraph D containing a $u - v$ path, the **directed distance** $\vec{d}(u, v)$ from u to v is the length of a shortest $u - v$ path in D. The distances $\vec{d}(u, v)$ and $\vec{d}(v, u)$ are defined for all pairs u, v of vertices in a digraph D if and only if D is strong. This distance is not a metric, in general. Although directed distance satisfies the triangle inequality, it is not symmetric unless D is symmetric, in which case D can be considered a graph. Eccentricity can be defined as before, as well as radius and diameter in a strong digraph D. The **eccentricity** $e(u)$ of a vertex u in D is the distance from u to a vertex farthest from u. The minimum eccentricity of the vertices of D is the **radius** $\text{rad}(D)$ of D, while the **diameter** $\text{diam}(D)$ is the greatest eccentricity.

Each vertex of the strong digraph D of Figure 7.8 is labeled with its eccentricity. Observe that $\text{rad}(D) = 2$ and $\text{diam}(D) = 5$, so it is not true, in general, that $\text{diam}(D) \leq 2\,\text{rad}(D)$, as is the case with graphs.

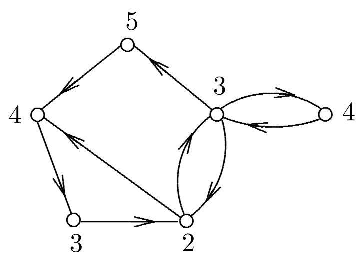


Figure 7.8: Eccentricities in a strong digraph

7.2 Strong Digraphs

We saw that there are two types of connectedness for digraphs, namely weakly connected (or, more simply, connected) digraphs and strongly connected (or simply strong) digraphs. In this section, we explore strong digraphs in more detail.

The following theorem is the digraph analogue of Theorem 2.1 and its proof is analogous as well (see Exercise 15).

Theorem 7.3 *Let u and v be two vertices in a digraph D. For every $u - v$ walk W in D, there exists a $u - v$ path P such that every arc of P belongs to W.*

Strong digraphs are characterized in the following theorem.

Theorem 7.4 *A digraph D is strong if and only if D contains a closed spanning walk.*

Proof. Assume that $W = (u_1, u_2, \ldots, u_k, u_1)$ is a closed spanning walk in D. Let $u, v \in V(D)$. Then $u = u_i$ and $v = u_j$ for some i, j with $1 \leq i, j \leq k$ and $i \neq j$. Without loss of generality, assume that $i < j$. Then $W_1 = (u_i, u_{i+1}, \ldots, u_j)$ is a $u_i - u_j$ walk in D and $W_2 = (u_j, u_{j+1}, \ldots, u_k, u_1, \ldots, u_i)$ is a $u_j - u_i$ walk in D. By Theorem 7.3, D contains both a $u_i - u_j$ path and a $u_j - u_i$ path in D and so D is strong.

Conversely, assume that D is a nontrivial strong digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$ and consider the cyclic sequence $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. Since D is strong, D contains a $v_i - v_{i+1}$ path P_i for $i = 1, 2, \ldots, n$. Then the sequence $P_1, P_2, \ldots, P_n$ of paths produces a closed spanning walk in D. ■

The **converse** $\vec{D}$ of a digraph D is obtained from D by reversing the direction of every arc of D. Thus, D is strong if and only if its converse $\vec{D}$ is strong (see Exercise 16).

Robbins' Theorem

We saw that an orientation of a graph G is a digraph obtained by assigning a direction to each edge of G. Herbert Robbins (1922–2001) studied those graphs having a strong orientation. Certainly, if G has a strong orientation, then G must be connected. Also, if G has a bridge, then it is impossible to produce a strong orientation of G. Robbins [205] showed that this is all that's required for G to have a strong orientation.

Theorem 7.5 (**Robbins' Theorem**) *A nontrivial graph G has a strong orientation if and only if G is connected and bridgeless.*

Proof. We have already observed that if a graph G has a strong orientation, then G is connected and bridgeless. Suppose that the converse is false. Then there exists a connected and bridgeless graph G that has no strong orientation. Among the subgraphs of G, let H be one of maximum order that has a strong orientation. Such a subgraph exists since for each $v \in V(G)$, the subgraph $G[\{v\}]$ trivially has a strong orientation. Thus, $|V(H)| < |V(G)|$, since, by assumption, G has no strong orientation.

Assign directions to the edges of H so that the resulting digraph D is strong, but assign no directions to the edges of $G - E(H)$. Let $u \in V(H)$ and let $v \in V(G) - V(H)$. Since G is connected and bridgeless, it follows by Theorem 4.18 that G contains two edge-disjoint $u - v$ paths. Let P be one of these $u - v$ paths and let Q be the $v - u$ path that results from the other $u - v$ path. Further, let u_1 be the last vertex of P that belongs to H, and let v_1 be the first vertex of Q belonging to H. Next, let P_1 be the $u_1 - v$ subpath of P and let Q_1 be the $v - v_1$ subpath of Q. Direct the edges of P_1 from u_1 toward v, producing the directed path P'_1 and direct the edges of Q_1 from v toward v_1, producing the directed path Q'_1.

Define the digraph D' by

$$V(D') = V(D) \cup V(P'_1) \cup V(Q'_1) \text{ and } E(D') = E(D) \cup E(P'_1) \cup E(Q'_1).$$

Since D is strong, so too is D', contradicting the choice of H. ■

As we mentioned, Theorem 7.5 is due to Robbins. The paper in which this theorem appears is titled "A theorem on graphs, with an application to a problem of traffic control" and was published in 1939 in the *American Mathematical Monthly*, only a year after Robbins received his Ph.D. from Harvard University in topology, under the direction of Hassler Whitney. This was only Robbins' second publication of what would become a long and impressive list. Also in 1939, at age 24, Robbins began work on the classic book *What is Mathematics?* with Richard Courant. Robbins classified this book as a literary work rather than a scientific work. This book discussed mathematics as it existed at that time. A few years later, Robbins became interested in and devoted his research to mathematical statistics, to which he made major contributions.

7.3 Eulerian and Hamiltonian Digraphs

Eulerian and Hamiltonian graphs have natural analogues for digraphs. In both instances, these are strong digraphs.

Eulerian Digraphs

An **Eulerian circuit** in a connected digraph D is a circuit that contains every arc of D (necessarily exactly once); while an **Eulerian trail** in D is an open trail that contains every arc of D. A connected digraph that contains an Eulerian circuit is an **Eulerian digraph**. The next theorem gives a characterization of Eulerian digraphs whose statement and proof are similar to that of Theorem 5.1 (see Exercise 21).

Theorem 7.6 *Let D be a nontrivial connected digraph. Then D is Eulerian if and only if* $\operatorname{od} v = \operatorname{id} v$ *for every vertex v of D.*

With the aid of Theorem 7.6, a characterization of digraphs containing an Eulerian trail can be given (see Exercise 23).

Theorem 7.7 *Let D be a nontrivial connected digraph. Then D contains an Eulerian trail if and only if D contains two vertices u and v such that*

$$\operatorname{od} u = \operatorname{id} u + 1 \quad \text{and} \quad \operatorname{id} v = \operatorname{od} v + 1,$$

while $\operatorname{od} w = \operatorname{id} w$ *for all other vertices w of D. Furthermore, each Eulerian trail of D begins at u and ends at v.*

Thus, the digraph D_1 of Figure 7.9 contains an Eulerian circuit, D_2 contains an Eulerian $u - v$ trail and D_3 contains neither an Eulerian circuit nor an Eulerian trail.

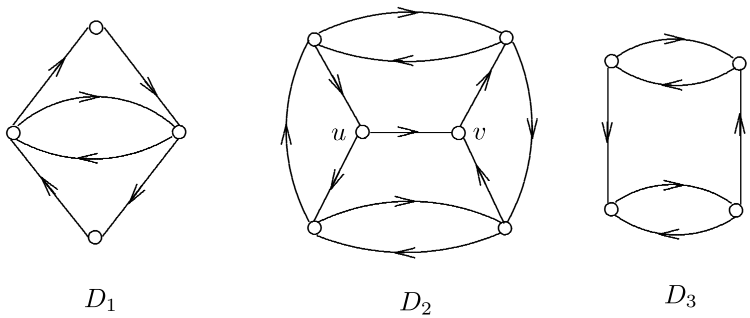


Figure 7.9: Eulerian circuits and trails in digraphs

Hamiltonian Digraphs

A digraph D is **Hamiltonian** if D contains a spanning cycle. Such a cycle is called a **Hamiltonian cycle** of D. As with Hamiltonian graphs, no characterization of Hamiltonian digraphs exists. Indeed, if anything, the situation for Hamiltonian digraphs is even more complex than it is for Hamiltonian graphs. There are sufficient conditions for a digraph to be Hamiltonian, however, that are analogues of the simpler sufficient conditions for graphs to be Hamiltonian. The proofs of these results, unlike their graphical counterparts, are quite lengthy and, for this reason, are not given here.

The following result of Henri Meyniel [170] gives a sufficient condition (much like that in Theorem 6.4 for graphs) for a digraph to be Hamiltonian.

Theorem 7.8 (**Meyniel's Theorem**) *If D is a nontrivial strong digraph of order n such that*

$$\deg u + \deg v \geq 2n - 1$$

for every pair u, v of nonadjacent vertices, then D is Hamiltonian.

Among the consequences of Theorem 7.8 is a result obtained by Douglas Woodall [259].

Corollary 7.9 *If D is a nontrivial digraph of order n such that*

$$\operatorname{od} u + \operatorname{id} v \geq n$$

whenever u and v are distinct vertices with $(u, v) \notin E(D)$, then D is Hamiltonian.

The proof of the following theorem (due to Alain Ghouila-Houri [104]) is an immediate consequence of Theorem 7.8.

Corollary 7.10 *If D is a strong digraph of order n such that* $\deg v \geq n$ *for every vertex v of D, then D is Hamiltonian.*

Corollary 7.10 also has a corollary. We provide a proof of this result.

Corollary 7.11 *If D is a digraph of order n such that*

$$\operatorname{od} v \geq n/2 \quad \textit{and} \quad \operatorname{id} v \geq n/2$$

for every vertex v of D, then D is Hamiltonian.

Proof. Suppose that the theorem is false. Since the theorem is clearly true for $n = 2$ and $n = 3$, there exists some integer $n \geq 4$ and a digraph D of order n that satisfies the hypothesis but which is not Hamiltonian. Let C be a cycle in D of maximum length k. It follows from Theorem 7.2 and the assumption that D is not Hamiltonian that $1 + n/2 \leq k < n$. Also, let P be a path of maximum length such that no vertex of P lies on C. Suppose that P is a $u - v$ path of length $\ell \geq 0$. Therefore, $k + \ell + 1 \leq n$. (See Figure 7.10.)

Since

$$\ell \leq n - k - 1 \leq n - \left(1 + \frac{n}{2}\right) - 1 = \frac{n}{2} - 2,$$

it follows that $\ell \leq n/2 - 2$ and that there are at least two vertices adjacent to u which do not lie on P. Since P is a longest path all of whose vertices do not lie on C, it follows that there are at least two vertices that lie on C that are adjacent to u and at least two vertices adjacent from v which lie on C.

Let a denote the number of vertices on C that are adjacent to u. Thus, $a \geq 2$. For every vertex x on C that is adjacent to u, the $\ell + 1$ vertices immediately following x on C are not adjacent from v, for otherwise, D has a cycle of length exceeding k. Since C contains vertices adjacent from v, there must be a vertex

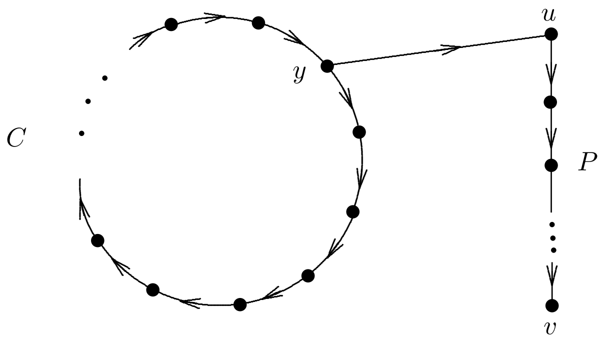


Figure 7.10: A step in the proof of Corollary 7.11

y on C that is adjacent to u such that none of the $\ell + 1$ vertices immediately following y on C are adjacent to u or adjacent from v.

For each of the $a - 1$ vertices on C that are distinct from y and adjacent to u, the vertex immediately following it cannot be adjacent from v. Hence, at least $(a - 1) + (\ell + 1) = a + \ell$ vertices on C are not adjacent from v, for otherwise again, D has a cycle of length exceeding k. Since P is a longest path in D containing no vertices of C, every vertex adjacent to u is either on C or on P.

Because $\text{id}\, u \geq n/2$ and the only vertices of D that can be adjacent to u belong to C or P, it follows that $a+\ell \geq n/2$. Therefore, v is adjacent to at most $(n-1)-(a+\ell) \leq (n-1)-n/2 = n/2-1$ vertices, producing a contradiction. ■

7.4 Tournaments

There are sporting events involving teams (or individuals) that require every two teams to compete against each other exactly once. This is referred to as a round robin tournament. Men's soccer has been part of the Summer Olympic Games since 1900. Teams from 16 countries participate, divided into four pools of four teams each. In each pool, a round robin tournament takes place, in which the top two teams in each pool advance to play for Olympic medals. This also occurs during the World Cup for soccer supremacy when 32 countries participate, divided into eight pools of four teams each.

Round robin tournaments give rise quite naturally to a class of digraphs that we mentioned earlier. Recall that a **tournament** is an orientation of a complete graph. Therefore, a tournament can be defined as a digraph such that for every pair u, v of distinct vertices, exactly one of (u, v) and (v, u) is an arc. A tournament T then models a round robin tournament in which no ties are permitted. The vertices of T are the teams in the round robin tournament and (u, v) is an arc in T if team u defeats team v.

Figure 7.11 shows two tournaments of order 3. In fact, these are the only two tournaments of order 3. The number of non-isomorphic tournaments increases sharply with their orders. For example, there is only one tournament of order 1 and one of order 2. As we just observed, the tournaments T_1 and T_2 in Figure 7.11 are the only two tournaments of order 3. There are four tournaments of order 4, 12 of order 5, 56 of order 6 and over 154 billion of order 12.

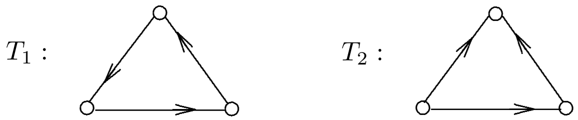


Figure 7.11: The tournaments of order 3

Since the size of a tournament of order n is $\binom{n}{2}$, it follows from Theorem 7.1 that

$$\sum_{v \in V(T)} \operatorname{od} v = \sum_{v \in V(T)} \operatorname{id} v = \binom{n}{2}.$$

Transitive Tournaments

A tournament T is **transitive** if whenever (u, v) and (v, w) are arcs of T, then (u, w) is also an arc of T. The tournament T_2 of Figure 7.11 is transitive while T_1 is not. The following result gives an elementary property of transitive tournaments. An **acyclic digraph** is a digraph having no cycles.

Theorem 7.12 *A tournament is transitive if and only if it is acyclic.*

Proof. Let T be an acyclic tournament and suppose that (u, v) and (v, w) are arcs of T. Since T is acyclic, $(w, u) \notin E(T)$. Therefore, $(u, w) \in E(T)$ and T is transitive.

Conversely, suppose that T is a transitive tournament and assume that T contains a cycle, say $C = (v_1, v_2, \ldots, v_k, v_1)$, where $k \geq 3$. Since (v_1, v_2) and (v_2, v_3) are arcs of the transitive tournament T, it follows that (v_1, v_3) is also an arc of T. Since (v_1, v_3) and (v_3, v_4) are arcs, if $k \geq 4$, then (v_1, v_4) is an arc. Similarly, (v_1, v_5), (v_1, v_6), ..., (v_1, v_k) are arcs of T. However, this contradicts the fact that (v_k, v_1) is an arc of T. Thus, T is acyclic. ■

Suppose that a tournament T of order n with vertex set $V(T)=\{v_1, v_2, \ldots, v_n\}$ represents a round robin tournament involving competition among n teams $v_1, v_2, \ldots, v_n$. If team v_i defeats team v_j, then (v_i, v_j) is an arc of T. The number of victories by team v_i is the outdegree of v_i. For this reason, the outdegree of the vertex v_i in a tournament is also referred to as the **score** of v_i. A sequence $s_1, s_2, \ldots, s_n$ of nonnegative integers is called a **score sequence of**

a tournament if there exists a tournament T of order n whose vertices can be labeled $v_1, v_2, \ldots, v_n$ such that $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, n$.

Figure 7.12 shows transitive tournaments of order n for $n = 3, 4, 5$. The score sequence of every transitive tournament has an interesting property. The following result describes precisely which sequences are score sequences of transitive tournaments.

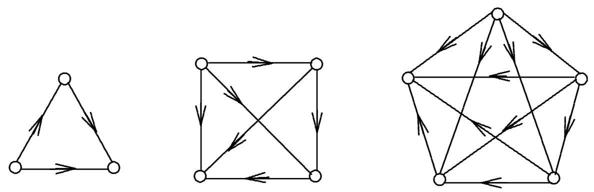


Figure 7.12: Transitive tournaments of orders 3, 4, 5

Theorem 7.13 *A nondecreasing sequence π of n nonnegative integers is a score sequence of a transitive tournament of order n if and only if π is the sequence* $0, 1, \ldots, n-1$.

Proof. First we show that $\pi : 0, 1, \ldots, n-1$ is a score sequence of a transitive tournament of order n. Let T be the tournament with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(T) = \{(v_i, v_j) : 1 \leq i < j \leq n\}$. We claim that T is transitive. Let (v_i, v_j) and (v_j, v_k) be arcs of T. Then $i < j < k$. Since $i < k$, (v_i, v_k) is an arc of T and so T is transitive. For $1 \leq i \leq n$, $\operatorname{od} v_i = n - i$. Therefore, a score sequence of T is $\pi : 0, 1, \ldots, n-1$.

Next, we show that if T is a transitive tournament of order n, then $0, 1, \ldots, n-1$ is a score sequence of T. This is equivalent to showing that every two vertices of T have distinct scores. Let u and w be two vertices of T. Assume, without loss of generality, that (u, w) is an arc of T. Let W be the set of vertices of T to which w is adjacent. Therefore, $\operatorname{od} w = |W|$. For each $x \in W$, (w, x) is an arc of T. Since T is transitive, (u, x) is also an arc of T. However then, $\operatorname{od} u \geq |W| + 1$ and so $\operatorname{od} u \neq \operatorname{od} w$. ■

The proof of Theorem 7.13 shows that the structure of a transitive tournament is uniquely determined.

Corollary 7.14 *For every positive integer n, there is exactly one transitive tournament of order n.*

Combining this corollary with Theorem 7.12, we arrive at yet another corollary.

Corollary 7.15 *For every positive integer n, there is exactly one acyclic tournament of order n.*

Although there is only one transitive tournament of each order n, in a certain sense, which we now describe, every tournament has the structure of a transitive tournament. Let T be a tournament. We define a relation on $V(T)$ by u is related to v if there is both a $u-v$ path and a $v-u$ path in T. This relation is an equivalence relation and, as such, this relation partitions $V(T)$ into equivalence classes $V_1, V_2, \ldots, V_k$ $(k \geq 1)$. Let $S_i = T[V_i]$ for $i = 1, 2, \ldots, k$. Then each subdigraph S_i is a strong tournament and, indeed, is maximal with respect to the property of being strong. The subdigraphs $S_1, S_2, \ldots, S_k$ are called the **strong components** of T. So the vertex sets of the strong components of T produce a partition of $V(T)$.

Let T be a tournament with strong components $S_1, S_2, \ldots, S_k$, and let $\widetilde{T}$ denote that digraph whose vertices $u_1, u_2, \ldots, u_k$ are in one-to-one correspondence with these strong components (where u_i corresponds to S_i, $i = 1, 2, \ldots, k$) such that (u_i, u_j) is an arc of $\widetilde{T}$, $i \neq j$, if and only if some vertex of S_i is adjacent to some vertex of S_j. If (u_i, u_j) is an arc of $\widetilde{T}$, then because S_i and S_j are distinct strong components of T, it follows that every vertex of S_i is adjacent to *every* vertex of S_j. Hence, $\widetilde{T}$ is obtained by identifying the vertices of S_i for $i = 1, 2, \ldots, k$. A tournament T and its associated digraph $\widetilde{T}$ are shown in Figure 7.13.

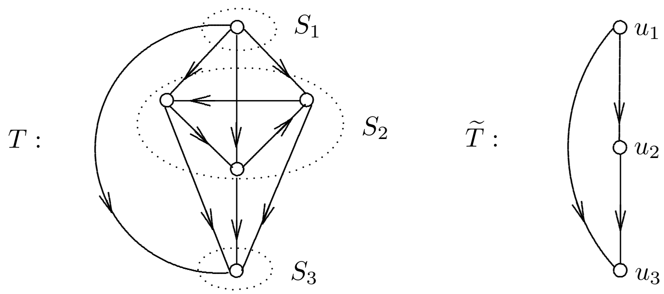


Figure 7.13: A tournament T and its associated transitive tournament $\widetilde{T}$

Observe that for the tournament T of Figure 7.13, $\widetilde{T}$ is itself a tournament, indeed a transitive tournament. That this always occurs follows from Theorem 7.16. (See Exercise 37.)

Theorem 7.16 *If T is a tournament with exactly k strong components, then $\widetilde{T}$ is the transitive tournament of order k.*

Since for every tournament T, the tournament $\widetilde{T}$ is transitive, it follows that if T is a tournament that is not strong, then $V(T)$ can be partitioned as $\{V_1, V_2, \ldots, V_k\}$ $(k \geq 2)$ such that $T[V_i]$ is a strong tournament for each i, and if $v_i \in V_i$ and $v_j \in V_j$, where $i < j$, then $(v_i, v_j) \in E(T)$. This decomposition is often useful when studying the properties of tournaments that are not strong.

We already noted that there are four tournaments of order 4. Of course, one of these is transitive, which consists of four trivial strong components

S_1, S_2, S_3, S_4, where the vertex of S_i is adjacent to the vertex of S_j if and only if $i < j$. There are two tournaments of order 4 containing two strong components S_1 and S_2, depending on whether S_1 or S_2 is the strong component of order 3. (No strong component has order 2.) Since there are four tournaments of order 4, there is exactly one strong tournament of order 4. These tournaments are depicted in Figure 7.14. The arcs not drawn in the tournaments T_1, T_2 and T_3 that are not strong are all directed downward, as indicated by the double arrow.

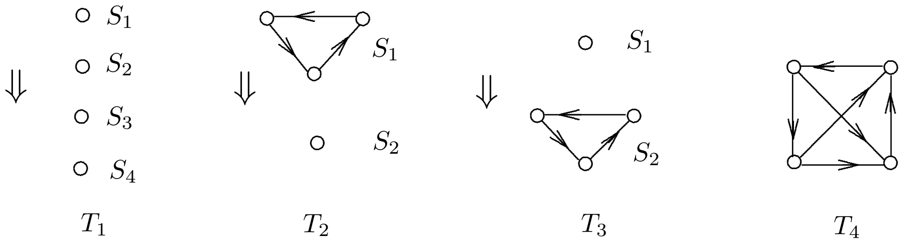


Figure 7.14: The four tournaments of order 4

We also stated that there are 12 tournaments of order 5. There are six tournaments of order 5 that are not strong, shown in Figure 7.15. Again all arcs that are not drawn are directed downward. Thus, there are six strong tournaments of order 5.

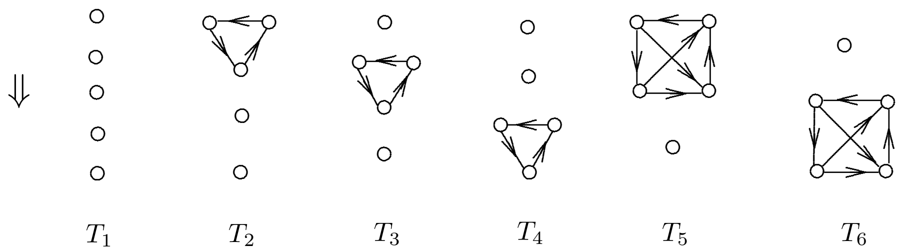


Figure 7.15: The six tournaments of order 5 that are not strong

Score Sequences of Tournaments

Theorem 7.13 characterizes score sequences of transitive tournaments. We next investigate score sequences of tournaments in general. We begin with a theorem similar to Theorem 1.12.

Theorem 7.17 *A nondecreasing sequence* $\pi : s_1, s_2, \ldots, s_n$ $(n \geq 2)$ *of nonnegative integers is a score sequence of a tournament if and only if the sequence* $\pi_1 : s_1, s_2, \ldots, s_{s_n}, s_{s_n+1} - 1, \ldots, s_{n-1} - 1$ *is a score sequence of a tournament.*

Proof. Assume that π_1 is a score sequence of a tournament. Then there exists a tournament T_1 of order $n-1$ having π_1 as a score sequence. Hence the vertices of T_1 can be labeled as $v_1, v_2, \ldots, v_{n-1}$ such that

$$\text{od}\, v_i = \begin{cases} s_i & \text{for } 1 \le i \le s_n \\ s_i - 1 & \text{for } i > s_n. \end{cases}$$

We construct a tournament T by adding a vertex v_n to T_1 where v_n is adjacent to v_i if $1 \le i \le s_n$ and v_n is adjacent from v_i otherwise. The tournament T then has π as a score sequence.

For the converse, we assume that π is a score sequence. Hence there exist tournaments of order n whose score sequence is π. Among all such tournaments, let T be one such that $V(T) = \{v_1, v_2, \ldots, v_n\}$, $\text{od}\, v_i = s_i$ for $i = 1, 2, \ldots, n$ and the sum of the scores of the vertices adjacent from v_n is minimum. We claim that v_n is adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Assume, to the contrary, that v_n is not adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Necessarily, then, there exist vertices v_j and v_k with $j < k$ and $s_j < s_k$ such that v_n is adjacent to v_k and v_n is adjacent from v_j. Since the score of v_k exceeds the score of v_j, there exists a vertex v_t such that v_k is adjacent to v_t, and v_t is adjacent to v_j (Figure 7.16(a)). Thus, a 4-cycle $C = (v_n, v_k, v_t, v_j, v_n)$ is produced. If we reverse the directions of the arcs of C, a tournament T' is obtained also having π as a score sequence (Figure 7.16(b)). However, in T', the vertex v_n is adjacent to v_j rather than v_k. Hence the sum of the scores of the vertices adjacent from v_n is smaller in T' than in T, which is impossible. Thus, as claimed, v_n is adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Then $T - v_n$ is a tournament having score sequence π_1. ■

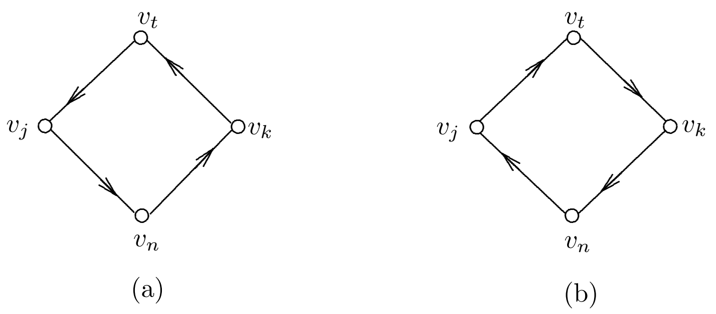


Figure 7.16: A step in the proof of Theorem 7.17

As an illustration of Theorem 7.17, we consider the sequence

$$\pi : 1, 2, 2, 3, 3, 4.$$

In this case, s_n (actually s_6) has the value 4; thus, we delete the last term, repeat the first $s_n = 4$ terms, and subtract 1 from the remaining terms, obtaining

$$\pi_1' : 1, 2, 2, 3, 2.$$

Rearranging, we have

$$\pi_1 : 1, 2, 2, 2, 3.$$

Repeating this process twice more, we have

$$\begin{aligned} \pi_2' &: 1, 2, 2, 1 \\ \pi_2 &: 1, 1, 2, 2 \\ \pi_3 &: 1, 1, 1. \end{aligned}$$

The sequence π_3 is clearly a score sequence of a tournament. By Theorem 7.17, π_2 is as well, as are π_1 and π. We can use this information to construct a tournament with score sequence π. The sequence π_3 is the score sequence of the tournament T_3 of Figure 7.17. Proceeding from π_3 to π_2, we add a new vertex to T_3 and join it *to* two vertices of T_3 and *from* the other, producing a tournament T_2 with score sequence π_2. To proceed from π_2 to π_1, we add a new vertex to T_2 and join it *to* vertices having scores 1, 2 and 2 and *from* the remaining vertex of T_2, producing a tournament T_1 with score sequence π_1. Continuing in the same fashion, we finally produce a desired tournament T with score sequence π by adding a new vertex to T_1 and joining it *to* vertices having scores 1, 2, 2 and 3, and joining it *from* the other vertex.

The sociologist Hyman Garshin Landau [152] characterized those sequences of nonnegative integers that are score sequences of tournaments. The proof we present of his theorem is due to Carsten Thomassen [233].

Theorem 7.18 *A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ of nonnegative integers is a score sequence of a tournament if and only if for each integer k with $1 \le k \le n$,*

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2}, \tag{7.1}$$

with equality holding when $k = n$.

Proof. Suppose first that $\pi : s_1, s_2, \ldots, s_n$ is a score sequence of a tournament of order n. Then there exists a tournament T with $V(T) = \{v_1, v_2, \ldots, v_n\}$ such that $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, n$. For an integer k with $1 \le k \le n$ and $S = \{v_1, v_2, \ldots, v_k\}$, the subdigraph $T_1 = T[S]$ induced by S is a tournament of order k and size $\binom{k}{2}$. Since $\operatorname{od}_T v_i \ge \operatorname{od}_{T_1} v_i$ for $1 \le i \le k$, it follows that

$$\sum_{i=1}^{k} s_i = \sum_{i=1}^{k} \operatorname{od}_T v_i \ge \sum_{i=1}^{k} \operatorname{od}_{T_1} v_i = \binom{k}{2}.$$

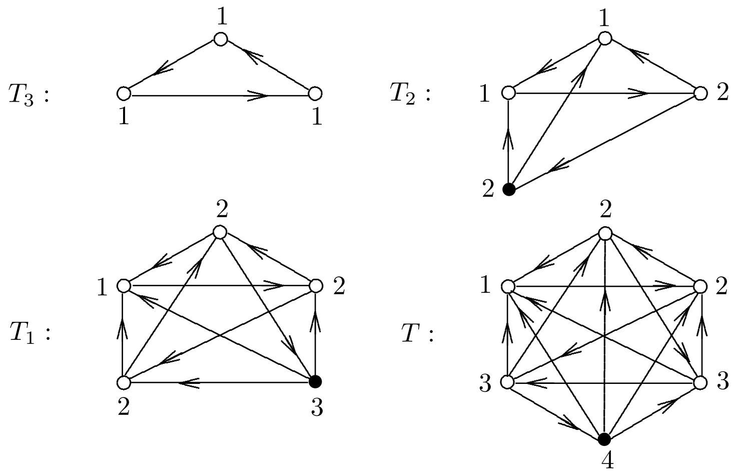


Figure 7.17: Construction of a tournament with a given score sequence

We now verify the converse. Suppose that the converse is false. Then among all counterexamples for which n is minimum, let $\pi : s_1, s_2, \ldots, s_n$ be one for which s_1 is minimum. Suppose first that there exists an integer k with $1 \leq k \leq n-1$ such that

$$\sum_{i=1}^{k} s_i = \binom{k}{2}. \tag{7.2}$$

Since $k < n$, it follows that $\pi_1 : s_1, s_2, \ldots, s_k$ is a score sequence of a tournament T_1 of order k.

Let $\tau : t_1, t_2, \ldots, t_{n-k}$ be the sequence, where $t_i = s_{k+i} - k$ for $i = 1, 2, \ldots, n-k$. Since

$$\sum_{i=1}^{k+1} s_i \geq \binom{k+1}{2},$$

it follows from (7.2) that

$$s_{k+1} = \sum_{i=1}^{k+1} s_i - \sum_{i=1}^{k} s_i \geq \binom{k+1}{2} - \binom{k}{2} = k.$$

Since π is a nondecreasing sequence,

$$t_i = s_{k+i} - k \geq s_{k+1} - k \geq 0$$

for $i = 1, 2, \ldots, n-k$ and so τ is a nondecreasing sequence of nonnegative integers. We now show that τ satisfies (7.1).

For each integer r with $1 \leq r \leq n-k$, we have

$$\sum_{i=1}^{r} t_i = \sum_{i=1}^{r} (s_{k+i} - k) = \sum_{i=1}^{r} s_{k+i} - rk = \sum_{i=1}^{r+k} s_i - \sum_{i=1}^{k} s_i - rk.$$

Since

$$\sum_{i=1}^{r+k} s_i \geq \binom{r+k}{2}$$

and

$$\sum_{i=1}^{k} s_i = \binom{k}{2},$$

it follows that

$$\sum_{i=1}^{r} t_i \geq \binom{r+k}{2} - \binom{k}{2} - rk = \binom{r}{2},$$

with equality holding for $r = n - k$. Thus, τ satisfies (7.1). Since $n - k < n$, there is a tournament T_2 of order $n - k$ having score sequence τ.

Let T be the tournament with $V(T) = V(T_1) \cup V(T_2)$ and

$$E(T) = E(T_1) \cup E(T_2) \cup \{(u, v) : u \in V(T_2), v \in V(T_1)\}.$$

Then π is a score sequence for T, contrary to our assumption. Consequently,

$$\sum_{i=1}^{k} s_i > \binom{k}{2}$$

for $k = 1, 2, \ldots, n-1$. In particular, $s_1 > 0$.

We now consider the sequence $\pi' : s_1 - 1, s_2, s_3, \ldots, s_{n-1}, s_n + 1$. Then π' is a nondecreasing sequence of nonnegative integers satisfying (7.1). By the minimality of s_1, there is a tournament T' of order n having score sequence π'. Let x and y be vertices of T' such that $\text{od}_{T'} x = s_n + 1$ and $\text{od}_{T'} y = s_1 - 1$. Since $\text{od}_{T'} x \geq \text{od}_{T'} y + 2$, there is a vertex $w \neq x, y$ such that $(x, w) \in E(T')$ and $(w, y) \in E(T')$. Thus, $P = (x, w, y)$ is a path in T'.

Let T be a tournament obtained from T' by reversing the directions of the arcs in P. Then π is a score sequence for T, producing a contradiction. ■

Frank Harary and Leo Moser [121] obtained a related characterization of sequences of nonnegative integers that are score sequences of strong tournaments (see Exercise 42).

Theorem 7.19 *A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ of nonnegative integers is a score sequence of a strong tournament if and only if*

$$\sum_{i=1}^{k} s_i > \binom{k}{2}$$

for $1 \leq k \leq n-1$ and

$$\sum_{i=1}^{n} s_i = \binom{n}{2}.$$

Furthermore, every tournament whose score sequence satisfies these conditions is strong.

7.5 Kings in Tournaments

While tournaments can be used to represent the results of round robin tournaments (especially among teams participating in a sports event), they can be used to model any collection of objects where in each pair of objects, one is preferred over the other in some manner. An example of this occurs in a flock of chickens. In a pair of chickens, one chicken will dominate the other. The dominant chicken in the pair asserts this dominance by pecking the other on its head and neck. (This is what led to the term *pecking order.*) It is rare when this dominance is transitive; that is, if the first chicken pecks a second chicken and the second pecks a third, it does not mean that the first necessarily pecks the third. The question then arises: Which chicken (or chickens) should be considered most dominant in the flock? Any such chicken is referred to as a *king chicken.* Landau [152] defined a chicken K in a flock $\mathcal{F}$ of chickens to be *king* if for every chicken C in $\mathcal{F}$, either K pecks C or K pecks a chicken that pecks C.

This situation can be modeled by a tournament and leads to a concept involving tournaments. A vertex u in a tournament T is a **king** in T if for every vertex w different from u, either $u \to w$ or there is a vertex v such that $u \to v \to w$. Landau then proved the following.

Theorem 7.20 *Every tournament contains a king.*

Proof. Let T be a tournament and let u be a vertex having maximum outdegree in T. We show that u is a king. If this is not the case, then there is a vertex w in T for which u is neither adjacent to w nor adjacent to any vertex that is adjacent to w. Then w is adjacent to every vertex to which u is adjacent and adjacent to u as well. Thus, $\operatorname{od} w > \operatorname{od} u$, a contradiction. ■

A vertex u in a tournament of order n is called an **emperor** if $\operatorname{od} u = n - 1$. Since no vertex is adjacent to u, we have the following observation.

Theorem 7.21 *If a tournament T has an emperor u, then u is the unique king in T.*

While it's possible for a tournament to have exactly one king, it is not possible for a tournament to contain exactly two kings.

Theorem 7.22 *Every tournament containing no emperor contains at least three kings.*

Proof. Let T be a tournament containing no emperor and let u be a vertex of maximum outdegree in T. By the proof of Theorem 7.20, it follows that u is a king of T.

Among all vertices adjacent to u, let v be one of maximum outdegree. We claim that v is a king of T. Assume, to the contrary, that v is not a king in T. Then there is a vertex x of T such that v is neither adjacent to x nor adjacent to a vertex that is adjacent to x. Thus, x is adjacent to both u and v. Furthermore, x is adjacent to every vertex to which v is adjacent. However then, $\operatorname{od} x > \operatorname{od} v$, a contradiction. Thus, v is a king of T.

Next, among the vertices adjacent to v, let w be one of maximum outdegree. We claim that w is also a king of T. Assume, to the contrary, that w is not a king. Then there is a vertex y of T such that w is neither adjacent to y nor adjacent to a vertex that is adjacent to y. Thus, y is adjacent to v and w. In addition, y is adjacent to every vertex to which w is adjacent. However then, $\operatorname{od} y > \operatorname{od} w$, a contradiction. Hence w is a king of T. ■

7.6 Hamiltonian Tournaments

The large number of arcs in a tournament often produce paths and cycles of varying lengths. Perhaps the most basic result of this type was a property of tournaments first observed by László Rédei [197] in 1934, resulting in the first theoretical result on tournaments. A path in a digraph D containing every vertex of D is a **Hamiltonian path**.

Theorem 7.23 *Every tournament contains a Hamiltonian path.*

Proof. Let T be a tournament of order n and let $P = (v_1, v_2, \ldots, v_k)$ be a longest path in T. If P is not a Hamiltonian path of T, then $1 \leq k < n$ and there is a vertex v of T not on P. Since P is a longest path, $(v, v_1), (v_k, v) \notin E(T)$, and so $(v_1, v), (v, v_k) \in E(T)$. This implies that there is a largest integer i $(1 \leq i < k)$ such that $(v_i, v) \in E(T)$. So $(v, v_{i+1}) \in E(T)$ (see Figure 7.18). But then

$$(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k)$$

is a path whose length exceeds that of P, producing a contradiction. ■

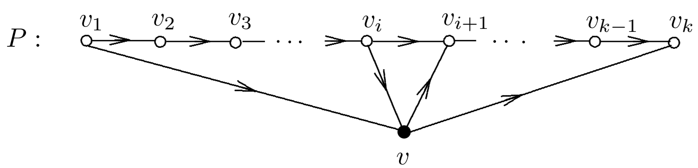


Figure 7.18: A step in the proof of Theorem 7.23

A simple but useful consequence of Theorem 7.23 concerns transitive tournaments.

Corollary 7.24 *Every transitive tournament contains exactly one Hamiltonian path.*

The preceding corollary is a special case of a result found independently by Rédei [197] and Tibor Szele [231], who showed that every tournament contains an odd number of Hamiltonian paths.

Figure 7.19 shows a tournament of order 5 consisting of three strong components S_1, S_2 and S_3, where S_1 and S_3 consists of a single vertex and S_2 is a 3-cycle. This tournament has three Hamiltonian paths, namely $P_1 = (u, v, w, x, y)$, $P_2 = (u, w, x, v, y)$ and $P_3 = (u, x, v, w, y)$.

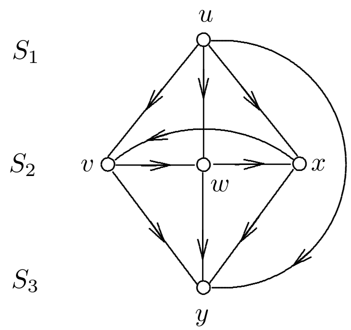


Figure 7.19: A tournament with three Hamiltonian paths

While each transitive tournament contains exactly one Hamiltonian path, there are, not surprisingly, tournaments with many Hamiltonian paths. The next result, also due to Szele [231], establishes the existence of such tournaments and provides a lower bound on the number of Hamiltonian paths in them. The proof of this result, considered the first application of the probabilistic method in combinatorics, will be presented in Chapter 21 (see Theorem 21.3).

Theorem 7.25 *For each integer $n \geq 2$, there exists a tournament of order n containing at least $n!/2^{n-1}$ Hamiltonian paths.*

While every tournament contains a Hamiltonian path, certainly not every tournament contains a Hamiltonian cycle. Indeed, by Theorem 7.12, every transitive tournament is acyclic. If a tournament T contains a Hamiltonian cycle, then T is strong by Theorem 7.4. Paul Camion [41] showed that the converse is true as well.

Theorem 7.26 *A nontrivial tournament T is Hamiltonian if and only if T is strong.*

Proof. We have already seen that every Hamiltonian tournament is strong. For the converse, assume that T is a nontrivial strong tournament. Thus, T contains cycles. Let C be a cycle of maximum length in T. If C contains all of the vertices of T, then C is a Hamiltonian cycle. So, assume that C is not Hamiltonian, say

$$C = (v_1, v_2, \ldots, v_k, v_1),$$

where $3 \leq k < n$. If T contains a vertex v that is adjacent to some vertex of C and adjacent from some vertex of C, then there must be a vertex v_i of C that is adjacent to v such that v_{i+1} is adjacent from v. In this case,

$$C' = (v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k, v_1)$$

is a cycle whose length is greater than that of C, producing a contradiction. Hence, every vertex of T that is not on C is either adjacent to every vertex of C or adjacent from every vertex of C. Since T is strong, there must be vertices of each type.

Let U be the set of all vertices of T that are not on C and such that each vertex of U is adjacent from every vertex of C, and let W be the set of those vertices of T that are not on C such that every vertex of W is adjacent to each vertex of C (see Figure 7.20). Then $U \neq \emptyset$ and $W \neq \emptyset$.

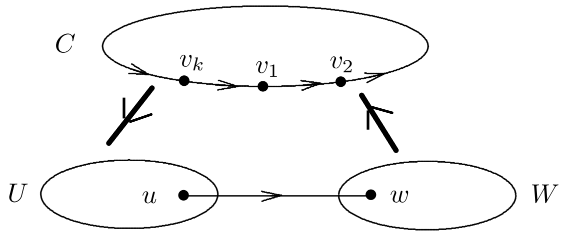


Figure 7.20: A step in the proof of Theorem 7.26

Since T is strong, there is a path from every vertex of C to every vertex of W. Since no vertex of C is adjacent to any vertex of W, there must be a vertex $u \in U$ that is adjacent to a vertex $w \in W$. However then,

$$C'' = (v_1, v_2, \ldots, v_k, u, w, v_1)$$

is a cycle whose length is greater than the length of C, a contradiction. ■

If T is a Hamiltonian tournament, then, of course, every vertex of T lies on every Hamiltonian cycle of T. Actually, every vertex of T lies on a triangle of T as well.

Theorem 7.27 *Every vertex in a nontrivial strong tournament belongs to a triangle.*

Proof. Let v be a vertex in a nontrivial strong tournament T. By Theorem 7.26, T is Hamiltonian. Thus, T contains a Hamiltonian cycle $(v = v_1, v_2, \ldots, v_n, v_1)$. Since v is adjacent to v_2 and adjacent from v_n, there is a vertex v_i with $2 \leq i < n$ such that (v, v_i) and (v_{i+1}, v) are arcs of T. Thus, (v, v_i, v_{i+1}, v) is a triangle of T containing v. ■

It is perhaps surprising that if a tournament is Hamiltonian, then it must possess significantly stronger properties. A digraph D of order $n \geq 3$ is **pancyclic** if it contains a cycle of every possible length, that is, D contains a cycle of length ℓ for each $\ell = 3, 4, \ldots, n$ and is **vertex-pancyclic** if each vertex v of D lies on a cycle of every possible length. Frank Harary and Leo Moser [121] showed that every nontrivial strong tournament is pancyclic, while John W. Moon [173] went one step further by obtaining the following result. The proof given here is due to Carsten Thomassen.

Theorem 7.28 *Every nontrivial strong tournament is vertex-pancyclic.*

Proof. Let T be a strong tournament of order $n \geq 3$, and let v_1 be a vertex of T. We show that v_1 lies on an ℓ-cycle for each $\ell = 3, 4, \ldots, n$. We proceed by induction on ℓ.

Since T is strong, it follows by Theorem 7.27 that v_1 lies on a 3-cycle. Assume that v_1 lies on an ℓ-cycle $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $3 \leq \ell \leq n-1$. We show that v_1 lies on an $(\ell+1)$-cycle.

Case 1. *There is a vertex v not on C that is adjacent to at least one vertex of C and is adjacent from at least one vertex of C.* This implies that for some i $(1 \leq i \leq \ell)$, both (v_i, v) and (v, v_{i+1}) are arcs of T (where all subscripts are expressed modulo ℓ). Thus, v_1 lies on the $(\ell+1)$-cycle

$$(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_\ell, v_1).$$

Case 2. *No vertex v exists as in Case* 1. Let U denote the set of all vertices in $V(T) - V(C)$ that are adjacent from every vertex of C, and let W be the set of all vertices in $V(T) - V(C)$ that are adjacent to every vertex of C. Then $U \cup W = V(T) - V(C)$. Since T is strong, neither U nor W is empty and there is a vertex u in U and a vertex w in W such that $(u, w) \in E(T)$. Thus, v_1 lies on the $(\ell+1)$-cycle

$$(u, w, v_1, v_2, \ldots, v_{\ell-1}, u),$$

completing the proof. ■

Corollary 7.29 *Every nontrivial strong tournament is pancyclic.*

Exercises for Chapter 7

Section 7.1. Introduction to Digraphs

1. (a) We saw in Theorem 1.14 that there exists no graph whose vertices have distinct degrees. Show that there exists a digraph of order 5 whose vertices have distinct outdegrees and distinct indegrees.

   (b) Does there exist a digraph of order 5 whose vertices have distinct outdegrees but the same indegree?

2. Determine all digraphs of order 4 and size 4.

3. Show that for every positive integer k, there exists a digraph of even order, half of whose vertices have outdegree a and half have outdegree b and $a - b = k$.

4. If all vertices of a digraph D of order 5 have distinct outdegrees except for two vertices that have the same outdegree a, then what are the possible values of a?

5. Prove or disprove: No digraph contains an odd number of vertices of odd outdegree or an odd number of vertices of odd indegree.

6. Prove or disprove: If D_1 and D_2 are two digraphs with $V(D_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$ such that $\mathrm{id}_{D_1} u_i = \mathrm{id}_{D_2} v_i$ and $\mathrm{od}_{D_1} u_i = \mathrm{od}_{D_2} v_i$ for $i = 1, 2, \ldots, n$, then $D_1 \cong D_2$.

7. Prove that there exist regular tournaments of every odd order but there are no regular tournaments of even order.

8. Let T be a tournament with $V(T) = \{v_1, v_2, \ldots, v_n\}$. We know that

$$\sum_{i=1}^{n} \mathrm{od}\, v_i = \sum_{i=1}^{n} \mathrm{id}\, v_i = \binom{n}{2}.$$

   (a) Prove that $\sum_{i=1}^{n} (\mathrm{od}\, v_i)^2 = \sum_{i=1}^{n} (\mathrm{id}\, v_i)^2$.

   (b) Prove or disprove: $\sum_{i=1}^{n} (\mathrm{od}\, v_i)^3 = \sum_{i=1}^{n} (\mathrm{id}\, v_i)^3$.

9. The **adjacency matrix** $A(D)$ of a digraph D with $V(D) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $[a_{ij}]$ defined by $a_{ij} = 1$ if $(v_i, v_j) \in E(D)$ and $a_{ij} = 0$ otherwise.

   (a) What information do the row sums and column sums of the adjacency matrix of a digraph provide?

   (b) Characterize matrices that are adjacency matrices of digraphs.

10. (a) Prove Theorem 7.2: *If D is a digraph such that $\mathrm{od}\, v \geq k \geq 1$ for every vertex v of D, then D contains a cycle of length at least $k+1$.*

(b) Prove that if D is a digraph such that $\text{id}\, v \geq k \geq 1$ for every vertex v of D, then D contains a cycle of length at least $k+1$.

11. Let G be a connected graph of order $n \geq 3$. Prove that there is an orientation of G containing no directed path of length 2 if and only if G is bipartite.

12. Prove that for every two positive integers a and b with $a \leq b$, there exists a strong digraph D with $\text{rad}(D) = a$ and $\text{diam}(D) = b$.

13. The **center** $\text{Cen}(D)$ of a strong digraph D is the subdigraph induced by those vertices v with $e(v) = \text{rad}(D)$. Prove that for every oriented graph D_1, there exists a strong oriented graph D such that $\text{Cen}(D) = D_1$.

14. Prove that every digraph D contains a set S of vertices with the properties (1) no two vertices in S are adjacent in D and (2) for every vertex v of D not in S, there exists a vertex u in S such that $\vec{d}(u, v) \leq 2$.

Section 7.2. Strong Digraphs

15. Prove Theorem 7.3: *Let u and v be two vertices in a digraph D. For every $u - v$ walk W in D, there exists a $u - v$ path P such that every arc of P belongs to W.*

16. Show that a digraph D is strong if and only if its converse $\vec{D}$ is strong.

17. Let G be a nontrivial connected graph without bridges.

    (a) Show that for every edge e of G and for every orientation of e, there exists an orientation of the remaining edges of G such that the resulting digraph is strong.

    (b) Show that (a) need not be true if we begin with an orientation of two edges of G.

18. Let G be a connected graph with cut-vertices. Show that an orientation D of G is strong if and only if the subdigraph of D induced by the vertices of each block of G is strong.

19. According to Theorem 7.5, a nontrivial graph G has a strong orientation if and only if G is connected and contains no bridges.

    (a) Prove that if G is a nontrivial connected graph with at most two bridges, then there exists an orientation D of G having the property that if u and v are any two vertices of D, there is either a $u - v$ path or a $v - u$ path.

    (b) Show that the statement (a) is false if G contains three bridges.

20. Let D be a digraph of order $n \geq 2$. Prove that if $\text{od}\, v \geq (n-1)/2$ and $\text{id}\, v \geq (n-1)/2$ for every vertex v of D, then D is strong.

Section 7.3. Eulerian and Hamiltonian Digraphs

21. Prove Theorem 7.6: *Let D be a nontrivial connected digraph. Then D is Eulerian if and only if $\text{od}\, v = \text{id}\, v$ for every vertex v of D.*

22. Prove that a graph G has an Eulerian orientation if and only if G is Eulerian.

23. Prove Theorem 7.7: *Let D be a nontrivial connected digraph. Then D contains an Eulerian trail if and only if D contains two vertices u and v such that $\text{od}\, u = \text{id}\, u + 1$ and $\text{id}\, v = \text{od}\, v + 1$, while $\text{od}\, w = \text{id}\, w$ for all other vertices w of D. Furthermore, each Eulerian trail of D begins at u and ends at v.*

24. Prove that if D is a connected digraph containing two vertices u and v such that $\text{od}\, u = \text{id}\, u+k$ and $\text{id}\, v = \text{od}\, v+k$ for some positive integer k and $\text{od}\, w = \text{id}\, w$ for all other vertices w of D, then D contains k arc-disjoint $u-v$ paths.

25. Let D be a digraph with an Eulerian trail. Then D contains two vertices u and v such that $\text{od}\, u = \text{id}\, u + 1$ and $\text{id}\, v = \text{od}\, v + 1$, where $\text{od}\, w = \text{id}\, w$ for all other vertices w of D.

    (a) Let T' be a $u-x$ trail in D that cannot be extended to a longer trail. Must $x = v$?

    (b) If T is a $u-v$ trail in D, must T be an Eulerian trail?

26. Prove that a nontrivial connected digraph D is Eulerian if and only if $E(D)$ can be partitioned into subsets E_i, $1 \leq i \leq k$, where the subdigraph $D[E_i]$ induced by the set E_i is a cycle for each i.

27. Prove that if D is a connected digraph such that $\sum_{v\in V(D)} |\,\text{od}\, v - \text{id}\, v| = 2t$, where $t \geq 1$, then $E(D)$ can be partitioned into subsets E_i, $1 \leq i \leq t$, so that the subgraph $G[E_i]$ induced by E_i is an open trail for each i.

28. Let D be a connected digraph of order n with $V(D) = \{v_1, v_2, \ldots, v_n\}$. Prove that if $\text{od}\, v_i \geq \text{id}\, v_i$ for $1 \leq i \leq n$, then D is Eulerian.

29. A vertex v in a digraph D is said to be **reachable** from a vertex u in D if D contains a $u-v$ path. Let D be a digraph and for each vertex u of D, let $R(u)$ be the set of vertices reachable from u and let $r(u) = |R(u)|$. Since $u \in R(u)$ for every vertex u of D, it follows that $r(u) \geq 1$. Prove that if $r(x) \neq r(y)$ for every two distinct vertices x and y of D, then D contains a Hamiltonian path.

30. By Corollary 7.9, if D is a nontrivial digraph of order n such that $\text{od}\, u + \text{id}\, v \geq n$ when $(u, v) \notin E(D)$, then D is Hamiltonian. Show that if D is a nontrivial digraph of order n such that $\text{od}\, u + \text{id}\, v \geq n - 1$ when $(u, v) \notin E(D)$, then D is strong but may not be Hamiltonian.

31. Show for infinitely many positive integers n that there exists a digraph D of order n such that $\text{od}\, v \geq (n-1)/2$ and $\text{id}\, v \geq (n-1)/2$ for every vertex v of D but D is not Hamiltonian.

32. Corollary 7.10 states: *If D is a strong digraph of order n such that* $\deg v \geq$ *n for every vertex v of D, then D is Hamiltonian.* Show that if the digraph D is not required to be strong, then D need not be Hamiltonian.

Section 7.4. Tournaments

33. Give an example of two non-isomorphic strong tournaments of order 5.

34. How many tournaments of order 7 are there that are not strong?

35. Determine those positive integers n for which there exist regular tournaments of order n.

36. Give an example of two non-isomorphic regular tournaments of the same order.

37. Prove Theorem 7.16. *If T is a tournament with exactly k strong components, then $\widetilde{T}$ is the transitive tournament of order k.*

38. (a) Show that if two vertices u and v have the same score in a tournament T, then u and v belong to the same strong component of T.
    (b) Prove that every regular tournament is strong.

39. Which of the following sequences are score sequences of tournaments? For each sequence that is a score sequence, construct a tournament having the given sequence as a score sequence.
    (a) $0, 1, 1, 4, 4$
    (b) $1, 1, 1, 4, 4, 4$
    (c) $1, 3, 3, 3, 3, 3, 5$
    (d) $2, 3, 3, 4, 4, 4, 4, 5$.

40. Show that if $\pi : s_1, s_2, \ldots, s_n$ is a score sequence of a tournament, then $\pi_1 : n-1-s_1, n-1-s_2, \ldots, n-1-s_n$ is a score sequence of a tournament.

41. What tournament T of order n has a score sequence $s_1, s_2, \ldots, s_n$ such that equality holds in (7.1) for every integer k with $1 \leq k \leq n$?

42. Prove Theorem 7.19: *A nondecreasing sequence* $\pi : s_1, s_2, \ldots, s_n$ *of nonnegative integers is a score sequence of a strong tournament if and only if* $\sum_{i=1}^{k} s_i > \binom{k}{2}$ *for* $1 \leq k \leq n-1$ *and* $\sum_{i=1}^{n} s_i = \binom{n}{2}$*. Furthermore, every tournament whose score sequence satisfies these conditions is strong.*

43. We have seen that there is exactly one transitive tournament of each order. A tournament of order $n \geq 3$ is defined to be **circular** if whenever (u, v) and (v, w) are arcs of T, then (w, u) is an arc of T.

    (a) How many circular tournaments of order 3 are there?

    (b) Show that in a tournament of order 3 or more, every vertex, with at most two exceptions, has positive outdegree and positive indegree.

    (c) How many circular tournaments of order 4 or more are there?

44. For each positive integer k, there exist round robin tournaments containing $2k$ teams with no ties permitted in which k of these teams win r games and the remaining k of these teams win s games for some r and s with $r \neq s$. What is the minimum value of s for which this is possible?

45. Prove that if u and v are vertices of a tournament such that $\vec{d}(u, v) = k$, then $\operatorname{id} u \geq k - 1$.

46. For a tournament T of order n, let

$$\Delta = \max\{\operatorname{od} v : v \in V(T)\} \text{ and } \delta = \min\{\operatorname{od} v : v \in V(T)\}.$$

    Prove that if $\Delta - \delta < \frac{n}{2}$, then T is strong.

47. Let (u, v) be an arc of a tournament T. Show that if $\operatorname{od} v > \operatorname{od} u$, then (u, v) lies on a triangle of T.

48. Show that a tournament can contain three vertices of outdegree 1 but can never contain four vertices of outdegree 1.

49. Let T be a tournament of order $n \geq 10$. Suppose that T contains two vertices u and v such that when the arc joining u and v is removed, the resulting digraph D contains neither a $u - v$ path nor a $v - u$ path. Show that $\operatorname{od}_D u = \operatorname{od}_D v$.

50. Let u and v be two vertices in a tournament T. Prove that if $\vec{d}(u, v) = k \geq 2$, then T contains a cycle of length ℓ for each integer ℓ with $3 \leq \ell \leq k+1$.

51. Let u and v be two vertices in a tournament T. Prove that if u and v do not lie on a common cycle, then $\operatorname{od} u \neq \operatorname{od} v$.

52. Let T be a tournament with the property that every vertex of T belongs to a directed 3-cycle. Let u and v be distinct vertices of T. Prove that if $|\operatorname{od} u - \operatorname{od} v| \leq 1$, then T contains both a directed $u - v$ path and a directed $v - u$ path.

Section 7.5. Kings in Tournaments

53. Show that every vertex in a nontrivial regular tournament is a king.

54. A tournament T of order n can only be regular if n is odd and so $\text{od}\, v = (n-1)/2$ for every vertex v of T. By Exercise 53, every vertex of T is a king. Prove or disprove: There exists an even integer $n \geq 6$ such that for every tournament T of order n for which $\text{od}\, v \geq (n-2)/2$ for each $v \in V(T)$, every vertex of T is a king.

55. Show that there exists a tournament of order 4 having exactly three kings.

56. Show that there exists a tournament of order 5 having exactly four kings.

57. Show that there is an infinite class of tournaments in which every vertex except one is a king.

58. A vertex z in a nontrivial tournament is called a **serf** if for every vertex x distinct from z, either x is adjacent to z or x is adjacent to a vertex that is adjacent to z. Prove that every nontrivial tournament has at least one serf.

Section 7.6. Hamiltonian Tournaments

59. Prove that if T is a tournament that is not transitive, then T has at least three Hamiltonian paths.

60. (a) It has been mentioned that every tournament has an odd number of Hamiltonian paths. If T is a tournament of order 5 that is not strong, then what is the maximum number of Hamiltonian paths that T can have?
    (b) A tournament T of order 9 has no strong components of order 5 or more and contains k Hamiltonian paths. What are the possible values of k?

61. A tournament T of order 10 contains k Hamiltonian paths and consists of two strong components S_1 and S_2 of order 5. The strong component S_1 has $V(S_1) = \{v_1, v_2, \ldots, v_5\}$ and for $1 \leq i \leq 5$, (v_i, v_j) is an arc of S if $j = i+1$ or $j = i+2$ (addition modulo 5). Determine the number of Hamiltonian paths in S_2 in terms of k.

62. Prove or disprove: If every vertex of a tournament T belongs to a cycle in T, then T is strong.

63. (a) Prove or disprove: Every arc of a nontrivial strong tournament T lies on a Hamiltonian cycle of T.
    (b) A digraph D is **Hamiltonian-connected** if for every pair u, v of vertices of D, there exists a Hamiltonian $u - v$ path. Prove or disprove: Every vertex-pancyclic tournament is Hamiltonian-connected.

64. Show that if a tournament T has an ℓ-cycle, then T has an s-cycle for $s = 3, 4, \ldots, \ell$.

65. A tournament T of order n contains a k-cycle C for some $k \geq 4$, no $(k+1)$-cycle, a $(k-1)$-cycle C' having no vertex on C, a $(k-1)$-cycle C'' having a vertex on C and k vertices lying on no cycle of T. What is the minimum value of n in terms of k.

Chapter 8

Flows in Networks

Networks are special digraphs that are useful in modeling certain types of real-world problems. They can also be used to study problems of connectedness that occur in digraphs. It is this class of digraphs that is discussed here.

8.1 Networks

A **network** N is a digraph D with two distinguished vertices u and v, called the **source** and **sink**, respectively, together with a nonnegative real-valued function c on $E(D)$. The digraph D is called the **underlying digraph** of N and the function c is called the **capacity function** of N. The value $c(a) = c(x, y)$ of an arc $a = (x, y)$ of D is called the **capacity** of a. Any vertex of N distinct from u and v is called an **intermediate vertex** of N.

The source u of N can be thought of as the location from which material is shipped and then transported through N, eventually reaching its destination, namely the sink v of N. The capacity of an arc (x, y) in N may be thought of as the maximum amount of material that can be transported from x to y along (x, y) per unit time. For example, the capacity of the arc (x, y) may represent the number of seats available on a direct flight from city x to city y in some airline network, or perhaps $c(x, y)$ is the capacity of a pipeline from city x to city y in some oil network, or perhaps $c(x, y)$ is the maximum weight of items that can be transported by truck from city x to city y in some highway network. The problem then is to maximize the *flow* of material that can be transported from the source u to the sink v without exceeding the capacity of any arc.

A network N can be represented by drawing the underlying digraph D of N and labeling each arc of D with its capacity. A network is shown in Figure 8.1. In this network, the capacity of the arc (x, y) is then $c(x, y) = 4$. While, in general, there may be more than one source from which material originates and more than one sink providing destinations of the material, it suffices to consider a network with a single source and a single sink (see Exercise 17).

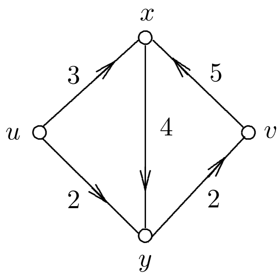


Figure 8.1: A network

For a vertex x in a digraph D, recall that the out-neighborhood $N^+(x)$ and the in-neighborhood $N^-(x)$ of x are defined by

$$\begin{aligned} N^+(x) &= \{y \in V(D): \ (x,y) \in E(D)\} \ \text{ and} \\ N^-(x) &= \{y \in V(D): \ (y,x) \in E(D)\}. \end{aligned}$$

Thus $|N^+(x)| = \operatorname{od} x$ and $|N^-(x)| = \operatorname{id} x$.

For a digraph D and a real-valued function g defined on $E(D)$, it is convenient to introduce some notation. For subsets X and Y of $V(D)$, define the set $[X, Y]$ and the number $g(X, Y)$ by

$$[X, Y] = \{(x, y) : x \in X, y \in Y\}$$

and

$$g(X, Y) = \sum_{(x,y)\in[X,Y]} g(x, y),$$

where $g(X, Y) = 0$ if $[X, Y] = \emptyset$. For $x \in V(D)$,

$$g^+(x) = \sum_{y\in N^+(x)} g(x, y) \ \text{ and } \ g^-(x) = \sum_{y\in N^-(x)} g(y, x). \tag{8.1}$$

More generally, for $X \subseteq V(D)$,

$$g^+(X) = \sum_{x\in X} g^+(x) \ \text{ and } \ g^-(X) = \sum_{x\in X} g^-(x).$$

Network Flows

A **flow** in a network N with underlying digraph D, source u, sink v and capacity function c is a real-valued function f on $E(D)$ satisfying

$$0 \le f(a) \le c(a) \text{ for every arc } a \text{ of } D \tag{8.2}$$

such that

$$f^+(x) = f^-(x) \text{ for each intermediate vertex } x \text{ of } D. \tag{8.3}$$

If f is a function on $E(D)$ defined by $f(a) = 0$ for every arc a of D, then f satisfies both (8.2) and (8.3) and so f is a flow, called the **zero flow**.

The value $f(a) = f(x, y)$ of an arc $a = (x, y)$ is called the **flow along the arc** a and can be interpreted as the rate at which material is transported along a under the flow f. Condition (8.2) requires that the flow along a cannot exceed the capacity of a. Condition (8.3) is referred to as the **conservation equation** and states that the rate at which material is transported into an intermediate vertex x equals the rate at which material is transported out of x.

For a flow f in a network N, the **net flow out of a vertex** x is defined by

$$f^+(x) - f^-(x), \tag{8.4}$$

while the **net flow into** x is

$$f^-(x) - f^+(x). \tag{8.5}$$

By the conservation equation (8.3), it follows that for every intermediate vertex x of D, the net flow out of x equals the net flow into x and the common value of (8.4) and (8.5) is 0.

If $f(a) = c(a)$ for an arc a in a network N, then the arc a is said to be **saturated** with respect to the flow f. On the other hand, if $f(a) < c(a)$, then the arc a is **unsaturated**. An example of a flow in a network is shown in Figure 8.2. The first number associated with an arc is its capacity and for each arc of N, the capacity of the arc is a fixed number, while the second number is the flow along the arc. In general, many flows are possible for a given network. While the arc (x, t) is saturated for the flow f shown in the network in Figure 8.2 since $f(x, t) = c(x, t)$, the arc (w, y) is unsaturated since $f(w, y) < c(w, y)$. In this example, the net flow out of the source u is 3 and the net flow into the sink v is also 3. As we will soon see, that these two numbers are equal is true in general.

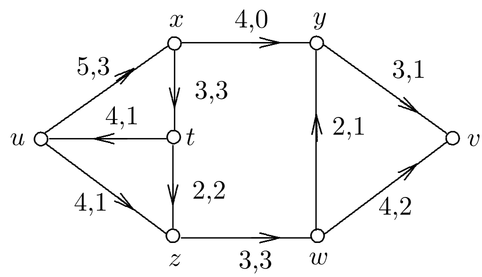


Figure 8.2: A flow in a network

Theorem 8.1 *Let u and v be the source and sink, respectively, of a network N with underlying digraph D and let f be a flow defined on N. Then the net flow out of u equals the net flow into v, that is,*

$$f^+(u) - f^-(u) = f^-(v) - f^+(v).$$

Proof. Since $f^+(V(D)) = f^-(V(D))$, it follows that

$$\sum_{x \in V(D)} f^+(x) = \sum_{x \in V(D)} f^-(x). \tag{8.6}$$

By (8.3),

$$f^+(x) = f^-(x) \text{ when } x \neq u, v. \tag{8.7}$$

By (8.6) then,

$$f^+(u) + f^+(v) = f^-(u) + f^-(v),$$

giving the desired result. ■

Maximum Flows

The **value of a flow** f in a network N, denoted by $\text{val}(f)$, is defined as the net flow out of the source of N. By Theorem 8.1, $\text{val}(f)$ is also the net flow into the sink of N. For the flow f defined on the network in Figure 8.2, we have $\text{val}(f) = 3$.

There are certain flows of particular and obvious interest to us. A flow in a network N whose value is maximum among all flows that can be defined on N is called a **maximum flow**. Thus, a flow f defined on N is a maximum flow if $\text{val}(f) \geq \text{val}(f')$ for every flow f' defined on N. For a given network, a major goal is to find a maximum flow. For the purpose of doing this, it will be convenient to introduce another concept.

Let N be a network with underlying digraph D, source u, sink v and capacity function c. For a set X of vertices in D, let $\overline{X} = V(D) - X$. A **cut** in N is a set of arcs of the form $[X, \overline{X}]$, where $u \in X$ and $v \in \overline{X}$. If $K = [X, \overline{X}]$ is a cut in N, then the **capacity** of K, denoted by $\text{cap}(K)$, is

$$\text{cap}(K) = c(X, \overline{X}) = \sum_{(x,y) \in [X, \overline{X}]} c(x, y).$$

For the network N of Figure 8.2 and $X = \{u, x\}$, the cut

$$K = [X, \overline{X}] = \{(u, z), (x, y), (x, t)\}$$

in N (see Figure 8.3) has capacity

$$\text{cap}(K) = c(u, z) + c(x, y) + c(x, t) = 4 + 4 + 3 = 11.$$

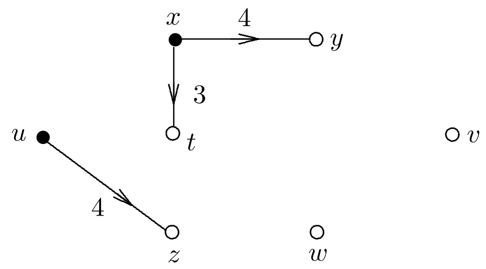


Figure 8.3: The cut $[X, \overline{X}]$ in the network N
of Figure 8.2 for $X = \{u, x\}$

If K is a cut in a network N, then any path from the source u to the sink v must contain at least one arc of K. Consequently, if all arcs of K were removed from the underlying digraph D of N, then there would be no path from u to v. So just as a vertex-cut in a graph G separates some pair of vertices in G and an edge-cut in G separates some pair of vertices in G, a cut in the underlying digraph D of a network N separates u and v in a certain sense.

Let N be a network with underlying digraph D, source u and sink v. For a set X of vertices of D with $u \in X$ and $v \in \overline{X}$ and a flow f defined on N, the **net flow out of** X is $f^+(X) - f^-(X)$ and the **net flow into** X is $f^-(X) - f^+(X)$. It then follows (see Exercise 8) that

$$f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X). \tag{8.8}$$

For the set $X = \{u, x, t\}$ in the network N in Figure 8.2, $f^+(X) = 10$ and $f^-(X) = 7$. For this network then, the net flow out of X is $f^+(X) - f^-(X) = 10 - 7 = 3 = \text{val}(f)$. We now show that for a network N and any cut $K = [X, \overline{X}]$ in N, the value of any flow in N is the net flow out of X and that this value never exceeds the capacity of K.

Theorem 8.2 *Let f be a flow in a network N and let $K = [X, \overline{X}]$ be a cut in N. Then*

$$\text{val}(f) = f^+(X) - f^-(X) \le \text{cap}(K).$$

Proof. Let D be the underlying digraph of N, let u and v be the source and sink, respectively, of N and let c be the capacity function of N.

Since $f^+(x) - f^-(x) = 0$ for every $x \in X - \{u\}$, it follows that

$$\sum_{x \in X} (f^+(x) - f^-(x)) = f^+(u) - f^-(u) = \text{val}(f).$$

Furthermore,

$$\sum_{x \in X} (f^+(x) - f^-(x)) = \sum_{x \in X} f^+(x) - \sum_{x \in X} f^-(x) = f^+(X) - f^-(X)$$

and so $\text{val}(f) = f^+(X) - f^-(X)$. Since $0 \le f(a) \le c(a)$ for every arc a of N, it follows that

$$\begin{aligned} \text{val}(f) &= f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X) \\ &\le f(X, \overline{X}) \le c(X, \overline{X}) = \text{cap}(K), \end{aligned}$$

giving the desired result. ∎

Minimum Cuts

There are, in general, many cuts in a network N and each cut has a capacity. Any cut in N whose capacity is minimum among all cuts in N is called a **minimum cut**. That is, a cut K in N is a minimum cut if $\text{cap}(K) \le \text{cap}(K')$ for every cut K' in N. The following two corollaries provide some important information about minimum cuts and maximum flows.

Corollary 8.3 *If f is a flow in a network N and K is a cut in N such that* $\text{val}(f) = \text{cap}(K)$, *then f is a maximum flow and K is a minimum cut in N.*

Proof. If f^* is a maximum flow in N and K^* is a minimum cut, then $\text{val}(f^*) \le \text{cap}(K^*)$ by Theorem 8.2. Consequently,

$$\text{val}(f) \le \text{val}(f^*) \le \text{cap}(K^*) \le \text{cap}(K). \tag{8.9}$$

Since $\text{val}(f) = \text{cap}(K)$, it follows that there is equality throughout (8.9) and so $\text{val}(f) = \text{val}(f^*)$ and $\text{cap}(K^*) = \text{cap}(K)$, that is, f is a maximum flow and K is a minimum cut. ∎

Corollary 8.4 *If f is flow in a network N with capacity function c and $[X, \overline{X}]$ is a cut in N such that*

$$f(a) = c(a) \text{ for all } a \in [X, \overline{X}]$$

and

$$f(a) = 0 \text{ for all } a \in [\overline{X}, X],$$

then f is a maximum flow in N and $[X, \overline{X}]$ is a minimum cut.

Corollary 8.4 (see Exercise 9) suggests how the values of a flow f should be defined on the arcs of a minimum cut in order for f to be a maximum flow.

A network N with source u and sink v is shown in Figure 8.4 together with a flow f defined on N. As always, the first number associated with an arc a is its capacity $c(a)$ and the second number is the flow $f(a)$. If $X = \{u, x, y\}$, then $K = [X, \overline{X}]$ is a cut in N. Since $f(a) = c(a)$ for all $a \in [X, \overline{X}]$ and $f(a) = 0$ for all $a \in [\overline{X}, X]$, it follows by Corollary 8.4 that f is a maximum flow and K is a minimum cut. Since $\text{cap}(K) = 4$, the value of the maximum flow f is 4.

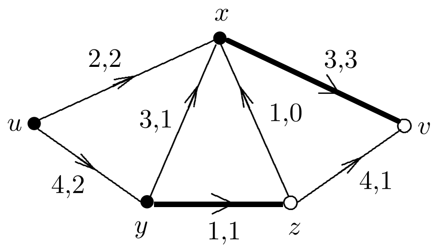


Figure 8.4: A cut $K = [X, \overline{X}]$ in a network, where $X = \{u, x, y\}$

According to Corollary 8.3, if it should ever occur that the value of some flow f in a network N equals the capacity of some cut K in N, then f must be a maximum flow and K is a minimum cut. We next show for any maximum flow f and any minimum cut K that $\text{val}(f) = \text{cap}(K)$. In preparation for proving this, some additional terminology is useful.

For a digraph D, an $x - y$ **semipath** in D is an alternating sequence

$$P = (x = w_0, a_1, w_1, a_2, w_2, \ldots, w_{k-1}, a_k, w_k = y)$$

of distinct vertices and arcs of D beginning with x and ending with y such that either $a_i = (w_{i-1}, w_i)$ or $a_i = (w_i, w_{i-1})$ for each i $(1 \leq i \leq k)$. In this case, (w_{i-1}, w_i) is called a **forward arc** of P and (w_i, w_{i-1}) is a **backward arc** of P. Hence, when proceeding from x to y along the semipath P in D, we move in the direction of a forward arc on P and move opposite to the direction of a backward arc on P.

f-Augmenting Semipaths

Let N be a network with underlying digraph D and capacity function c and on which is defined a flow f. Recall that an arc a is unsaturated if $f(a) < c(a)$. A semipath

$$P = (w_0, a_1, w_1, a_2, \ldots, w_{k-1}, a_k, w_k)$$

in D is said to be **f-unsaturated** if for each i $(1 \leq i \leq k)$,

(i) a_i is unsaturated whenever a_i is a forward arc and

(ii) $f(a_i) > 0$ whenever a_i is a backward arc.

A trivial semipath in D is vacuously f-unsaturated. If P is an f-unsaturated $u - v$ semipath in D where u and v are the source and sink, respectively, then P is called an **f-augmenting semipath**. As we will see, given an f-augmenting semipath in the underlying digraph D of a network N, it is possible to augment (alter) the values of the flow f on each arc of P to obtain a new flow f' whose value exceeds that of f. For example, consider the flow f in the network N in Figure 8.2, shown again in Figure 8.5. Then

$$P = (u, (t, u), t, (x, t), x, (x, y), y, (y, v), v)$$

is an f-augmenting semipath.

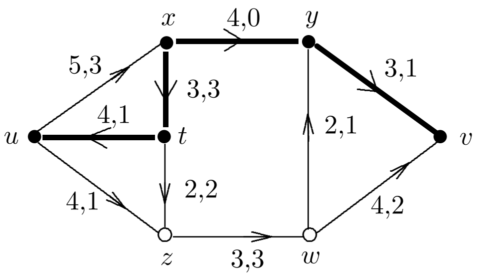


Figure 8.5: An f-augmenting semipath in a network

Recall that the value of the flow f defined on the network N in Figure 8.5 is $f(u, x) + f(u, z) - f(t, u) = 3 + 1 - 1 = 3$. Let f' be the function defined on $E(D)$ by redefining the flow of each arc on the f-augmenting semipath as follows:

$$f'(a) = \begin{cases} f(a) + 1 & \text{if } a \in \{(x, y), (y, v)\} \\ f(a) - 1 & \text{if } a \in \{(t, u), (x, t)\} \\ f(a) & \text{otherwise.} \end{cases}$$

Then f' is also a flow in N and $\text{val}(f') = 4$. Consequently, f is not a maximum flow. In fact, f' is not a maximum flow either. That f is not a maximum flow and that N contains an f-augmenting semipath is not a coincidence, as Lester Randolph Ford, Jr. and Delbert Ray Fulkerson [93] showed.

Theorem 8.5 *Let N be a network with underlying digraph D. A flow f in N is a maximum flow if and only if there is no f-augmenting semipath in D.*

Proof. Let u and v be the source and sink, respectively, of N and let c be the capacity function. Suppose first that D contains an f-augmenting semipath

$$P = (u = w_0, a_1, w_1, a_2, w_2, \ldots, w_{k-1}, a_k, w_k = v).$$

Then, for $1 \le i \le k$,

(a) $f(a_i) < c(a_i)$ whenever a_i is a forward arc and

(b) $f(a_i) > 0$ whenever a_i is a backward arc.

If there is at least one forward arc $a_i = (w_{i-1}, w_i)$ on P, let p_1 be the minimum value of $c(a_i) - f(a_i)$ over all forward arcs a_i. If there is at least one backward arc $a_i = (w_i, w_{i-1})$ on P, let p_2 be the minimum value of $f(a_i)$ over all such backward arcs a_i. If only one of p_1 and p_2 is defined, then denote this number by p; otherwise, let $p = \min(p_1, p_2)$.

Define a function f' on $E(D)$ by

$$f'(a) = \begin{cases} f(a) + p & \text{if } a \text{ is a forward arc on } P \\ f(a) - p & \text{if } a \text{ is a backward arc on } P \\ f(a) & \text{if } a \notin E(P). \end{cases}$$

Then f' is also a flow. Since $\text{val}(f') = \text{val}(f) + p$, it follows that f is not a maximum flow.

We now turn to the converse. Assume that there is no f-augmenting semipath in D. Let X be the set of all vertices x in D for which there exists an f-unsaturated $u - x$ semipath. Then $u \in X$ and, by assumption, $v \notin X$. Thus, $K = [X, \overline{X}]$ is a cut in N.

Let (y, z) be an arc in K. Since $y \in X$, there exists an f-unsaturated $u - y$ semipath P in D. Since $z \in \overline{X}$, there is no f-unsaturated $u - z$ semipath in D, which implies that $f(y, z) = c(y, z)$. Similarly, if $(w, x) \in [\overline{X}, X]$, then $f(w, x) = 0$. Since $K = [X, \overline{X}]$ is a cut such that $f(a) = c(a)$ for every arc $a \in [X, \overline{X}]$ and $f(a) = 0$ for every arc $a \in [\overline{X}, X]$, it follows by Corollary 8.4 that f is a maximum flow. ∎

8.2 The Max-Flow Min-Cut Theorem

With the aid of Theorem 8.5, Ford and Fulkerson [93] proved a famous result in 1956 that is known as the *Max-Flow Min-Cut Theorem.* Independently, and also in 1956, Peter Elias, Amiel Feinstein and Claude Elwood Shannon [77] discovered and proved the very same result.

Theorem 8.6 (**The Max-Flow Min-Cut Theorem**) *In any network, the value of a maximum flow equals the capacity of a minimum cut.*

Proof. Let f be a maximum flow in a network N having capacity function c. By Theorem 8.5, there is no f-augmenting semipath in the underlying digraph D of N. By the proof of Theorem 8.5, since D contains no f-augmenting semipath, there is a minimum cut $K = [X, \overline{X}]$ such that

$$f(a) = \begin{cases} c(a) & \text{if } a \in K \\ 0 & \text{if } a \in [\overline{X}, X]. \end{cases}$$

Therefore, by (8.8),

$$\begin{aligned} \text{val}(f) &= f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X) \\ &= c(X, \overline{X}) - 0 = \text{cap}(K), \end{aligned}$$

as desired. ∎

The proof of Theorem 8.5 provides the basis of an algorithm, also due to Ford and Fulkerson [94], for finding a maximum flow in a network.

Algorithm 8.7 (**The Ford–Fulkerson Algorithm**) *For a network N with underlying digraph D, source u, sink v and capacity function c,*

1. Let f be a flow on D. (The zero flow may be used.)

2. Find an f-augmenting semipath.

3. If there is no f-augmenting semipath, then f is a maximum flow and stop.

   If there is an f-augmenting semipath P, then augment P as in the proof of Theorem 8.5 to produce a new flow f'.

4. Set $f = f'$ and return to Step 3.

This is a somewhat simplified version of the algorithm given by Ford and Fulkerson in [94]. They provided a scheme for finding an f-augmenting semipath for step 2. However, the version we presented contains the two inherent issues. First of all, Ford and Fulkerson themselves showed that their procedure might not terminate if the capacities are irrational. Their complex example can be found in [94]. The second issue is that even if it terminates (as in the case of only rational capacities), the algorithm may not be efficient. Consider the network N in Figure 8.6, where the labels on the arcs indicate their capacities and C is any constant.

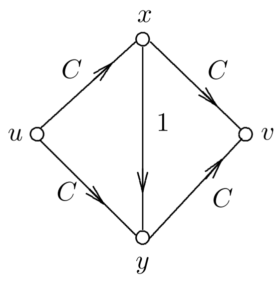


Figure 8.6: A network N

If we unwisely always choose augmenting semipaths that include the only arc with capacity equal to 1 (which is possible in the original Ford–Fulkerson Algorithm), then the maximum flow is $2C$, no matter what constant C we choose, and the number of iterations of the algorithm is $2C$ if we start with the zero flow. Of course, optimally, the algorithm takes just two iterations.

The Edmonds–Karp Algorithm

A slight refinement of the Ford–Fulkerson algorithm, due to the Russian scientist Efim A. Dinic [69], was first published in 1970 and published independently by Jack Edmonds and Richard M. Karp [74] in 1972. The Edmonds-Karp algorithm, which we describe next, searches for a shortest f-augmenting semipath in a network with a given flow f. This is an efficient algorithm, having complexity $O(nm^2)$, where the underlying digraph of the network has order n and size m.

Let u and v be the source and sink of a network N with underlying digraph D and let f be a given flow in N (perhaps the zero flow). The algorithm proceeds by constructing a sequence of labelings of the vertices of D. A vertex w in D is assigned a label only if there is an f-unsaturated $u-w$ semipath P in D. The label assigned to w is an ordered pair. If x is the vertex immediately preceding w on P, then the first coordinate of the label is either $x+$ or $x-$, according to whether the arc immediately preceding w is (x, w) or (w, x). The second coordinate of the label is a positive number that reflects the potential change in f along P, which we will soon describe. As we proceed through the algorithm, a list of labeled vertices is created. At a certain stage of the algorithm, a list L consisting of some labeled vertices of D is formed. At some point, the first vertex on L is examined (scanned) and removed from L to determine whether this vertex is adjacent to certain unlabeled vertices possessing a particular property, in which case all such vertices are then labeled and added to L. If the sink v is labeled, then a new flow of greater value can be obtained. This process is then repeated. On the other hand, if the sink v is not labeled, then f is a maximum flow. In this case, the labels can be used to find a minimum cut.

Throughout the course of the algorithm, each vertex of D is considered to be in one of the following three states:

(1) unlabeled, (2) labeled and unscanned, (3) labeled and scanned.

Prior to the implementation of the algorithm, all vertices are unlabeled. Once a vertex is assigned a label, it is placed at the end of a list L consisting of the labeled and unscanned vertices. The vertices of L are scanned on a *first-labeled first-scanned* basis, which will guarantee the selection of a shortest f-augmenting semipath. The Edmonds–Karp algorithm is now stated.

Algorithm 8.8 (**The Edmonds–Karp Algorithm**) *For a network N with underlying digraph D, source u, sink v and capacity function c,*

1. Let f be a flow on D and label each arc of D with the value of f.

2. Label the source u with the ordered pair $(-, \infty)$ and add u to the list L of labeled and unscanned vertices.

3. If L is empty, then stop. Otherwise, scan and remove the first element of L, say x, having the label $(z+, \epsilon(x))$ or $(z-, \epsilon(x))$.

   3.1 Assign to each unlabeled vertex y for which
   $$(x, y) \in E(D) \text{ and } f(x, y) < c(x, y)$$
   the label $(x+, \epsilon(y))$, where
   $$\epsilon(y) = \min\{\epsilon(x), c(x, y) - f(x, y)\}$$
   and add y to the end of L.

   3.2 Assign to each unlabeled vertex y for which
   $$(y, x) \in E(D) \text{ and } f(y, x) > 0$$
   the label $(x-, \epsilon(y))$, where
   $$\epsilon(y) = \min\{\epsilon(x), f(y, x)\}$$
   and add y to the end of L.

4. If v has been labeled, go to Step 5; otherwise return to Step 3.

5. The labels describe an f-augmenting semipath
   $$(u = w_0, a_1, w_1, a_2, w_2, \ldots, w_{k-1}, a_k, w_k = v)$$
   where, for $1 \leq i \leq k$, w_i is labeled
   $$(w_{i-1}+, \epsilon(w_i)) \text{ if } a_i = (w_{i-1}, w_i) \text{ is a forward arc}$$
   or w_i is labeled
   $$(w_{i-1}-, \epsilon(w_i)) \text{ if } a_i = (w_i, w_{i-1}) \text{ is a backward arc.}$$
   In the first case, replace $f(w_{i-1}, w_i)$ by $f(w_{i-1}, w_i) + \epsilon(v)$; while in the second case, replace $f(w_i, w_{i-1})$ by $f(w_i, w_{i-1}) - \epsilon(v)$.

6. Discard all labels, remove all vertices from L and return to Step 2.

If the capacity of every arc in a network is an integer, then the value of an integer-valued flow increases by 1 or more with each iteration of Algorithm 8.8 and the algorithm terminates after finitely many iterations. This is also the case when the capacity of every arc is a rational number and a given flow is rational-valued. Since networks commonly encountered in discrete mathematics have integer capacities, it is integer-valued flows that we seek. For the remainder of our discussion on flows in networks, in particular in the proof of Theorem 8.9, we assume that the capacities and flows in networks are integer-valued. It should be emphasized, however, that Edmonds and Karp [74] proved that if the underlying digraph of the network has order n and size m, then their algorithm terminates in at most $O(nm^2)$ steps, even with irrational capacities.

Theorem 8.9 *Algorithm* 8.8 *terminates with a maximum flow f in N. Furthermore, if X is the set of labeled vertices upon termination, then $[X, \overline{X}]$ is a minimum cut.*

Proof. By Theorem 8.5, each time that Step 5 is completed, a new flow f having a larger value is constructed. If, in Step 3, the list L is empty, then there is no f-augmenting semipath and so, by Theorem 8.5, f is a maximum flow and $[X, \overline{X}]$ is a minimum cut. This process must terminate since for every flow f' in N,

$$\text{val}(f') \leq c(u, V(D))$$

and so Step 5 can be repeated at most $c(u, V(D))$ times. ■

We now illustrate Algorithm 8.8 for the network shown in Figure 8.7. It is not difficult to find the flow f shown in Figure 8.7, which we take as the initial flow. As always, each arc a is labeled with the pair $c(a)$, $f(a)$, where $c(a)$ is the capacity of a and $f(a)$ is the flow along a.

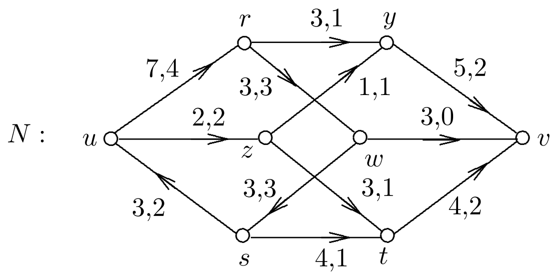


Figure 8.7: A flow f in a network

Since an initial flow has been given to the network N, Step 1 of Algorithm 8.8 has been completed and we have the situation shown in Figure 8.7. Initially, the source u is assigned the label $(-, \infty)$ and the list L of labeled but unscanned vertices now consists of the vertex u only. Thus Step 2 of Algorithm 8.8 is now completed.

We now move on to Step 3. Since L is not empty, we do not stop. We now scan and remove the first element of L from this list. In this case, only u belongs to L; so, u is now removed from L. Thus, L is now empty. We search for all vertices x of N such that either (i) $(u, x) \in E(D)$ and $f(u, x) < c(u, x)$ or (ii) $(x, u) \in E(D)$ and $f(x, u) > 0$. There are two vertices with one of these two properties, namely r and s. We consider r first. Since $(u, r) \in E(D)$, the first coordinate of the label of r is $u+$. Since $c(u, r) - f(u, r) = 3$, the second coordinate of the label of r is 3 and so $\epsilon(r) = 3$. That is, r is assigned the label $(u+, 3)$. The vertex r is then placed at the end of L. Since L was previously empty, we now have $L : r$. We now turn to the vertex s. Because $(s, u) \in E(D)$, the first coordinate of the label of s is $u-$. Because $f(s, u) = 2$, the second coordinate of the label of s is 2 and so $\epsilon(s) = 2$. Thus, s is assigned the label $(u-, 2)$. The vertex s is then added to the end of L (and so s follows r in the list L). At this moment, we now have three labeled vertices, while the list L consists of two vertices, namely

labeled vertices: $u\ (-, \infty)$, $r\ (u+, 3)$, $s\ (u-, 2)$ and list $L : r, s$.

Since we have now completed Step 3, we move on to Step 4. Since v has not been labeled, we return to Step 3.

Since L is not empty, we do not stop and, instead, we scan and remove the first element of L, namely r, from this list and search for all unlabeled vertices x of N such that either (i) $(r, x) \in E(D)$ and $f(r, x) < c(r, x)$ or (ii) $(x, r) \in E(D)$ and $f(x, r) > 0$. Only the vertex y has this property, namely $(r, y) \in E(D)$ and $f(r, y) < c(r, y)$. Therefore,

$$\epsilon(y) = \min\{\epsilon(r),\ c(r, y) - f(r, y)\} = \min\{3, 2\} = 2$$

and so y is assigned the label $(r+, 2)$ and y is placed at the end of L, resulting in $L : s, y$. Since Step 3 has been completed, we move on to Step 4. Since v has not been labeled, we return to Step 3.

Since the list L is not empty, we scan and remove the first vertex, namely s, from L and search for all unlabeled vertices x of N such that either (i) $(s, x) \in E(D)$ and $f(s, x) < c(s, x)$ or (ii) $(x, s) \in E(D)$ and $f(x, s) > 0$. The vertices w and t satisfy these properties. Since $(w, s) \in E(D)$ and $f(w, s) > 0$, it follows that

$$\epsilon(w) = \min\{\epsilon(s),\ f(w, s)\} = \min\{2, 3\} = 2$$

and so the vertex w is assigned the label $(s-, 2)$ and placed at the end of L. Since $(s, t) \in E(D)$ and $f(s, t) < c(s, t)$, it follows that

$$\epsilon(t) = \min\{\epsilon(s),\ c(s, t) - f(s, t)\} = \min\{2, 3\} = 2$$

and so the vertex t is assigned the label $(s+, 2)$ and placed at the end of L. Thus, we currently have the following:

labeled vertices: $u\ (-, \infty)$, $r\ (u+, 3)$, $s\ (u-, 2)$, $y\ (r+, 2)$, $w\ (s-, 2)$, $t\ (s+, 2)$

list $L : y, w, t$.

Since Step 3 has been completed, we turn to Step 4. Since v has not been labeled, we once again return to Step 3. The list L is not empty, so the vertex y on L is removed and scanned. Only one unlabeled vertex is adjacent to or from y, namely the sink v. Because $(y, v) \in E(D)$ and $f(y, v) < c(y, v)$, the vertex v is assigned a label. The first coordinate is $y+$ and the second coordinate is

$$\epsilon(v) = \min\{\epsilon(y),\ c(y, v) - f(y, v)\} = \min\{2, 3\} = 2.$$

Thus, the vertex v is assigned the label $(y+, 2)$. Since this completes Step 3, we move on to Step 4. Since v is labeled, we now move on to Step 5. Working in reverse and beginning with the sink v, we observe that the first coordinate of the label of v is $y+$. This tells us that the f-augmenting semipath being constructed ends with $y, (y, v), v$. Because the first coordinate of the label of y is $r+$, the semipath being constructed ends with $r, (r, y), y, (y, v), v$. Since the first coordinate of the label of r is $u+$, we obtain the following f-augmenting semipath:

$$P = (u, (u, r), r, (r, y), y, (y, v), v).$$

Since $\epsilon(v) = 2$, a new flow f' (see Figure 8.8) is obtained from f by augmenting the flow of each arc of P by 2.

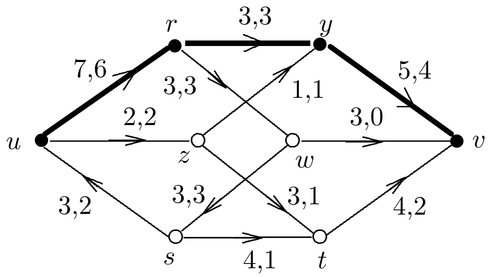


Figure 8.8: The f-augmenting semipath and the new flow f' in the network

Since Step 5 has been completed, we go to Step 6, when the labels of all vertices of D are removed and the vertices on the list L are removed. The table below summarizes what has transpired.

u is labeled $(-, \infty)$	$L : u$
u is scanned r is labeled $(u+, 3)$ s is labeled $(u-, 2)$	$L :$ $L : r$ $L : r, s$
r is scanned y is labeled $(r+, 2)$	$L : s$ $L : s, y$
s is scanned w is labeled $(s-, 2)$ t is labeled $(s+, 2)$	$L : y$ $L : y, w$ $L : y, w, t$
y is scanned v is labeled $(y+, 2)$	$L : w, t$ $L : w, t, v$

We now apply Algorithm 8.8 to the flow f' in the network N shown in Figure 8.8. The table below summarizes what occurs in this case.

u is labeled $(-, \infty)$	$L : u$
u is scanned r is labeled $(u+, 1)$ s is labeled $(u-, 2)$	$L :$ $L : r$ $L : r, s$
r is scanned	$L : s$
s is scanned w is labeled $(s-, 2)$ t is labeled $(s+, 2)$	$L :$ $L : w$ $L : w, t$
w is scanned v is labeled $(w+, 2)$	$L : t$ $L : t, v$

In this application of Algorithm 8.8, the f'-augmenting semipath

$$P' = (u, (s, u), s, (w, s), w, (w, v), v)$$

is obtained. Since $\epsilon(v) = 2$, a new flow f'' is obtained from f' by augmenting the flow of each arc of P' by 2, as shown in Figure 8.9.

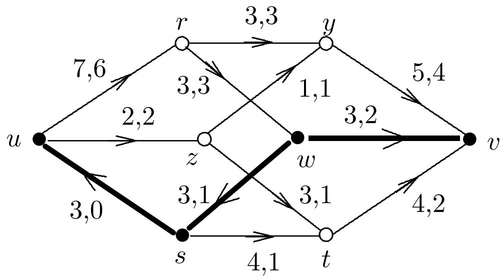


Figure 8.9: The f'-augmenting semipath and the new flow f'' in the network

Once again, Algorithm 8.8 is applied, this time to the flow f'' in the network N shown in Figure 8.9. The table below shows what happens here.

u is labeled $(-, \infty)$	$L : u$
u is scanned r is labeled $(u+, 1)$	$L :$ $L : r$
r is scanned	$L :$

The algorithm stops and the current flow f'' is a maximum flow with $\text{val}(f'') = 8$. Let X consist of the labeled vertices of D, namely $X = \{u, r\}$. So $K = [X, \overline{X}]$, $\text{cap}(K) = 8$ and K is a minimum cut in the network.

8.3 Menger Theorems for Digraphs

Since Menger's theorem (Theorem 4.10) and its edge analogue (Theorem 4.17) both deal with sets that separate two vertices in a graph, it is perhaps not surprising that these two theorems are closely related to the Max-Flow Min-Cut Theorem which deals with deleting the arcs of a cut, thereby separating the source and sink of a network

Recall that Menger's theorem states that if u and v are nonadjacent vertices of a graph G, then the maximum number of internally disjoint $u-v$ paths in G equals the minimum number of vertices that separate u and v. That is, Menger's theorem is a "max-min" theorem. As it turns out, there are other forms of Menger's theorem. We saw one of these in Theorem 4.17, which is the edge form of Menger's theorem. The vertex and edge forms of Menger's theorem have natural analogues for digraphs. All four of these results can be proved, either directly or indirectly, using the Max-Flow Min-Cut Theorem. In each case, the object is to construct an appropriate network from the given graph or digraph. For example, for two nonadjacent vertices u and v in a given graph G, Algorithm 8.8 can be used to determine the minimum number of vertices that separate u and v as well as such a set of vertices.

While we have defined separating sets of vertices and separating sets of edges in a graph, we need analogous terminology for digraphs. Let D be a digraph and let u and v be two nonadjacent vertices of D. A set $S \subseteq V(D) - \{u, v\}$ is said to be a $u-v$ **separating set of vertices** if every $u-v$ path in P contains at least one vertex of S. A set S of arcs in D is a $u-v$ **separating set of arcs** if every $u-v$ path in P contains at least one arc of S. We now present the arc form of Menger's theorem.

The Arc Form of Menger's Theorem

Theorem 8.10 (**The Arc Form of Menger's Theorem**) *For distinct vertices u and v in a digraph D, the maximum number of pairwise arc-disjoint $u-v$ paths in D equals the minimum number of arcs in a $u-v$ separating set of arcs.*

Proof. Suppose that the maximum number of $u-v$ paths in a collection of pairwise arc-disjoint $u-v$ paths in D is k and the minimum number of arcs in a $u-v$ separating set S is ℓ. Since each of these $u-v$ paths contains at least one arc of S and no arc in S belongs to more than one such $u-v$ path, it follows that $k \le \ell$. It remains to show that $\ell \le k$.

We now construct a network N with underlying digraph D, source u and sink v by defining a capacity function c on $E(D)$ such that $c(a) = 1$ for each arc a of D. By the Max-Flow Min-Cut Theorem (Theorem 8.6), the value of a maximum flow in N equals the capacity of a minimum cut. Let f be a maximum flow in N and K a minimum cut. Thus, $\text{cap}(K) = \text{val}(f)$. We next show that $\ell \leq \text{cap}(K)$ and $\text{val}(f) \leq k$, from which it will follow that $\ell \leq k$ and so $\ell = k$.

Since K is a minimum cut in N, the set K is a $u - v$ separating set of arcs in D. Therefore, $\ell \leq |K| = \text{cap}(K)$.

Since f is an integer-valued function defined on $E(D)$ such that $0 \leq f(a) \leq c(a)$ for every arc a of D, it follows that either $f(a) = 0$ or $f(a) = 1$ for every arc a of D. Let D_1 be the digraph obtained from D by deleting all arcs a from D for which $f(a) = 0$. Consequently, $f(a) = 1$ for each arc a of D_1. Since f is a flow in D, it follows by (8.7) that

$$f^+(x) = f^-(x) \text{ for each } x \in V(D) - \{u, v\}$$

and that

$$f^+(u) - f^-(u) = \text{val}(f) = f^-(v) - f^+(v).$$

However, for each vertex x in D,

$$f^+(x) = \text{od}_{D_1} x \text{ and } f^-(x) = \text{id}_{D_1} x.$$

Therefore,

$$\text{od}_{D_1} w = \text{id}_{D_1} w \text{ if } w \in V(D) - \{u, v\}$$

and

$$\text{od}_{D_1} u - \text{id}_{D_1} u = \text{val}(f) = \text{id}_{D_1} v - \text{od}_{D_1} v.$$

Since $\text{od}_{D_1} u = \text{id}_{D_1} u + \text{val}(f)$ and $\text{id}_{D_1} v = \text{od}_{D_1} v + \text{val}(f)$, it follows by Exercise 24 in Chapter 7 that the digraph D_1 and D as well contain $\text{val}(f)$ arc-disjoint $u - v$ paths and so $k \geq \text{val}(f)$. Therefore, $\ell = k$. ■

With the aid of the proof of Theorem 8.10, we use Algorithm 8.8 to determine the maximum number of arc-disjoint $u - v$ paths in the digraph D of Figure 8.10(a). Define a capacity function c on $E(D)$ such that $c(a) = 1$ for each arc a of D. We then have a network N with underlying digraph D, source u, sink v and capacity function c. Applying Algorithm 8.8, we obtain a maximum flow f and minimum cut $K = [X, \overline{X}]$, where $X = \{u, u_4, u_5\}$, so that $\text{val}(f) = \text{cap}(K) = 3$. See Figure 8.10(b). Hence,

$$K = \{(u, u_1), (u, u_2), (u_5, v)\}.$$

Consequently, the maximum number of arc-disjoint $u - v$ paths in D is 3. Three such paths are

$$P = (u, u_1, u_2, u_3, v),\ P' = (u, u_2, v) \text{ and } P'' = (u, u_5, v).$$

We now present the digraph analogue of Menger's theorem itself, which we refer to as the directed vertex form of Menger's theorem.

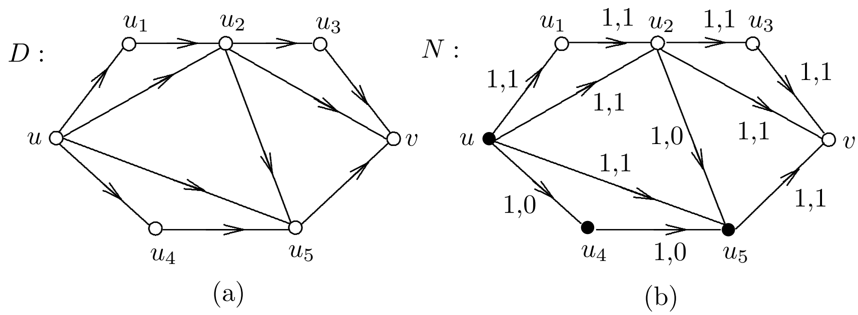


Figure 8.10: Determining the maximum number of internally disjoint $u-v$ paths in a digraph

The Directed Vertex Form of Menger's Theorem

Theorem 8.11 (**The Directed Vertex Form of Menger's Theorem**) *Let D be a digraph and let u and v be distinct vertices of D such that $(u,v) \notin E(D)$. Then the maximum number of internally disjoint $u-v$ paths in D equals the minimum number of vertices in a $u-v$ separating set of vertices of D.*

Proof. Suppose that the maximum number of $u-v$ paths in a collection of internally disjoint $u-v$ paths in D is k and the minimum number of vertices in a $u-v$ separating set S of vertices of D is ℓ. Since each of these $u-v$ paths contains a vertex of S and no vertex of S lies on more than one such path, it follows that $k \leq \ell$. It remains to show that $\ell \leq k$.

A new digraph D' is constructed from D by replacing each vertex $t \neq u, v$ by two new vertices t' and t'' and the arc (t', t''). Suppose that x and y are two vertices of D such that neither x nor y is u or v, that is, $x, y \notin \{u, v\}$. If (x,y) is an arc of D, then (x,y) is replaced by (x'', y'). If (u,x) is an arc of D, then (u,x) is replaced by (u,x'). If (x,u) is an arc of D, then (x,u) is replaced by (x'',u). If (x,v) is an arc of D, then (x,v) is replaced by (x'',v); while if (v,x) is an arc of D, then (v,x) is replaced by (v,x'). This is summarized in the table below.

For $x, y \in V(D)$ with $\{x,y\} \cap \{u,v\} = \emptyset$,

arcs in D	replaced by	arcs in D'
(x,y)		(x'',y')
(u,x)		(u,x')
(x,u)		(x'',u)
(x,v)		(x'',v)
(v,x)		(v,x')

This is illustrated for the digraph D shown in Figure 8.11.

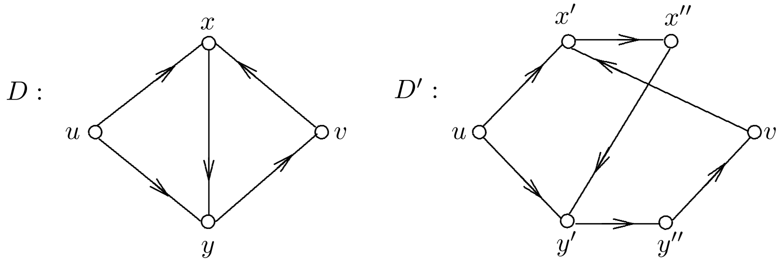


Figure 8.11: Constructing a digraph D' from a digraph D

Suppose that k' is the maximum number of pairwise arc-disjoint $u-v$ paths in D' and ℓ' is the minimum number of arcs in a $u-v$ separating set of arcs in D'. By Theorem 8.10, $\ell' = k'$. We show that $\ell \leq \ell'$ and $k' \leq k$, which will give us the desired inequality $\ell \leq k$.

Let A be a $u-v$ separating set of arcs in D' with $|A| = \ell'$. Thus, A contains no arc of the form (z, u) or (v, z) where $z \neq u, v$. Each arc $a \in A$ is either of the form

$$a = (u, x'),\ a = (x', x''),\ a = (y'', x') \text{ or } a = (x'', v)$$

for some vertices $x, y \in V(D) - \{u, v\}$. Regardless of what the arc a is, denote the vertex x involved in a by w_a. Furthermore, let

$$W = \{w_a : \ a \in A\} \text{ so that } W \subseteq V(D) - \{u, v\}$$

and $|W| \leq |A| = \ell'$. Since A is a $u-v$ separating set of arcs in D', the set W is a $u-v$ separating set of vertices in D. Thus, $\ell \leq |W| \leq \ell'$ and so $\ell \leq \ell'$.

Next, let $P_1', P_2', \ldots, P_{k'}'$ be a collection of k' pairwise arc-disjoint $u-v$ paths in D'. Then each path P_i' $(1 \leq i \leq k)$ is of the form

$$P_i' = (u, x_1', x_1'', x_2', x_2'', \ldots, x_r', x_r'', v)$$

and gives rise to the path

$$P_i = (u, x_1, x_2, \ldots, x_r, v).$$

From the manner in which the digraph D' is constructed and the fact that $P_1', P_2', \ldots, P_{k'}'$ are pairwise arc-disjoint, it follows that $P_1, P_2, \ldots, P_{k'}$ are internally disjoint $u-v$ paths in D. Thus, $k \geq k'$. Hence, $\ell \leq \ell' = k' \leq k$ and so $\ell = k$. ■

As an illustration of the proof of Theorem 8.11, consider the digraph D of Figure 8.12. We seek the maximum number of internally disjoint $u-v$ paths in D. Although this is rather easy to do in this case, we construct the digraph D' described in the proof of Theorem 8.11 and determine the maximum number of pairwise arc-disjoint $u-v$ paths in D'. This number is 2 and two such paths are

$$P_1' = (u, w', w'', z', z'', v) \text{ and } P_2' = (u, y', y'', x', x'', v).$$

As described in the proof of Theorem 8.11, this gives rise to the paths

$$P_1 = (u, w, z, v) \text{ and } P_2 = (u, y, x, v),$$

which constitute a maximum set of internally disjoint $u - v$ paths in D.

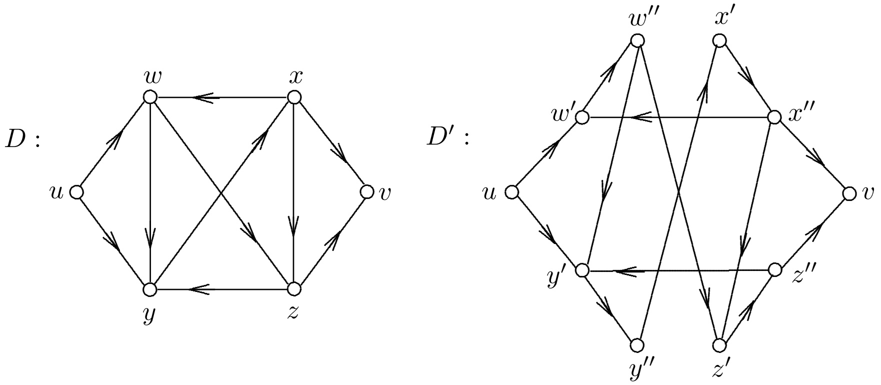


Figure 8.12: Illustrating the proof of Theorem 8.11

For a given graph G, an alternative proof of the edge form of Whitney's theorem (Theorem 4.18) can be obtained by applying Theorem 8.10 to the symmetric digraph D with underlying graph G. For vertices u and v in G, it remains only to observe that there is a one-to-one correspondence between the $u - v$ paths in G and the (directed) $u - v$ paths in D. A proof of Menger's theorem itself (the vertex form for graphs in Theorem 4.10) for a graph G can also be obtained by applying Theorem 8.11 to the symmetric digraph D whose underlying graph is G.

Exercises for Chapter 8

Section 8.1. Networks

1. Let N be the network with source u and sink v shown in Figure 8.13, where each arc is labeled with its capacity. A function f is defined on the arcs of N as follows:

$$\begin{array}{llll} f(u,s)=3 & f(s,t)=3 & f(t,v)=4 & f(u,x)=3 \\ f(x,y)=3 & f(y,v)=1 & f(x,t)=1 & f(w,u)=0 \\ f(y,w)=2 & f(w,v)=2. & & \end{array}$$

   Is f a flow?

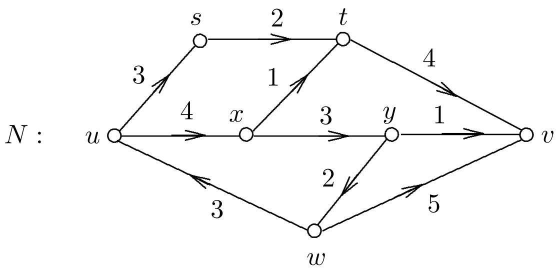


Figure 8.13: The network N in Exercise 1

2. Let N be the network with source u and sink v shown in Figure 8.14, where each arc is labeled with its capacity.

   (a) Show that no flow can have a value exceeding 9.

   (b) Give an example of a flow f on N such that $\text{val}(f) = 9$.

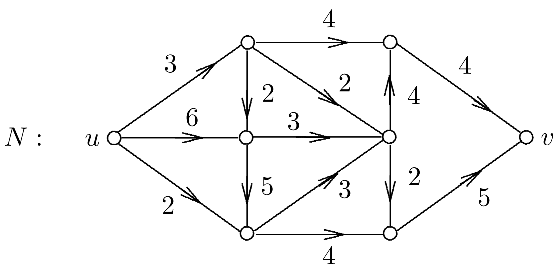


Figure 8.14: The network N in Exercise 2

3. For the network N shown in Figure 8.15 with source u and sink v, each arc has unlimited capacity. A flow f in the network is indicated by the labels on the arcs.

(a) Determine the missing flows a, b and c.

(b) Determine val(f).

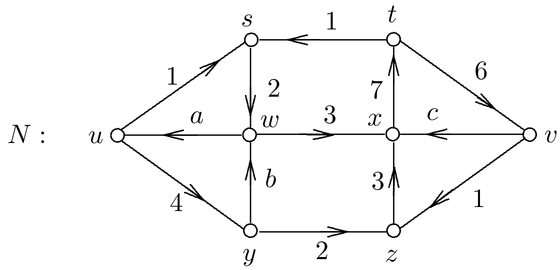


Figure 8.15: The network N in Exercise 3

4. Assume that the network N shown in Figure 8.16 has unlimited capacities. Give an example of a flow f on N where the flow along each arc is a positive integer and where the maximum of the flows along the arcs is as small as possible.

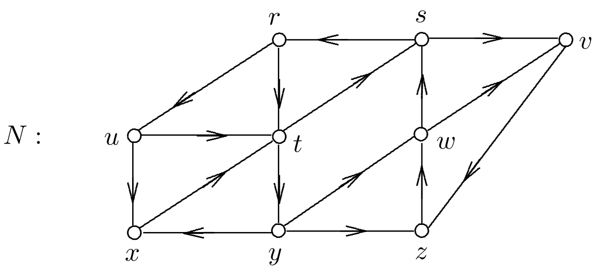


Figure 8.16: The network N in Exercise 4

5. Let N be a network with underlying digraph D, source u, sink v, capacity function c and let f be a flow on N. Suppose that $t \in V(D) - \{u, v\}$ such that $\text{id}\, t = 0$ and that N' is obtained from N by deleting t. Define a function f' on $E(D-t)$ by $f'(x, y) = f(x, y)$ for all arcs $(x, y) \in E(D-t)$. Show that f' is a flow on N' and that $\text{val}(f) = \text{val}(f')$. Show that the same conclusion holds if $\text{od}\, t = 0$.

6. A network N with source u and sink v is shown in Figure 8.17, where each arc is labeled with its capacity. Describe a flow on N where the flow along each arc is a positive integer.

(a) Determine the net flow out of each vertex.

(b) What is the maximum value of a flow f in the network N?

(c) Give an example of a minimum cut $[X, \overline{X}]$ and determine $f(X, \overline{X}) - f(\overline{X}, X)$.

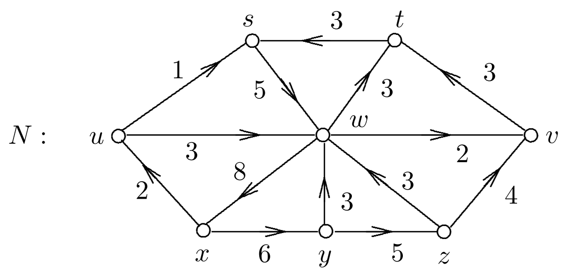


Figure 8.17: The network N in Exercises 6 and 10

7. Let u and v be two vertices of a digraph D and let A be a set of arcs of D such that every $u - v$ path in D contains at least one arc of A.

   (a) Show that there exists a set of arcs of the form $[X, \overline{X}]$ where $u \in X$ and $v \in \overline{X}$ and $[X, \overline{X}] \subseteq A$.

   (b) Show that $[X, \overline{X}]$ may be a proper subset of A.

8. Let N be a network with underlying digraph D, source u and sink v. For a set X of vertices of D with $u \in X$ and $v \in \overline{X}$ and a flow f defined on N, prove that $f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X)$.

9. (a) Prove Corollary 8.4: *If f is flow in a network N with capacity function c and $[X, \overline{X}]$ is a cut in N such that $f(a) = c(a)$ for all $a \in [X, \overline{X}]$ and $f(a) = 0$ for all $a \in [\overline{X}, X]$, then f is a maximum flow in N and $[X, \overline{X}]$ is a minimum cut.*

   (b) Show that the converse of Corollary 8.4 is true.

10. (a) Let N be the network N of Figure 8.17 in Exercise 6. Show that N has a flow f other than the zero flow with $\text{val}(f) = 0$.

    (b) Discuss a sufficient condition for a network to have a flow f other than the zero flow with $\text{val}(f) = 0$.

Section 8.2. The Max-Flow Min-Cut Theorem

11. Use Algorithm 8.8 to find a maximum flow f and a minimum cut K in the network N in Figure 8.18.

12. Use Algorithm 8.8 to find a maximum flow f and a minimum cut K in the network N in Figure 8.19.

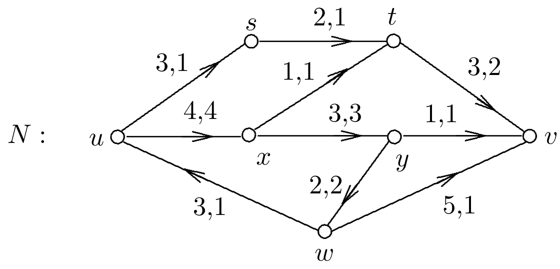


Figure 8.18: The network N in Exercise 11

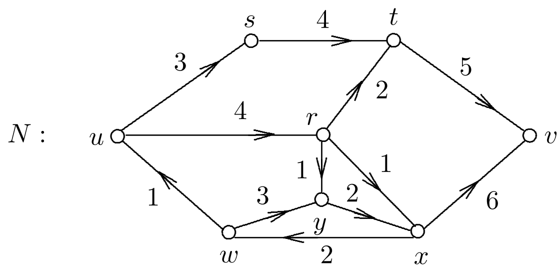


Figure 8.19: The network N in Exercise 12

13. Let N be a network with capacity function c and suppose that $[X, \overline{X}]$ is a minimum cut in N. Prove or disprove:

    (a) If f_1 and f_2 are flows in N that agree on $[X, \overline{X}]$ and $[\overline{X}, X]$, then f_1 and f_2 are maximum flows in N.

    (b) If f_1 and f_2 are maximum flows in N, then f_1 and f_2 agree on $[X, \overline{X}]$ and $[\overline{X}, X]$.

14. Define a **generalized network** N to be a digraph D with two distinguished vertices u and v called the *source* and *sink*, respectively, together with two nonnegative integer-valued functions c_1 and c_2 on $E(D)$. A *flow* in N is a real-valued function f on $E(D)$ that satisfies (8.3) (that is, $f^+(x) = f^-(x)$ for each intermediate vertex x of D) as well as

$$c_1(a) \leq f(a) \leq c_2(a) \text{ for every arc } a \text{ of } D. \tag{8.10}$$

    Give an example of a nontrivial generalized network N that has no (legal) flow.

Section 8.3. Menger Theorems for Digraphs

15. Use the proof of Theorem 8.11 to find the maximum number of internally disjoint $u - v$ paths in the digraph D shown in Figure 8.20.

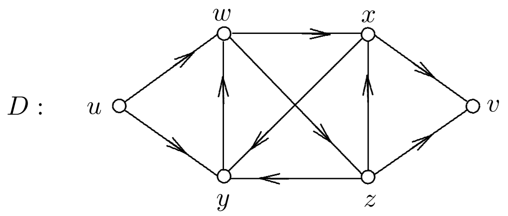


Figure 8.20: The digraph D in Exercise 15

16. Find the maximum number of internally disjoint $u-v$ paths in the digraph D shown in Figure 8.21.

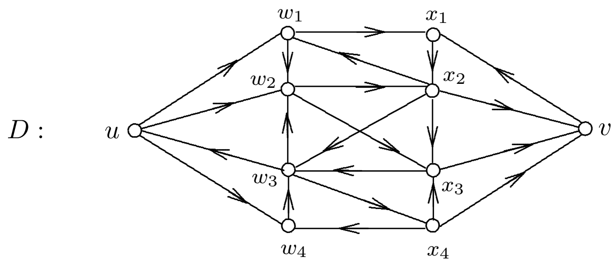


Figure 8.21: The digraph D in Exercise 16

17. Define a **multisource/multisink network** N to consist of a digraph D, two nonempty subsets S and T of vertices and a nonnegative real-valued function c defined on $E(D)$. Then D is called the *underlying digraph* of N, the vertices in S are called the *sources* of N, the vertices in T are called the *sinks* of N and c is called the *capacity function* of N. A *flow* in N is a real-valued function f on $E(D)$ satisfying (8.2) (that is, $0 \leq f(a) \leq c(a)$ for every arc a of D) and

$$f^+(x) = f^-(x) \text{ for each } x \in V(D) - S - T.$$

Using the natural definition for maximum flow and minimum cut in a multisource/multisink network, explain how the problem of determining a maximum flow can be reduced to the case of networks with a single source and sink.

Chapter 9

Automorphisms and Reconstruction

Determining whether a given graph possesses a property of interest depends on its structure. One way of studying the structure of graphs is by investigating their symmetries. A common method of doing this is by means of groups. A problem concerning how much of the structure of a graph can be determined from certain subgraphs of a graph is also described.

9.1 The Automorphism Group of a Graph

An **automorphism** of a graph G is an isomorphism from G to itself. Thus, an automorphism of G is a permutation of $V(G)$ that preserves adjacency (and nonadjacency). Of course, the identity function ϵ on $V(G)$ is an automorphism of G. The inverse of an automorphism of G is also an automorphism of G, as is the composition of two automorphisms of G. These observations lead us to the fact that the set of all automorphisms of a graph G form a group (under the operation of composition), called the **automorphism group** or simply the **group** of G, which is denoted by $\mathrm{Aut}(G)$.

The automorphism group of the graph G_1 of Figure 9.1 is cyclic of order 2, which we write as $\mathrm{Aut}(G_1) \cong \mathbb{Z}_2$ (the group of integers modulo 2). In addition to the identity permutation on $V(G_1)$, the group $\mathrm{Aut}(G_1)$ contains the *reflection* $\alpha = (uy)(vx)$, where α is expressed in terms of *permutation cycles*. That is,

$$\alpha(u) = y,\ \alpha(v) = x,\ \alpha(w) = w,\ \alpha(x) = v,\ \alpha(y) = u.$$

The graph G_2 of Figure 9.1 of order 6 has only the identity automorphism and so $\mathrm{Aut}(G_2) \cong \mathbb{Z}_1$. In fact, 6 is the smallest order of a nontrivial graph whose automorphism group consists only of the identity automorphism.

Every permutation of the vertex set of K_n is an automorphism and so $\mathrm{Aut}(K_n)$ is the symmetric group S_n of order $n!$. It is known that the symmetric

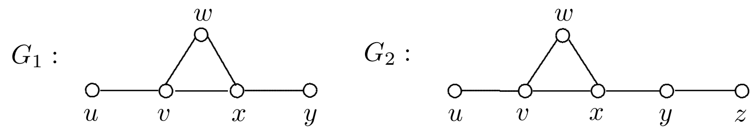
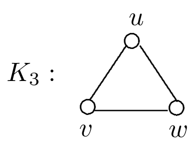


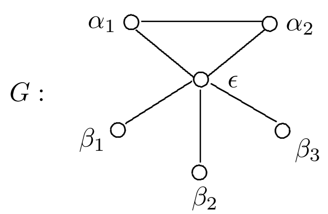
Figure 9.1: Graphs with automorphism groups of orders 2 and 1

group S_n is nonabelian only when $n \geq 3$. In fact, S_3 is the smallest nonabelian group. The complete graph K_3 is shown in Figure 9.2 along with the six elements of $\text{Aut}(K_3)$. Here, ϵ is the identity, $\alpha_1 = (u\ v\ w)$ and $\alpha_2 = (u\ w\ v)$ are rotations, and $\beta_1 = (v\ w)$, $\beta_2 = (u\ w)$ and $\beta_3 = (u\ v)$ are reflections. The group table of $\text{Aut}(K_3)$ is also shown in Figure 9.2. Associated with the group $\text{Aut}(K_3)$ is a graph G of order 6 where $V(G) = \text{Aut}(K_3)$ such that two vertices γ_1 and γ_2 of G are adjacent if and only if γ_1 and γ_2 commute (that is, $\gamma_1\gamma_2 = \gamma_2\gamma_1$) in $\text{Aut}(K_3)$. In $\overline{G}$ then, γ_1 and γ_2 are adjacent if and only if γ_1 and γ_2 do not commute. That is, G models commutativity in $\text{Aut}(K_3)$ and $\overline{G}$ models non-commutativity.



$\text{Aut}(K_3)$:

	ϵ	α_1	α_2	β_1	β_2	β_3
ϵ	ϵ	α_1	α_2	β_1	β_2	β_3
α_1	α_1	α_2	ϵ	β_3	β_1	β_2
α_2	α_2	ϵ	α_1	β_2	β_3	β_1
β_1	β_1	β_2	β_3	ϵ	α_1	α_2
β_2	β_2	β_3	β_1	α_2	ϵ	α_1
β_3	β_3	β_1	β_2	α_1	α_2	ϵ



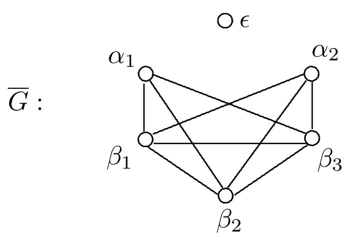
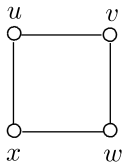


Figure 9.2: The graph K_3 and and its automorphism group

The automorphism group of C_n, $n \geq 3$, is the dihedral group D_n of order $2n$, consisting of n rotations and n reflections. The 4-cycle C_4 and the eight elements of its automorphism group are shown in Figure 9.3.

Next, we present a few basic facts concerning automorphism groups of graphs. We have already noted that every automorphism of a graph preserves both adjacency and nonadjacency. This leads to the following observation.



Automophisms:

ϵ, $\alpha_1 = (u\ v\ w\ x)$

$\alpha_2 = (u\ w)(v\ x)$, $\alpha_3 = (u\ x\ w\ v)$

$\phi_1 = (u\ w)$, $\phi_2 = (v\ x)$

$\phi_3 = (u\ v)(w\ x)$, $\phi_4 = (u\ x)(v\ w)$

Figure 9.3: A 4-cycle and the elements of its automorphism group

Theorem 9.1 *For every graph G,* $\mathrm{Aut}(G) \cong \mathrm{Aut}(\overline{G})$.

We mentioned previously that $\mathrm{Aut}(K_n) \cong S_n$ for every positive integer n. Certainly, if G is a graph of order n containing adjacent vertices as well as nonadjacent vertices, then $\mathrm{Aut}(G)$ is isomorphic to a proper subgroup of the symmetric group S_n. Combining this observation with Theorem 9.1 and Lagrange's Theorem on the order of a subgroup of a finite group (which states that the order of a subgroup of a finite group divides the order of the group), we arrive at the following.

Theorem 9.2 *The order $|\mathrm{Aut}(G)|$ of the automorphism group of a graph G of order n is a divisor of $n!$ and equals $n!$ if and only if $G = K_n$ or $G = \overline{K}_n$.*

Recall that two labelings of a graph G of order n from the same set of n labels are considered distinct if they do not produce the same edge set. With the aid of the automorphism group of a graph G of order n, it is possible to determine the number of distinct labelings of G.

Theorem 9.3 *The number of distinct labelings of a graph G of order n from a given set of n labels is $n!/|\mathrm{Aut}(G)|$.*

Proof. Let S be a set of n labels. Certainly, there exist $n!$ labelings of G using the elements of S without regard to which labelings are distinct. For a given labeling of G, each automorphism of G gives rise to an identical labeling of G; that is, each labeling of G from S determines $|\mathrm{Aut}(G)|$ identical labelings of G. Hence, there are $n!/|\mathrm{Aut}(G)|$ distinct labelings of G. ■

As an illustration of Theorem 9.3, consider the graph $G = P_3$ of Figure 9.4 and the set $S = \{1, 2, 3\}$. Since $\mathrm{Aut}(G) \cong \mathbb{Z}_2$, the number of distinct labelings of G is $3!/2 = 3$. The three distinct labelings of G from the set $\{1, 2, 3\}$ are shown in Figure 9.4.

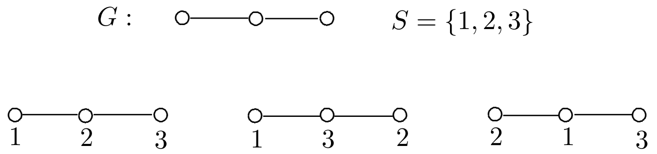


Figure 9.4: The distinct labelings of P_3

Similar Vertices

Let ϕ be an automorphism of a graph G. If a relation R is defined on the vertex set of G by $u\ R\ v$ if $\phi(u) = v$, then R has the properties of being reflexive, symmetric and transitive and therefore is an equivalence relation. The relation R then produces a partition of $V(G)$ into equivalence classes, referred to as the **orbits** of G. Two vertices belonging to the same orbit are called **similar vertices**. Therefore, two similar vertices have the same degree. The automorphism group of the graph G of Figure 9.5 is cyclic of order 3 and G has four orbits.

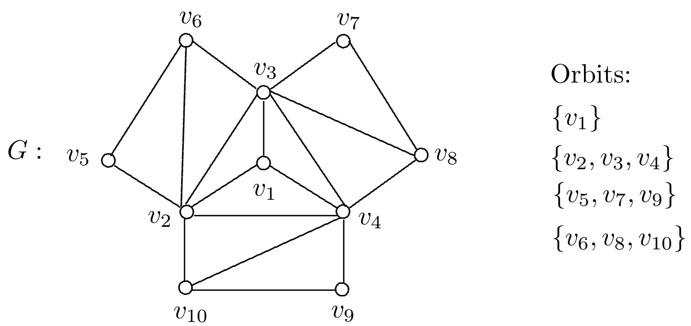


Figure 9.5: Orbits of a graph

In Chapter 2, we defined the eccentricity $e(v)$ of a vertex v in a connected graph G as the distance from v to a vertex farthest from v. For the graph G of Figure 9.6, we compute the eccentricity of each vertex of G. The eccentricity of each vertex is shown in Figure 9.6 as well. The orbits of this graph are $\{r_1, r_2\}$, $\{s\}$, $\{t_1, t_2\}$, $\{u\}$, $\{v\}$, $\{w_1, w_2\}$, $\{x\}$, $\{y\}$ and $\{z\}$. Since r_1 and r_2 are similar vertices, they necessarily have the same eccentricity, namely 7. The same can be said of t_1 and t_2, as well as of w_1 and w_2. This illustrates the fact that often, when evaluating the value of a certain parameter for each vertex of a graph, we can do this with fewer calculations by observing that some vertices are similar. Of course, it is possible for vertices that are not similar to have the same eccentricity, as is the case with u and x. For the graph G of Figure 9.6, $\mathrm{Aut}(G)$ is the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of three cyclic groups of order 2.

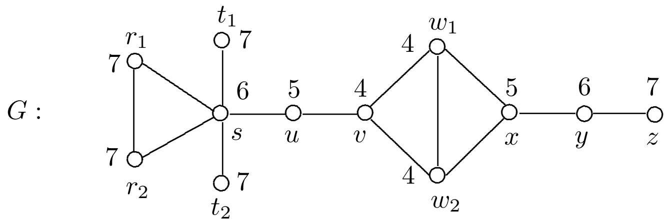


Figure 9.6: Similar vertices in a graph

Vertex-Transitive Graphs

A graph that contains a single orbit is called **vertex-transitive**. Thus, a graph G is vertex-transitive if and only if for every two vertices u and v of G, there exists an automorphism ϕ of G such that $\phi(u) = v$. Necessarily then, every vertex-transitive graph is regular. The graphs K_n $(n \geq 1)$, C_n $(n \geq 3)$ and $K_{r,r}$ $(r \geq 1)$ are all vertex-transitive. Also, the regular graphs $G_1 = C_5 \square K_2$ and $G_2 = K_{2,2,2}$, shown in Figure 9.7, are vertex-transitive. The regular graphs G_3 and G_4 in Figure 9.7 are not vertex-transitive, however (see Exercise 11).

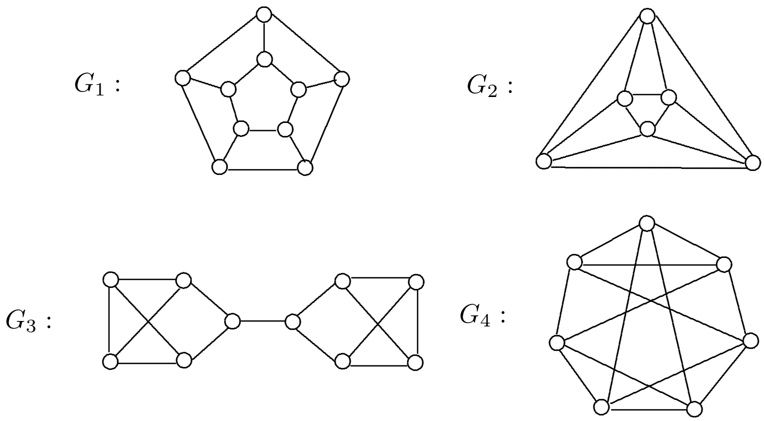


Figure 9.7: Vertex-transitive graphs and regular graphs that are not vertex-transitive

The two vertex-transitive graphs G_1 and G_2 of Figure 9.7 are Hamiltonian. In fact, there are many examples of vertex-transitive Hamiltonian graphs. Indeed, other than K_1 and K_2, there are only four known connected vertex-transitive graphs that are not Hamiltonian, namely the Petersen graph and the **Coxeter graph** (both shown in Figure 9.8) and the two graphs obtained from these by replacing each vertex by a triangle. These are called the **truncated Petersen graph** and the **truncated Coxeter graph**, also shown in

Figure 9.8. In fact, Gordon Royle made the following conjecture that these are the only connected vertex-transitive graphs that are not Hamiltonian.

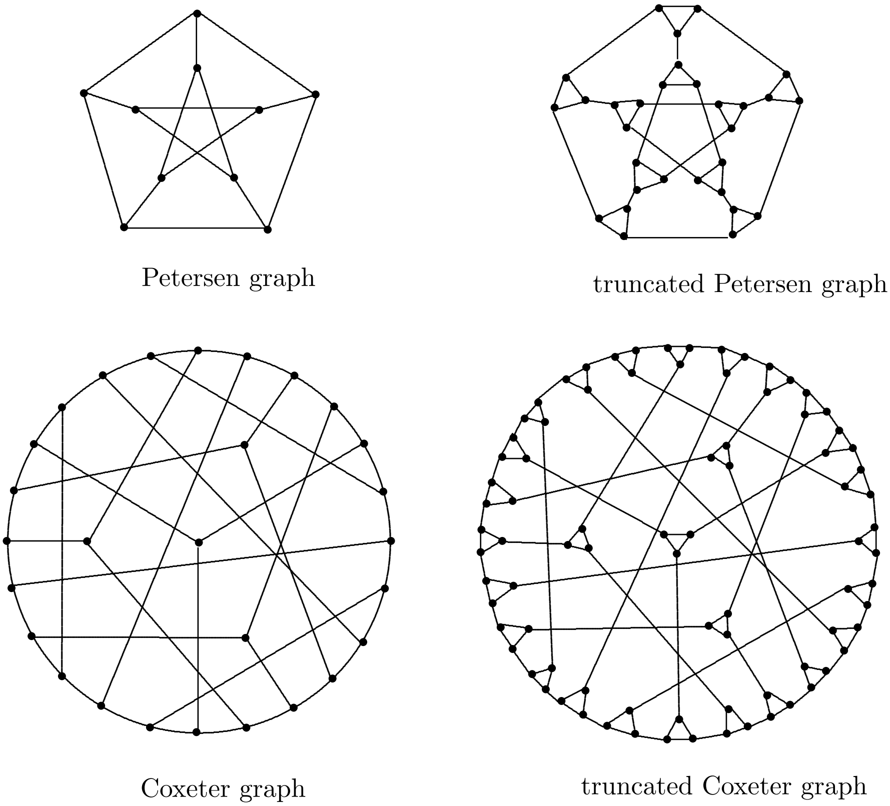


Figure 9.8: Connected vertex-transitive graphs that are not Hamiltonian

Royle's Conjecture *Every vertex-transitive graph G is Hamiltonian except when G is K_1, K_2, the Petersen graph, the truncated Petersen graph, the Coxeter graph or the truncated Coxeter graph.*

All four graphs in Figure 9.8 fail to contain a Hamiltonian cycle; yet all four contain a Hamiltonian path. Indeed, László Lovász made the following conjecture.

Lovász's Conjecture *Every connected vertex-transitive graph contains a Hamiltonian path.*

Every digraph also has an automorphism group. An **automorphism** of a digraph D is an isomorphism from D to itself; that is, an automorphism of D is a permutation α on $V(D)$ such that (u, v) is an arc of D if and only if $(\alpha u, \alpha v)$ is an arc of D. The set of all automorphisms under composition forms a group, called the **automorphism group** of D, which, as expected, is denoted by $\text{Aut}(D)$. While $\text{Aut}(G) \cong \mathbb{Z}_3$ for the graph G of Figure 9.5, there is an even simpler digraph D having $\text{Aut}(D) \cong \mathbb{Z}_3$. In particular, the digraphs D_1 and D_2 of Figure 9.9 have cyclic automorphism groups, namely $\text{Aut}(D_1) \cong \mathbb{Z}_3$ and $\text{Aut}(D_2) \cong \mathbb{Z}_5$.

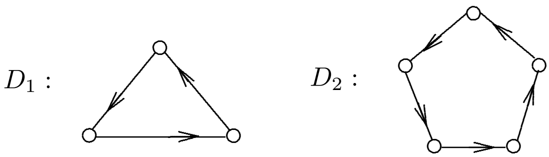


Figure 9.9: Digraphs with cyclic automorphism groups

9.2 Cayley Color Graphs

We have seen that with every graph and every digraph, there is a finite group that can be associated with it. We now consider the reverse question of associating a digraph and a graph with a given finite group.

A nontrivial group Γ is said to be **generated** by nonidentity elements $h_1, h_2, \ldots, h_k$ (and these elements are called **generators**) of Γ if every element of Γ can be expressed as a (finite) product of generators. Every nontrivial finite group has a finite **generating set** (often several such sets) since the set of all nonidentity elements of the group is always a generating set for Γ.

Let Γ be a given nontrivial finite group having the generating set $\Delta = \{h_1, h_2, \ldots, h_k\}$. We associate a digraph with Γ and Δ, commonly called the **Cayley color graph of Γ with respect to Δ** and denoted by $D_\Delta(\Gamma)$. The vertex set of $D_\Delta(\Gamma)$ is the set of group elements of Γ and so the order of $D_\Delta(\Gamma)$ is $|\Gamma|$. Each generator h_i is now regarded as a color. For $g_1, g_2 \in \Gamma$, there exists an arc (g_1, g_2) colored h_i in $D_\Delta(\Gamma)$ if $g_2 = g_1 h_i$. If h_i is a group element of order 2 (and is therefore self-inverse) and $g_2 = g_1 h_i$, then necessarily $g_1 = g_2 h_i$. When a Cayley color graph $D_\Delta(\Gamma)$ contains each of the arcs (g_1, g_2) and (g_2, g_1), both colored h_i, then, for simplicity, it is customary to represent this symmetric pair of arcs by the single edge $g_1 g_2$ colored h_i. As we have now seen, a Cayley color graph is actually a digraph, each arc of which is assigned a color, where a color is a generator in Δ. This digraph is named for the mathematician Arthur Cayley because of a famous theorem of his from group theory: *Every finite group is isomorphic to a group of permutations.*

Before proceeding further, let's illustrate the concepts just introduced. Let Γ denote the symmetric group S_3 of all permutations on the set $\{1, 2, 3\}$, and let $\Delta = \{a, b\}$, where $a = (123)$ and $b = (12)$. The six elements of Γ can then be expressed as a product of the elements a and b of Γ as follows:

the identity $e = b^2$, $a = a$, $b = b$, $(132) = a^2$, $(13) = ba$, $(23) = ab$.

The Cayley color graph $D_\Delta(\Gamma)$ in this case is shown in Figure 9.10.

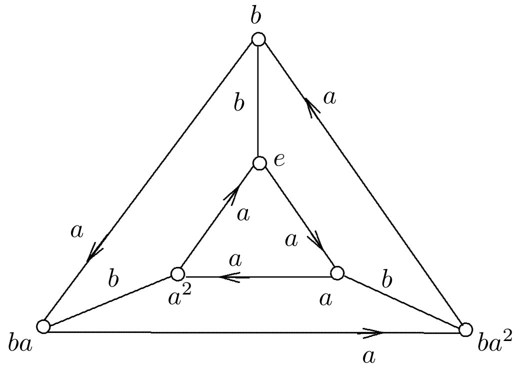


Figure 9.10: A Cayley color graph

If the generating set Δ of a given nontrivial finite group Γ with n elements is chosen to be the set of all nonidentity group elements, then for every two vertices g_1, g_2 of $D_\Delta(\Gamma)$, both (g_1, g_2) and (g_2, g_1) are arcs (although not necessarily of the same color) and $D_\Delta(\Gamma)$ is the complete symmetric digraph K_n^* of order n in this case.

Color-Preserving Automorphisms

Let Γ be a nontrivial finite group with generating set Δ. Every element α in the automorphic group $\mathrm{Aut}(D_\Delta(\Gamma))$ of the Cayley color graph $D_\Delta(\Gamma)$ has the property that if (g_1, g_2) is an arc of $D_\Delta(\Gamma)$, then $(\alpha(g_1), \alpha(g_2))$ is also an arc of $D_\Delta(\Gamma)$. If for every arc (g_1, g_2) of $D_\Delta(\Gamma)$, the arcs (g_1, g_2) and $(\alpha(g_1), \alpha(g_2))$ have the same color, then α is said to be **color-preserving**. For a given nontrivial finite group Γ with generating set Δ, the set of all color-preserving automorphisms of $D_\Delta(\Gamma)$ forms a subgroup of $\mathrm{Aut}(D_\Delta(\Gamma))$.

For example, let $\Gamma = \mathbb{Z}_4$ be a cyclic group of order 4 generated by the element a. Then we can write $\Gamma = \{g_1, g_2, g_3, g_4\}$ where $g_i = a^{i-1}$ for $i = 1, 2, 3, 4$. Thus, $g_1 = a^0 = e$ is the identity element of Γ. The set

$$\Delta = \{g_2, g_3, g_4\} = \{a, a^2, a^3\}$$

of nonidentity elements of Γ is a generating set for Γ. For this group Γ and this generating set Δ, the Cayley color graph $D_\Delta(\Gamma)$ is shown in Figure 9.11(a). Because there are symmetric pairs of arcs of the same color joining the pairs g_1, g_3 and g_2, g_4, this Cayley color graph is drawn as shown in Figure 9.11(b).

In this case, $D_\Delta(\Gamma)$ is the complete symmetric digraph K_4^* and so $\mathrm{Aut}(D_\Delta(\Gamma))$ is the symmetric group S_4. Since every permutation of the elements of Γ is an automorphism of $D_\Delta(\Gamma)$, the permutation $\alpha = (g_1\ g_2\ g_4)$ is an automorphism. Hence, α maps the arc (g_1, g_2) into the arc (g_2, g_4). Since the color of (g_1, g_2) is a and the color of (g_2, g_4) is α^2, it follows that α is not color-preserving. In this case, the subgroup of color-preserving automorphisms of $D_\Delta(\Gamma)$ consists of

$$\epsilon,\ (g_1\ g_2\ g_3\ g_4),\ \ (g_1\ g_3)(g_2\ g_4),\ \ (g_1\ g_4\ g_3\ g_2),$$

which is a cyclic group of order 4 and is, therefore, isomorphic to the group Γ. We are about to see that this is no coincidence.

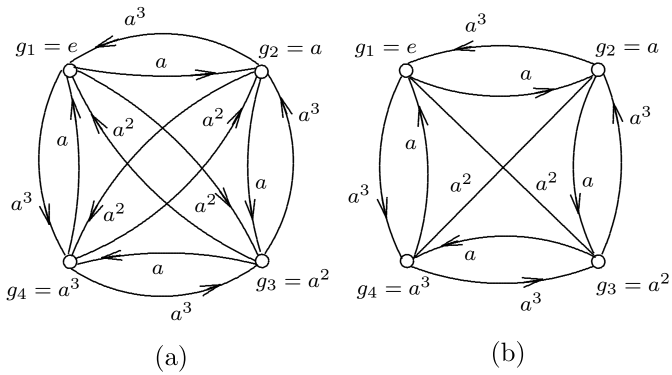


Figure 9.11: The Cayley color graph $D_\Delta(\Gamma)$ of a cyclic group Γ of order 4 with generating set $\Delta = \{a, a^2, a^3\}$

A useful characterization of color-preserving automorphisms is given in the next result (see Exercise 14).

Theorem 9.4 *Let Γ be a nontrivial finite group with generating set Δ and let α be a permutation of $V(D_\Delta(\Gamma))$. Then α is a color-preserving automorphism of $D_\Delta(\Gamma)$ if and only if $\alpha(gh) = (\alpha(g))h$ for every $g \in \Gamma$ and $h \in \Delta$.*

With the aid of Theorem 9.4, we can now prove the following result.

Theorem 9.5 *Let Γ be a nontrivial finite group with generating set Δ. Then the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to Γ.*

Proof. Let $\Gamma = \{g_1, g_2, \ldots, g_n\}$. For $i = 1, 2, \ldots, n$, define $\alpha_i : V(D_\Delta(\Gamma)) \to V(D_\Delta(\Gamma))$ by $\alpha_i(g_s) = g_i g_s$ for $1 \le s \le n$. Since Γ is a group, the mapping α_i is one-to-one and onto. Let $h \in \Delta$. Then for each i $(1 \le i \le n)$ and for each s $(1 \le s \le n)$,

$$\alpha_i(g_s h) = g_i(g_s h) = (g_i g_s)h = (\alpha_i(g_s))h.$$

Hence, by Theorem 9.4, α_i is a color-preserving automorphism of $D_\Delta(\Gamma)$.

Let α be an arbitrary color-preserving automorphism of $D_\Delta(\Gamma)$ and let g_1 be the identity of Γ. Suppose that $\alpha(g_1) = g_r$. Let $g_s \in \Gamma$. The element g_s of Γ can be expressed as a product of generators, say $g_s = h_1 h_2 \cdots h_t$, where $h_j \in \Delta$ and $1 \le j \le t$. Therefore,

$$\begin{aligned} \alpha(g_s) &= \alpha(g_1 h_1 h_2 \cdots h_t) = \alpha(g_1 h_1 h_2 \cdots h_{t-1}) h_t \\ &= \alpha(g_1 h_1 \cdots h_{t-2}) h_{t-1} h_t = \cdots = \alpha(g_1) h_1 h_2 \cdots h_t = g_r g_s. \end{aligned}$$

Thus, $\alpha = \alpha_r$.

We now show that the mapping ϕ defined by $\phi(g_i) = \alpha_i$ is an isomorphism from Γ to the group of color-preserving automorphisms of $D_\Delta(\Gamma)$. The mapping ϕ is already one-to-one and onto. It remains to show that ϕ is operation-preserving, namely that $\phi(g_i g_j) = \phi(g_i)\phi(g_j)$ for $g_i, g_j \in \Gamma$. Let $g_i g_j = g_k$. Then $\phi(g_i g_j) = \phi(g_k) = \alpha_k$ and $\phi(g_i)\phi(g_j) = \alpha_i \alpha_j$. Now,

$$\alpha_k(g_s) = g_k g_s = (g_i g_j) g_s = g_i (g_j g_s) = \alpha_i (g_j g_s) = \alpha_i(\alpha_j(g_s)) = (\alpha_i \alpha_j) g_s$$

and so $\alpha_k = \alpha_i \alpha_j$. ■

Frucht's Theorem

In the early years of the 20th century, Germany had been known for its mathematicians who excelled in group theory. One of these was Issai Schur. While at the University of Berlin, Schur supervised several doctoral students. One of Schur's students was Roberto Frucht (1906–1997), who received his Ph.D. in 1930 in the area of group theory.

Finding a job as a mathematician in Germany was very difficult during those days. Frucht wasn't even able to be hired as a high school teacher since Frucht was a Czechoslovakian citizen and German citizenship was required. So, Frucht moved to Italy to work for an insurance company. Frucht stayed there until 1938. During the time he was in Italy, he had essentially become mathematically inactive. However, one day in 1936, he received a catalogue advertising a book on graph theory, the first book written exclusively on graph theory. Frucht ordered this book and became an enthusiastic graph theorist the very day that the book arrived.

On page 5 of the first section (Basic Concepts) of the first chapter (Foundations) of this book, the author Dénes König [148, p. 5] wrote (translated into English):

> *When can a given abstract group be interpreted as the group of a graph and if this is the case, how can the corresponding graph be constructed?*

König's question on automorphism groups immediately caught Frucht's attention. After several months of unsuccessfully trying to solve the problem, he found a solution that seemed rather easy (after he had found it). Frucht [97] proved that *every* finite group has this property. We are now in a position to describe Frucht's proof of this result.

If Γ is the trivial group, then for $G = K_1$, wh have $\mathrm{Aut}(G) \cong \Gamma$. Therefore, we may assume that Γ is nontrivial and so $\Gamma = \{g_1, g_2, \ldots, g_n\}$ where $n \geq 2$. Let $\Delta = \{h_1, h_2, \ldots, h_t\}$, $1 \leq t \leq n$, be a generating set for Γ. We first construct the Cayley color graph $D_\Delta(\Gamma)$ of Γ with respect to Δ, which, recall, is actually a digraph. By Theorem 9.5, the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to Γ. We now transform the digraph $D_\Delta(\Gamma)$ into a graph G by the following technique. Let (g_i, g_j) be an arc of $D_\Delta(\Gamma)$ colored h_k. Delete this arc and replace it by the *graphical* path $g_i, u_{ij}, u'_{ij}, g_j$. At the vertex u_{ij} we construct a new path P_{ij} of length $2k-1$ and at the vertex u'_{ij} we construct a path P'_{ij} of length $2k$. This construction is now performed for every arc of $D_\Delta(\Gamma)$. This is illustrated in Figure 9.12 for $k = 1, 2, 3$.

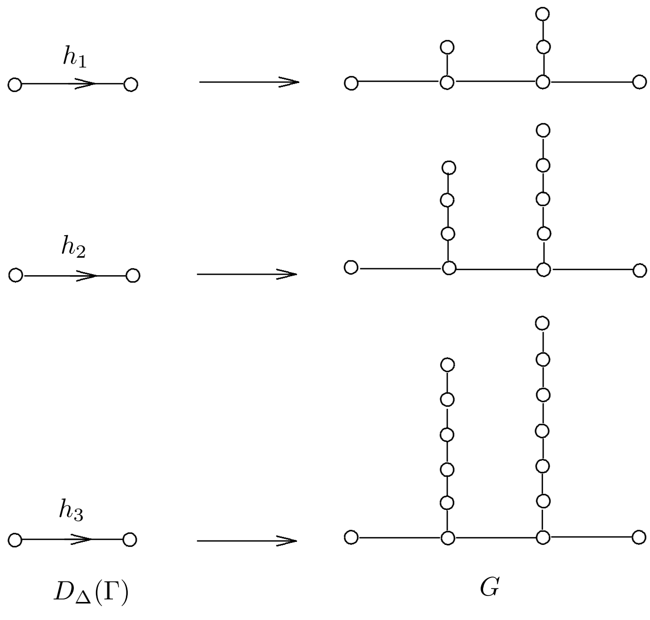


Figure 9.12: Constructing a graph G from a given group Γ

The addition of the paths P_{ij} and P'_{ij} in the formation of G is equivalent, in a sense, to assigning a direction and a color to each arc in the construction of $D_\Delta(\Gamma)$. Observing that every color-preserving automorphism of $D_\Delta(\Gamma)$ induces an automorphism of G, and conversely, results in a proof of Frucht's theorem.

Theorem 9.6 (**Frucht's Theorem**) *For every finite group* Γ*, there exists a graph* G *such that* $\mathrm{Aut}(G) \cong \Gamma$.

The condition of having a given group prescribed is not a particularly stringent one for graphs. For example, Herbert Izbicki [133] showed that for every finite group Γ and integer $r \geq 3$, there exists an r-regular graph G with $\text{Aut}(G) \cong \Gamma$.

Cayley Graphs

We have now seen that for every finite group Γ and generating set Δ, there is an associated digraph, namely the Cayley color graph $D_\Delta(\Gamma)$. The underlying graph of a Cayley color graph $D_\Delta(\Gamma)$ is called a **Cayley graph** and is denoted by $G_\Delta(\Gamma)$. Thus, a graph G is a Cayley graph if and only if there exists a finite group Γ and a generating set Δ for Γ such that $G \cong G_\Delta(\Gamma)$; that is, the vertices of G are the elements of Γ and two vertices g_1 and g_2 of G are adjacent if and only if either $g_1 = g_2 h$ or $g_2 = g_1 h$ for some $h \in \Delta$.

As observed earlier, the complete symmetric digraph K_n^* is a Cayley color graph; consequently, every complete graph is a Cayley graph. Since $K_2 \square K_3$ is the underlying graph of the Cayley color graph of Figure 9.10, $K_2 \square K_3$ is also a Cayley graph. Every Cayley graph is necessarily regular. Indeed, every Cayley graph is vertex-transitive. The converse is not true, however. The Petersen graph (Figure 9.8), for example, is vertex-transitive but it is not a Cayley graph.

9.3 The Reconstruction Problem

If ϕ is an automorphism of a nontrivial graph G and u is a vertex of G, then $G - u \cong G - \phi(u)$, that is, if u and v are similar vertices of a graph G, then $G - u \cong G - v$. The converse of this statement is not true, however. Indeed, the vertices u and v of the graph G of Figure 9.13 are not similar; yet $G - u \cong G - v$.

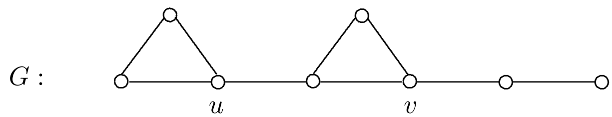


Figure 9.13: A graph with nonsimilar vertices whose vertex-deleted subgraphs are isomorphic

This comment brings up a question. Suppose that G and H are two graphs of the same order with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$, say. If it should occur that $G - v_1 \cong H - u_1$, then this does not imply that $G \cong H$. But what if, in addition to having $G - v_1 \cong H - u_1$, we also know that $G - v_2 \cong H - u_2$, $G - v_3 \cong H - u_3$ and so on, up to $G - v_n \cong H - u_n$? Can we then conclude that $G \cong H$?

This question is related to the problem of determining how much structure of a graph G can be recovered from its vertex-deleted subgraphs. This, in fact, brings us to a famous problem in graph theory.

Reconstructible Graphs

A graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$, is said to be **reconstructible** if for every graph H having $V(H) = \{u_1, u_2, \ldots, u_n\}$, $G - v_i \cong H - u_i$ for $i = 1, 2, \ldots, n$ implies that $G \cong H$. Hence, if G is a reconstructible graph, then the subgraphs $G - v$, $v \in V(G)$, uniquely determine G. It is believed by many but has never been verified that every graph of order at least 3 is reconstructible.

The Reconstruction Conjecture *Every graph of order at least* 3 *is reconstructible.*

This conjecture is believed to have been made in 1941 and is often attributed jointly to Paul J. Kelly and Stanislaw M. Ulam. In fact, Kelly was working on his Ph.D. at that time at the University of Wisconsin with a dissertation on this topic. At the same time, Ulam had a faculty position at the university. Kelly spent many years as a faculty member at the University of California at Santa Barbara. Ulam, born in Poland in 1909, became interested in astronomy, physics and mathematics while a teenager and taught himself calculus. He entered the Polytechnic Institute (now in Lvov, Ukraine) in 1927. One of his professors there was Kazimierz Kuratowski, whom we will encounter in Chapter 10. Ulam received his Ph.D. in 1933.

In 1940, Ulam acquired a faculty position at the University of Wisconsin. In 1943 John von Neumann asked to meet Ulam at a railroad station in Chicago. This resulted in Ulam going to the Los Alamos National Laboratory in New Mexico to work on the hydrogen bomb with the physicist Edward Teller. While at Los Alamos, Ulam developed the well-known Monte Carlo method for solving mathematical problems using a statistical sampling method with random numbers. Throughout his life, he made important contributions in many areas of mathematics.

The **Reconstruction Problem** is the problem of determining the truth or falsity of the Reconstruction Conjecture. The condition on the order in the Reconstruction Conjecture is necessary for if $G_1 = K_2$, then G_1 is not reconstructible. This is because if $G_2 = 2K_1$, then the subgraphs $G_1 - v$, $v \in V(G_1)$, and the subgraphs $G_2 - v$, $v \in V(G_2)$, are precisely the same. Thus, G_1 is not uniquely determined by its subgraphs $G_1 - v$, $v \in V(G_1)$. By the same reasoning, $G_2 = 2K_1$ is also not reconstructible. The Reconstruction Conjecture claims that K_2 and $2K_1$ are the only non-reconstructible graphs.

If there is a counterexample to the Reconstruction Conjecture, then it must have order at least 12, for, with the aid of computers, Brendan McKay [166] and Albert Nijenhuis [177] have shown that all graphs of order less than 12 (and greater than 2) are reconstructible. The graph G of Figure 9.14 is therefore reconstructible since its order is less than 12. Hence the graphs $G - v_i$ $(1 \leq i \leq 6)$ uniquely determine G. However, there exists a graph H with $V(H) = \{v_1, v_2, \ldots, v_6\}$ such that $G - v_i \cong H - v_i$ for $1 \leq i \leq 5$, but $G - v_6 \not\cong H - v_6$. Therefore, the graphs $G - v_i$ $(1 \leq i \leq 5)$ do not uniquely determine G. On the other hand, the graphs $G - v_i$ $(4 \leq i \leq 6)$ do uniquely determine G.

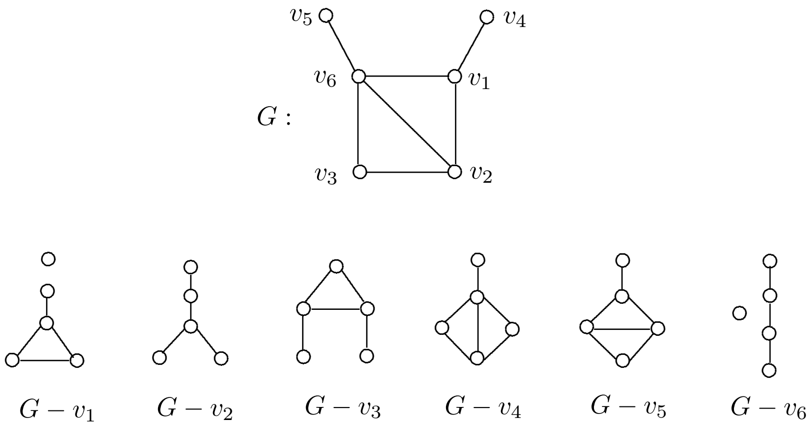


Figure 9.14: A reconstructible graph

Digraphs are not reconstructible, however. The vertex-deleted subdigraphs of the tournaments D_1 and D_2 of Figure 9.15 are the same; yet $D_1 \not\cong D_2$. Indeed, Paul K. Stockmeyer [227] showed that there are infinitely many pairs of counterexamples for digraphs (see [145, 228] as well).

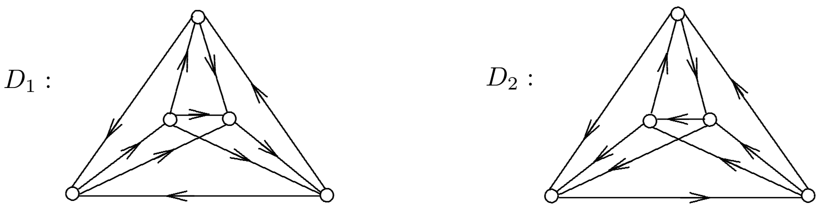


Figure 9.15: Two non-reconstructible digraphs

Recognizable Properties

There are several properties of a graph G that can be identified with the aid of the subgraphs $G - v$, $v \in V(G)$. We begin with the most elementary of these.

Theorem 9.7 *If G is a graph of order $n \geq 3$ and size m, then n and m as well as the degrees of the vertices of G are determined from the n subgraphs $G - v$, $v \in V(G)$.*

Proof. It is trivial to determine the number n, which is necessarily one greater than the order of any subgraph $G - v$. Also, n is equal to the number of subgraphs $G - v$. To determine m, label these subgraphs by G_i, $i = 1, 2, \ldots, n$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and suppose that $G_i = G - v_i$, where $v_i \in V(G)$. Let m_i denote the size of G_i. Consider an arbitrary edge e of G, say $e = v_j v_k$. Then e belongs to $n - 2$ of the subgraphs G_i, namely all except G_j and G_k. Since $\sum_{i=1}^{n} m_i$ counts each edge $n-2$ times, it follow that $\sum_{i=1}^{n} m_i = (n-2)m$ and so

$$m = \frac{\sum_{i=1}^{n} m_i}{n-2}. \tag{9.1}$$

The degrees of the vertices of G can be determined by simply noting that $\deg v_i = m - m_i$, $i = 1, 2, \ldots, n$. ■

We illustrate Theorem 9.7 with the six subgraphs $G-v$ shown in Figure 9.16 of some unspecified graph G. From these subgraphs we determine n, m and $\deg v_i$ for $i = 1, 2, \ldots, 6$. Clearly, $n = 6$. By calculating the integers m_i $(1 \leq i \leq 6)$, we find that $m = 9$. Thus, $\deg v_1 = \deg v_2 = 2$, $\deg v_3 = \deg v_4 = 3$ and $\deg v_5 = \deg v_6 = 4$.

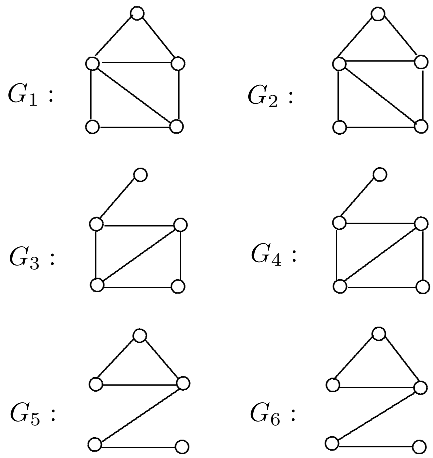


Figure 9.16: The subgraphs $G - v$ of a graph G

We say that a graphical parameter or graphical property is **recognizable** if, for each graph G of order at least 3, it is possible to determine the value of the parameter for G or whether G has the property from the subgraphs $G - v$, $v \in V(G)$. Theorem 9.7 thus states that for a graph of order at least 3, the order, the size and the degrees of its vertices are recognizable parameters. From Theorem 9.7, it also follows that the property of a graph being regular is recognizable; indeed, the degree of regularity is a recognizable parameter. For regular graphs, much more can be said.

Theorem 9.8 *Every regular graph of order at least* 3 *is reconstructible.*

Proof. As we have already mentioned, regularity and the degree of regularity are recognizable. Thus, without loss of generality, we may assume that G is an r-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$. It remains to show that G is uniquely determined by its subgraphs $G - v_i$, $i = 1, 2, \ldots, n$. Consider $G - v_1$, say. The graph $G - v_1$ then has order $n - 1$, where r vertices have degree $r - 1$ and the remaining $n - r - 1$ vertices have degree r. Adding the vertex v_1 to $G - v_1$ together with all those edges $v_1 v$ where $\deg_{G - v_1} v = r - 1$ produces the graph G. ∎

If G has order $n \geq 3$, then it can be determined whether G is connected from the n subgraphs $G - v$, $v \in V(G)$.

Theorem 9.9 *For graphs of order at least* 3*, connectedness is a recognizable property. In particular, if* G *is a graph with* $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, *then* G *is connected if and only if at least two of the subgraphs* $G - v_i$ *are connected.*

Proof. Let G be a connected graph. By Theorem 3.1, G contains at least two vertices that are not cut-vertices, implying the results.

Conversely, assume that there exist vertices $v_1, v_2 \in V(G)$ such that both $G - v_1$ and $G - v_2$ are connected. Thus, in $G - v_1$ and also in G, the vertex v_2 is connected to each vertex v_i for $i \geq 3$. Moreover, in $G - v_2$ (and thus in G), v_1 is connected to each vertex v_i for $i \geq 3$. Hence, every pair of vertices of G are connected and so G is connected. ∎

Since connectedness is a recognizable property, it is possible to determine from the subgraphs $G - v$, $v \in V(G)$, whether a graph G of order at least 3 is disconnected. We now show that disconnected graphs are reconstructible. There have been several proofs of this fact. The proof given here is due to Bennet Manvel [162].

Theorem 9.10 *Disconnected graphs of order at least* 3 *are reconstructible.*

Proof. We have already noted that disconnectedness in graphs of order at least 3 is a recognizable property. Thus, we assume without loss of generality that G is a disconnected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$. Further, let $G_i = G - v_i$ for $i = 1, 2, \ldots, n$. From Theorem 9.7, the degrees of the vertices v_i, $i = 1, 2, \ldots, n$, can be determined from the graphs $G - v_i$. Hence, if G contains an isolated vertex, then G is reconstructible. Assume then that G has no isolated vertices.

Since every component of G is nontrivial, it follows that $k(G_i) \geq k(G)$ for $i = 1, 2, \ldots, n$ and that $k(G_j) = k(G)$ for some integer j satisfying $1 \leq j \leq$

n. Hence, the number of components of G is $\min\{k(G_i) : \ i = 1, 2, \ldots, n\}$. Suppose that F is a component of G of maximum order. Necessarily, F is a component of maximum order among the components of the graphs G_i, that is, F is recognizable. Delete a vertex that is not a cut-vertex from F, obtaining F'.

Assume that there are r (≥ 1) components of G isomorphic to F. The number r is recognizable, as we shall see. Let

$$S = \{G_i : \ k(G_i) = k(G)\}$$

and let S' be the subset of S consisting of all those graphs G_i having a minimum number ℓ of components isomorphic to F. (Observe that if $r = 1$, then there exist graphs G_i in S containing no components isomorphic to F, that is, $\ell = 0$.) In general, then, $r = \ell + 1$. Next let S'' denote the set of those graphs G_i in S' having a maximum number of components isomorphic to F'.

Assume that $G_1, G_2, \ldots, G_t$ $(t \geq 1)$ are the elements of S''. Each graph G_i in S'' has $k(G)$ components. Since each graph G_i $(1 \leq i \leq t)$ has a minimum number of components isomorphic to F, each vertex v_i $(1 \leq i \leq t)$ belongs to a component F_i of G isomorphic to F, where the components F_i of G $(1 \leq i \leq t)$ are not necessarily distinct. Further, since each graph G_i $(1 \leq i \leq t)$ has a maximum number of components isomorphic to F', it follows that $F_i - v_i = F'$ for each $i = 1, 2, \ldots, t$. Hence, every two of the graphs $G_1, G_2, \ldots, G_t$ are isomorphic and G can be produced from G_1, say, by replacing a component of G_1 isomorphic to F' by a component isomorphic to F. ■

The Reconstruction Problem is often described in a somewhat playful manner as a problem involving a deck of cards. Suppose that we begin with two people, say you and a friend, and a graph G. However, only your friend knows what the graph G is. For each vertex v of G, the graph $G - v$ (unlabeled!) is drawn on a card. The set of all such cards is then referred to as a **deck**. This deck is then given to you. Your job is to find all graphs H whose set of vertex-deleted subgraphs are those appearing on the deck of cards. Any graph H satisfying this information is referred to as a **solution** to the deck. Of course, we know that there is at least one solution to the deck, namely the graph G. Furthermore, if the Reconstruction Conjecture is true, then the graph G is the *only* solution to the deck. However, knowing that such a deck has a solution (or maybe more than one solution) doesn't mean that finding a solution is easy.

To illustrate this idea, consider the deck of cards shown in Figure 9.17. The problem is to find all solutions of this deck. Each deck of cards uniquely determines the order n, size m and the degrees of the vertices of any solution of the deck.

Let m_i denote the size of $G_i = G - v_i$ $(1 \leq i \leq 6)$. Since there are six cards in the deck, $n = 6$. Furthermore, by (9.1),

$$m = \frac{\sum_{i=1}^{n} m_i}{n-2} = \frac{36}{4} = 9.$$

Therefore, the degrees $m - m_i$ $(1 \leq i \leq 6)$ of the vertices of a solution G of the deck are $2, 2, 3, 3, 4, 4$. Observe, for example, that the subgraph G_6 has size

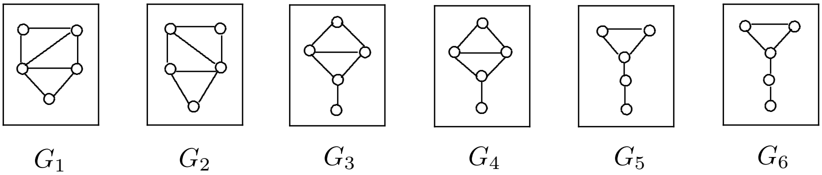


Figure 9.17: The subgraphs $G - v$ for one or more graphs G

5 and degree sequence $3, 2, 2, 2, 1$. Hence any solution G can be obtained by adding a vertex v_6 of degree 4 to G_6. Since each solution to the deck must have degree sequence $4, 4, 3, 3, 2, 2$, the vertex v_6 must be joined to all vertices of G_6 except one vertex of degree 2. If v_6 is joined to the two adjacent vertices of degree 2 in G_6, then any solution of the deck must contain K_4 as a subgraph. However, none of the graphs G_i $(1 \le i \le 6)$ contains K_4. Therefore, v_6 must be adjacent to only one of these vertices of degree 2 in G_6. Because these two vertices are similar, we conclude that the deck of cards shown in Figure 9.17 has the unique solution shown in Figure 9.18.

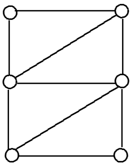


Figure 9.18: The solution of the deck of cards in Figure 9.17

It can be shown that (connected) graphs of order at least 3 whose complements are disconnected are reconstructible (Exercise 23). However, it remains to be shown that *all* connected graphs of order at least 3 are reconstructible.

Exercises for Chapter 9

Section 9.1. The Automorphism Group of a Graph

1. For the graphs G_1 and G_2 in Figure 9.19, describe the automorphisms of G_1 and of G_2 in terms of permutation cycles.

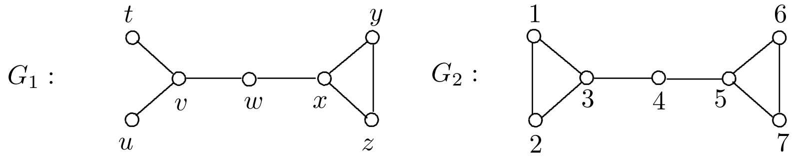


Figure 9.19: The graphs G_1 and of G_2 in Exercise 1

2. Figure 9.3 shows a 4-cycle C_4 and the elements of $\text{Aut}(C_4)$.
   (a) Construct the group table for $\text{Aut}(C_4)$.
   (b) Draw the graph G where $V(G) = \text{Aut}(C_4)$ such that two vertices γ_1 and γ_2 of G are adjacent if and only if γ_1 and γ_2 commute in $\text{Aut}(C_4)$.
   (c) Draw the graph $\overline{G}$ for the graph G in (b).

3. Does there exist a graph H of order 4 such that the graph G with $V(G) = \text{Aut}(H)$ where $\gamma_1\gamma_2 \in E(G)$ if and only if γ_1 and γ_2 commute in $\text{Aut}(H)$ has order 4?

4. Describe the elements of $\text{Aut}(C_5)$.

5. Find a nonseparable graph G whose automorphism group is isomorphic to the cyclic group of order 4.

6. Determine the number of distinct labelings of $K_{r,r}$.

7. For which pairs k, n of positive integers with $k \leq n$ does there exist a graph G of order n having k orbits?

8. For which pairs k, n of positive integers does there exist a graph G of order n and a vertex v of G such that there are exactly k vertices similar to v?

9. Show for every even integer $n \geq 2$ that there exists a graph G of order n such that G has $n/2$ pairs of similar vertices.

10. Let G be a graph that is not vertex-transitive and let H be the graph where $V(H) = V(G)$ and $xy \in E(H)$ if and only if x and y are similar vertices of G. Describe the complement $\overline{H}$ of H.

11. Show that the graphs G_3 and G_4 in Figure 9.7 are not vertex-transitive.

12. Describe the automorphism groups of the digraphs in Figure 9.20.

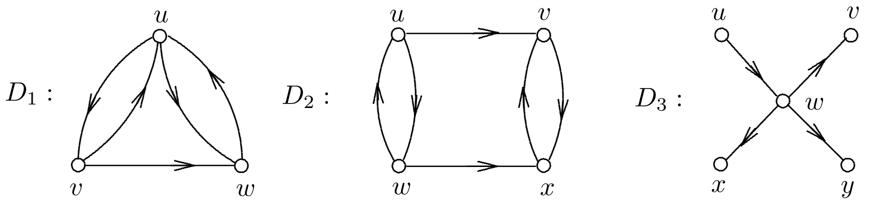


Figure 9.20: The digraphs in Exercise 12

Section 9.2. Cayley Color Graphs

13. Construct the Cayley color graph of the cyclic group of order 4 when the generating set Δ has (a) one element and (b) three elements.

14. Prove Theorem 9.4: *Let* Γ *be a nontrivial finite group with generating set* Δ *and let* α *be a permutation of* $V(D_\Delta(\Gamma))$. *Then* α *is a color-preserving automorphism of* $D_\Delta(\Gamma)$ *if and only if* $\alpha(gh) = (\alpha(g))h$ *for every* $g \in \Gamma$ *and* $h \in \Delta$.

15. Determine the group of color-preserving automorphisms for the Cayley color graph $D_\Delta(\Gamma)$ of Figure 9.21.

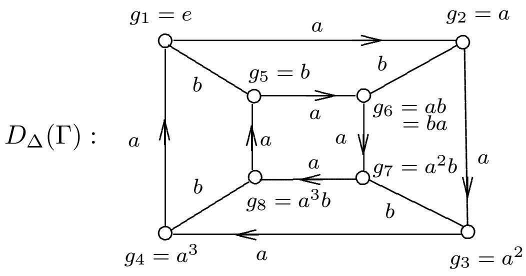


Figure 9.21: The Cayley color graph $D_\Delta(\Gamma)$ in Exercise 15

16. For a given finite group Γ, determine an infinite number of mutually nonisomorphic graphs whose groups are isomorphic to Γ.

17. Show that every n-cycle is a Cayley graph.

18. Show that the cube Q_3 is a Cayley graph.

Section 9.3. The Reconstruction Problem

19. Reconstruct the graph G whose subgraphs $G - v, v \in V(G)$ are given in Figure 9.16.

20. Reconstruct the graph G whose subgraphs $G - v, v \in V(G)$ are given in Figure 9.22.

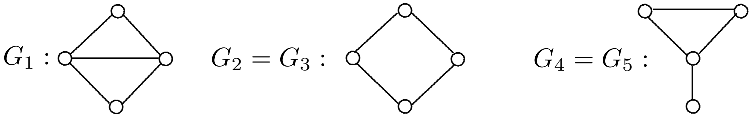


Figure 9.22: The subgraphs $G - v$ of the graph G in Exercise 20

21. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_7\}$ such that $G - v_i = K_{2,4}$ for $i = 1, 2, 3$ and $G - v_i = K_{3,3}$ for $i = 4, 5, 6, 7$. Show that G is reconstructible.

22. Show that the tournaments of Figure 9.15 are not isomorphic.

23. (a) Prove that if G is reconstructible, then $\overline{G}$ is reconstructible.
    (b) Prove that every graph of order n (≥ 3) whose complement is disconnected is reconstructible.

24. Prove that the property of being bipartite for a graph is recognizable.

25. Reconstruct the graph G whose subgraphs $G - v, v \in V(G)$ are given in Figure 9.23.

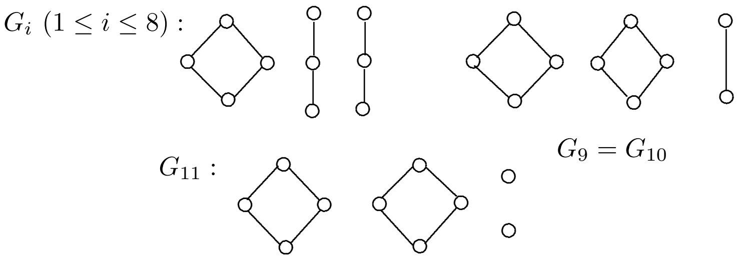


Figure 9.23: The subgraphs $G - v$ of the graph G in Exercise 25

26. Show that no graph of order at least 3 can be reconstructed from exactly two of the subgraphs $G - v, v \in V(G)$.

27. In the process of solving the deck of cards in Figure 9.17, we learned that the order of any solution is $n = 6$, the size is $m = 9$ and the degree sequence of a solution is $4, 4, 3, 3, 2, 2$. Was this enough information to obtain the solution shown in Figure 9.18?

28. Find all solutions to the deck of cards shown in Figure 9.24.

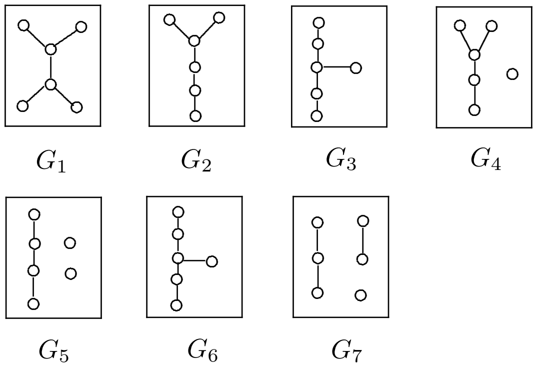


Figure 9.24: The deck of cards in Exercise 28

29. Find all solutions to the deck of cards shown in Figure 9.25.

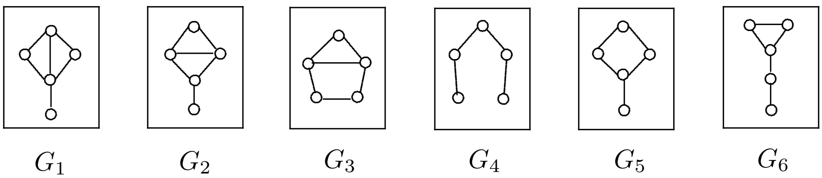


Figure 9.25: The deck of cards in Exercise 29

30. For the deck of cards shown in Figure 9.26, determine, without finding a solution, whether any solution G is Hamiltonian.

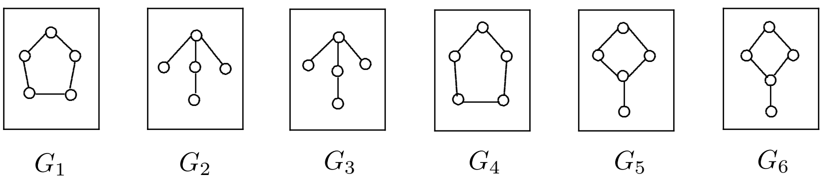


Figure 9.26: The deck of cards in Exercise 30

Chapter 10

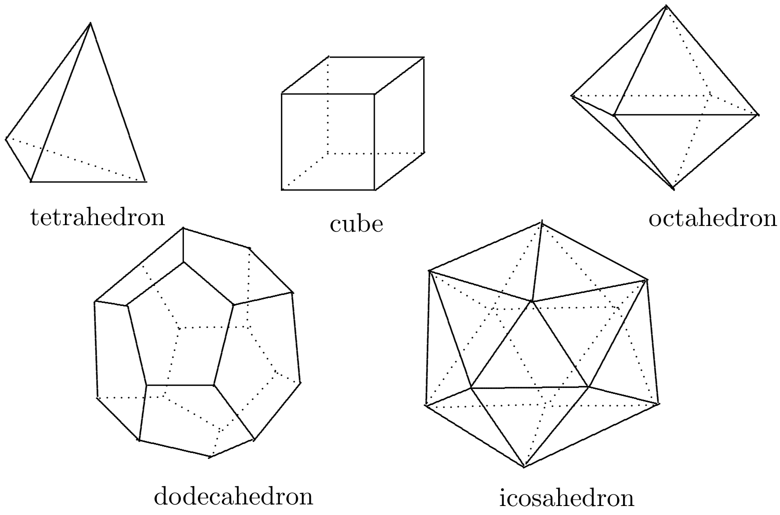
Planar Graphs

Of the methods we have described to represent a graph G, probably the most common and the most useful for determining whether G possesses particular properties of interest is that of presenting G by means of a drawing. There are occasions when the edges in a diagram may cross and other occasions when the edges in a diagram do not cross. If some pairs of edges in a diagram of a graph cross, then it may be that there are other drawings of this same graph when no edges cross – or perhaps this is not possible. It is the class of graphs that can be drawn in the plane without their edges crossing that will be of interest to us in this chapter. We will see that a result of Euler plays a central role in this study.

10.1 The Euler Identity

A **polyhedron** is a 3-dimensional object whose boundary consists of polygonal plane surfaces. These surfaces are typically called the **faces** of the polyhedron. The boundary of a face consists of the edges and vertices of the polygon. In this setting, the total number of faces in the polyhedron is commonly denoted by F, the total number of edges in the polyhedron by E and the total number of vertices by V. The best known polyhedra are the so-called **Platonic solids**: the **tetrahedron**, **cube** (**hexahedron**), **octahedron**, **dodecahedron** and **icosahedron**. These are shown in Figure 10.1, together with the values of V, E and F for these polyhedra.

During the 18th century, many letters (over 160) were exchanged between Leonhard Euler (who, as we saw in Chapter 5, essentially introduced graph theory to the world when he solved and then generalized the Königsberg Bridge Problem) and Christian Goldbach (well known for stating the conjecture that every even integer greater than 2 can be expressed as the sum of two primes). In a letter that Euler wrote to Goldbach on 14 November 1750, he stated a relationship that existed among the numbers V, E and F for a polyhedron and which would later become known as:



Platonic solid	V	E	F
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
dodecahedron	20	30	12
icosahedron	12	30	20

Figure 10.1: The five Platonic solids

The Euler Polyhedral Formula *If a polyhedron has V vertices, E edges and F faces, then*

$$V - E + F = 2.$$

That Euler was evidently the first mathematician to observe this formula (which is actually an identity rather than a formula) may be somewhat surprising in light of the fact that Archimedes and René Descartes both studied polyhedra long before Euler. A possible explanation as to why others had overlooked this identity might be due to the fact that geometry had primarily been a study of distances.

The Euler Polyhedral Formula appeared in print two years later (in 1752) in two papers by Euler [87, 88]. In the first of these two papers, Euler stated that he had been unable to prove the formula. However, in the second paper, he presented a proof by dissecting polyhedra into tetrahedra. Although his proof was clever, he nonetheless made some missteps. The first generally accepted proof was obtained by the French mathematician Adrien-Marie Legendre [153].

Each polyhedron can be converted into a map and then into a graph by inserting a vertex at each meeting point of the map (which is actually a vertex of the polyhedron). This is illustrated in Figure 10.2 for the cube.

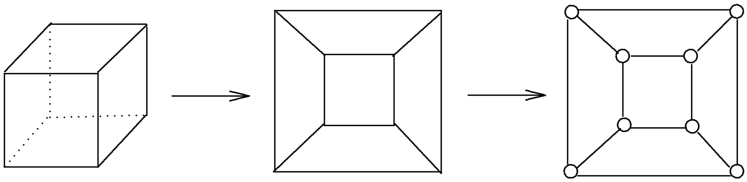


Figure 10.2: From a polyhedron to a map to a graph

The graphs obtained from the five Platonic solids are shown in Figure 10.3. These graphs have a property in which we will be especially interested: No two edges cross (intersect each other) in the graph.

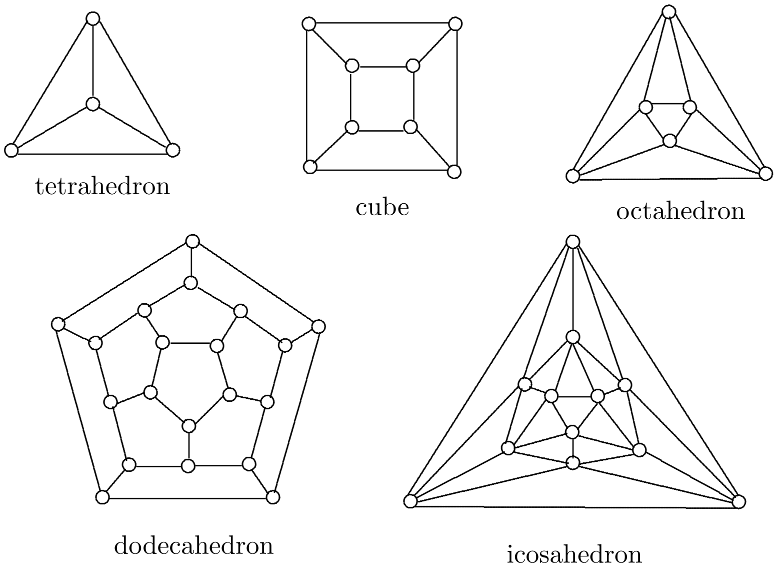


Figure 10.3: The graphs of the five Platonic solids

Planar Graphs

A graph G is called a **planar graph** if G can be drawn in the plane without any two of its edges crossing. Such a drawing is also called an **embedding of** G **in the plane**. In this case, the embedding is a **planar embedding**. A graph G that is already drawn in the plane in this manner is a **plane graph**. Certainly then, every plane graph is planar and every planar graph can be drawn as a plane graph. In particular, all five graphs of the Platonic solids are planar.

When those points in the plane that correspond to the vertices and edges of a plane graph G are removed from the plane, the resulting connected pieces of the plane are the **regions** of G. One of the regions is unbounded and is called the **exterior region** of G. For every planar embedding of a planar graph G and every region R in this planar embedding, there exists a planar embedding of G in which R is the exterior region. Consequently, for every edge e (or vertex v) of G, there is a planar embedding of G for which e (or v) lies on the boundary of the exterior region. When considering a plane graph G of a polyhedron, the faces of the polyhedron become the regions of G, one of which is the exterior region of G. On the other hand, a planar graph need not be the graph of any polyhedron. The plane graph H of Figure 10.4 is not the graph of any polyhedron. This graph has five regions, denoted by R_1, R_2, R_3, R_4 and R_5, where R_5 is the exterior region.

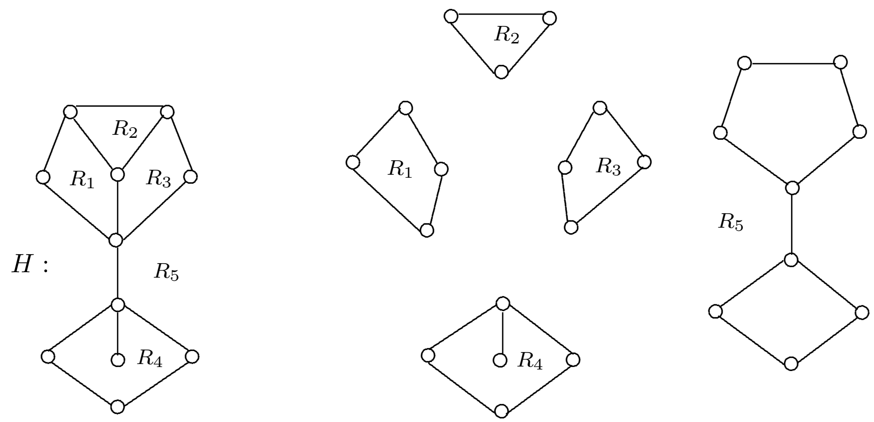


Figure 10.4: The boundaries of the regions of a plane graph

For a region R of a plane graph G, the vertices and edges incident with R form a subgraph of G called the **boundary** of R. Every edge of G that lies on a cycle belongs to the boundary of two regions of G, while every bridge of G belongs to the boundary of a single region. In Figure 10.4, the boundaries of the five regions of H are shown as well.

The five graphs G_1, G_2, G_3, G_4 and G_5 shown in Figure 10.5 are all planar, although G_1 and G_3 are not plane graphs. The graph G_1 can be drawn as G_2, while G_3 can be drawn as G_4. In fact, G_1 (and G_2) is the graph of the tetrahedron. For each graph, its order n, its size m and the number r of regions are shown as well.

Observe that $n - m + r = 2$ for each graph of Figure 10.5. Of course, this is not surprising for G_2 since this is the graph of a polyhedron (the tetrahedron) and $n = V$, $m = E$ and $r = F$. In fact, this identity holds for every connected plane graph.

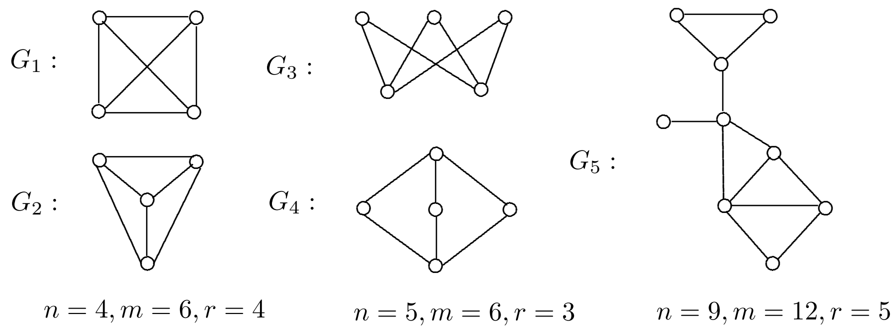


Figure 10.5: Planar graphs

Theorem 10.1 (**The Euler Identity**) *For every connected plane graph of order n, size m and having r regions,*

$$n - m + r = 2.$$

Proof. We proceed by induction on the size m of a connected plane graph. There is only one connected graph of size 0, namely K_1. In this case, $n = 1$, $m = 0$ and $r = 1$. Since $n - m + r = 2$, the base case of the induction holds.

Assume for a positive integer m that if H is a connected plane graph of order n' and size m', where $m' < m$ such that there are r' regions, then $n' - m' + r' = 2$. Let G be a connected plane graph of order n and size m with r regions. We consider two cases.

Case 1. *G is a tree.* In this case, $m = n - 1$ and $r = 1$. Thus $n - m + r = n - (n - 1) + 1 = 2$, producing the desired result.

Case 2. *G is not a tree.* Since G is connected and is not a tree, it follows by Theorem 3.10 that G contains an edge e that is not a bridge. In G, the edge e is on the boundaries of two regions. So in $G - e$ these two regions merge into a single region. Since $G - e$ has order n, size $m - 1$ and $r - 1$ regions and $m - 1 < m$, it follows by the induction hypothesis that $n - (m - 1) + (r - 1) = 2$ and so $n - m + r = 2$. ■

The Euler Polyhedron Formula is therefore a special case of Theorem 10.1. While Euler struggled with the verification of $V - E + F = 2$, he did not have the luxury of a developed graph theory at his disposal.

From Theorem 10.1, it follows that every two planar embeddings of a connected planar graph result in plane graphs having the same number of regions; thus one can speak of the number of regions of a connected planar graph. For planar graphs in general, we have the following result. (See Exercise 2.) Recall that $k(G)$ denotes the number of components of a graph G.

Corollary 10.2 *If G is a plane graph with n vertices, m edges and r regions, then*

$$n - m + r = 1 + k(G).$$

An Upper Bound for the Size of Planar Graphs

If G is a connected plane graph of order 4 or more, then the boundary of every region of G must contain at least three edges. This observation is helpful in showing that with respect to the order of a planar graph, its size cannot be too great.

Theorem 10.3 *If G is a planar graph of order $n \geq 3$ and size m, then*

$$m \leq 3n - 6.$$

Proof. Since the size of every graph of order 3 cannot exceed 3, the inequality holds for $n = 3$. So we may assume that $n \geq 4$. Futhermore, we may assume that the planar graphs under consideration are connected, for otherwise edges can be added to produce a connected graph. Suppose that G is a connected planar graph of order $n \geq 4$ and size m. Let there be given a planar embedding of G, resulting in r regions. By the Euler Identity, $n - m + r = 2$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and suppose that we denote the number of edges on the boundary of R_i $(1 \leq i \leq r)$ by m_i. Then $m_i \geq 3$ for $1 \leq i \leq r$. Since each edge of G is on the boundary of at most two regions of G, it follows that

$$3r \leq \sum_{i=1}^{r} m_i \leq 2m.$$

Hence,

$$6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m$$

and so $m \leq 3n - 6$. ■

By expressing Theorem 10.3 in its contrapositive form, we obtain the following reformulation of the theorem.

Theorem 10.4 *If G is a graph of order $n \geq 5$ and size m such that $m > 3n-6$, then G is nonplanar.*

There is an immediate consequence of this theorem.

Corollary 10.5 *Every planar graph contains a vertex of degree 5 or less.*

Proof. The result is obvious for planar graphs of order 6 or less. Let G be a graph of order n and size m all of whose vertices have degree 6 or more. Then $n \geq 7$ and

$$2m = \sum_{v \in V(G)} \deg v \geq 6n$$

and so $m \geq 3n$. By Theorem 10.4, G is nonplanar. ■

The Five Regular Polyhedra

We saw, by the Euler Polyhedron Formula, that if V, E and F are the number of vertices, edges and faces of a polyhedron, then

$$V - E + F = 2.$$

When dealing with a polyhedron P (as well as the graph of the polyhedron P), it is customary to represent the number of vertices of degree k by V_k and number of faces bounded by a k-cycle (k-sided faces) by F_k. It follows then that

$$2E = \sum_{k \geq 3} kV_k = \sum_{k \geq 3} kF_k. \tag{10.1}$$

By Corollary 10.5, every polyhedron has at least one vertex of degree 3, 4 or 5. As an analogue to this result, we have the following.

Theorem 10.6 *At least one face of every polyhedron is bounded by a k-cycle for some k where $k \in \{3, 4, 5\}$.*

Proof. Assume, to the contrary, that the statement is false. Then $F_3 = F_4 = F_5 = 0$. By equation (10.1),

$$2E = \sum_{k \geq 6} kF_k \geq \sum_{k \geq 6} 6F_k = 6 \sum_{k \geq 6} F_k = 6F.$$

Hence, $E \geq 3F$. Also,

$$2E = \sum_{k \geq 3} kV_k \geq \sum_{k \geq 3} 3V_k = 3 \sum_{k \geq 3} V_k = 3V.$$

By Theorem 10.1, $V - E + F = 2$ and so $3V - 3E + 3F = 6$. Hence, $6 = 3V - 3E + 3F \leq 2E - 3E + E = 0$, which is a contradiction. ■

Of the five Platonic solids shown in Figure 10.1, three are cubic polyhedra (the tetrahedron, cube and dodecahedron) as each vertex in these polyhedra has degree 3. The dodecahedron has twelve faces, all 5-sided. If the icosahedron (which has twelve vertices of degree 5 and twenty 3-sided faces) is “truncated”, replacing each vertex by a pentagon, another polyhedron results – namely a cubic polyhedron containing twelve 5-sided faces and twenty 6-sided faces. This is precisely what occurs with a soccer ball, which has 32 faces, 12 of which are pentagonal faces and 20 are hexagonal faces. Indeed, every cubic polyhedron containing only 5-sided and 6-sided faces must contain exactly twelve 5-sided faces. To see this, suppose that P is a cubic polyhedron with V vertices, E edges and F faces. Then $V = V_3$, $F = F_5 + F_6$ and, by the Euler Polyhedron Formula, $V - E + F = 2$. By (10.1), $2E = 3V_3 = 5V_5 + 6V_6$. Therefore,

$$\begin{aligned} 12 &= 6V - 6E + 6F = 6V_3 - 6E + 6(F_5 + F_6) \\ &= (10F_5 + 12F_6) - (15F_5 + 18F_6) + (6F_5 + 6F_6) = F_5. \end{aligned}$$

A **regular polyhedron** is a polyhedron whose faces are bounded by congruent regular polygons and whose polyhedral angles are congruent. In particular, for a regular polyhedron, $F = F_s$ for some s and $V = V_t$ for some t, where $s, t \in \{3, 4, 5\}$. For example, a cube is a regular polyhedron with $V = V_3$ and $F = F_4$. There are only four other regular polyhedra. These five regular polyhedra are the Platonic solids we saw in Figure 10.1. Over two thousand years ago, the Greeks were aware that there are only five such polyhedra.

Theorem 10.7 *There are exactly five regular polyhedra.*

Proof. Let P be a regular polyhedron and let G be an associated plane graph. Then $V - E + F = 2$, where V, E and F denote the number of vertices, edges and faces of P and the number of vertices, edges and regions of G. Therefore,

$$\begin{aligned} -8 &= 4E - 4V - 4F \\ &= 2E + 2E - 4V - 4F \\ &= \sum_{k\geq 3} kF_k + \sum_{k\geq 3} kV_k - 4\sum_{k\geq 3} V_k - 4\sum_{k\geq 3} F_k \\ &= \sum_{k\geq 3}(k-4)F_k + \sum_{k\geq 3}(k-4)V_k. \end{aligned} \tag{10.2}$$

Since G is regular, there exist integers s and t with $s, t \in \{3, 4, 5\}$ such that $F = F_s$ and $V = V_t$. Hence

$$-8 = (s-4)F_s + (t-4)V_t.$$

Moreover, $sF_s = 2E = tV_t$. If $s, t \geq 4$, then (10.2) yields $-8 = (s-4)F_s + (t-4)F_t \geq 0$, which is impossible. Hence either $s = 3$ or $t = 3$. This results in five possibilities for the pairs s, t.

Case 1. $s = 3$ *and* $t = 3$. Here we have

$$-8 = -F_3 - V_3 \quad \text{and} \quad 3F_3 = 3V_3;$$

so $F_3 = V_3 = 4$. Thus P is the *tetrahedron*. (That the tetrahedron is the only regular polyhedron with $V_3 = F_3 = 4$ follows from geometric considerations.)

Case 2. $s = 3$ *and* $t = 4$. Therefore,

$$-8 = -F_3 \quad \text{and} \quad 3F_3 = 4V_4.$$

Hence $F_3 = 8$ and $V_4 = 6$, implying that P is the *octahedron.*

Case 3. $s = 3$ *and* $t = 5$. In this case,

$$-8 = -F_3 + V_5 \quad \text{and} \quad 3F_3 = 5V_5,$$

so $F_3 = 20$, $V_5 = 12$ and P is the *icosahedron.*

Case 4. $s = 4$ *and* $t = 3$. We find here that

$$-8 = -V_3 \quad \text{and} \quad 4F_4 = 3V_3.$$

Thus $V_3 = 8$, $F_4 = 6$ and P is the *cube.*

Case 5. $s = 5$ *and* $t = 3$. For these values,

$$-8 = F_5 - V_3 \quad \text{and} \quad 5F_5 = 3V_3.$$

Solving for F_5 and V_3, we find that $F_5 = 12$ and $V_3 = 20$, so P is the *dodecahedron.* ■

The graphs of the five regular polyhedra are shown in Figure 10.3.

The Graphs K_5 and $K_{3,3}$

Theorem 10.4 provides us with a large class of nonplanar graphs.

Corollary 10.8 *The graph K_5 is nonplanar.*

Proof. The graph K_5 has order $n = 5$ and size $m = 10$. Since $m = 10 > 9 = 3n - 6$, it follows by Theorem 10.4 that K_5 is nonplanar. ■

Since it is evident that any graph containing a nonplanar subgraph is itself nonplanar, it follows that once we know that K_5 is nonplanar, we can conclude that K_n is nonplanar for every integer $n \geq 5$. Of course, K_n is planar for $1 \leq n \leq 4$.

We will soon see that K_5 is an especially important nonplanar graph. Another important nonplanar graph is $K_{3,3}$. Since $K_{3,3}$ has order $n = 6$ and size $m = 9$ but $m < 3n - 6$, Theorem 10.4 cannot be used to establish the nonplanarity of $K_{3,3}$, however. On the other hand, we can use the fact that $K_{3,3}$ is bipartite to establish this property.

Corollary 10.9 *The graph $K_{3,3}$ is nonplanar.*

Proof. Suppose that $K_{3,3}$ is planar. Let there be given a planar embedding of $K_{3,3}$, resulting in r regions. Thus, by the Euler Identity, $n-m+r = 6-9+r = 2$ and so $r = 5$. Let $R_1, R_2, \ldots, R_5$ be the five regions and let m_i be the number of edges on the boundary of R_i $(1 \le i \le 5)$. Since $K_{3,3}$ is bipartite, $K_{3,3}$ contains no triangles and so $m_i \ge 4$ for $1 \le i \le 5$. Since every edge of $K_{3,3}$ lies on the boundary of a cycle, every edge of $K_{3,3}$ belongs to the boundary of two regions. Thus,

$$20 = 4r \le \sum_{i=1}^{5} m_i = 2m = 18,$$

which is impossible. ■

10.2 Maximal Planar Graphs

A planar graph G is **maximal planar** if the addition to G of any edge joining two nonadjacent vertices of G results in a nonplanar graph. Necessarily then, if a maximal planar graph G of order $n \ge 3$ and size m is embedded in the plane resulting in r regions, then the boundary of every region of G is a triangle and so $3r = 2m$. It then follows by the proof of Theorem 10.3 that $m = 3n - 6$. All of the graphs shown in Figure 10.6 are maximal planar.

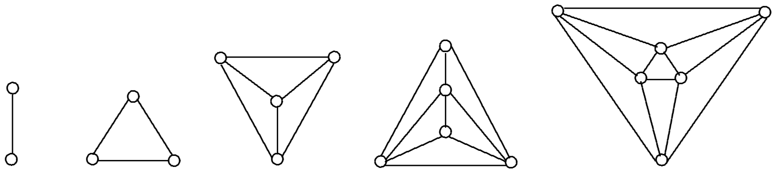


Figure 10.6: Maximal planar graphs

A graph G is **nearly maximal planar** if there exists a planar embedding of G such that the boundary of every region of G is a cycle, at most one of which is not a triangle. Thus, every maximal planar graph is nearly maximal planar (see Figure 10.7(a)). Also, the wheels $W_n = C_n \vee K_1$ $(n \ge 3)$ are nearly maximal planar (see Figure 10.7(b)). In addition, the graphs in Figures 10.7(c) and 10.7(d) (where the graph in Figure 10.7(d) is redrawn in Figure 10.7(e)) are nearly maximal planar.

We now derive some results concerning the degrees of the vertices of a maximal planar graph.

Theorem 10.10 *If G is a maximal planar graph of order 4 or more, then the degree of every vertex of G is at least 3.*

Proof. Let G be a maximal planar graph of order $n \ge 4$ and size m and let v be a vertex of G. Since $m = 3n - 6$, it follows that $G - v$ has order $n - 1$ and size $m - \deg v$. Since $G - v$ is planar and $n - 1 \ge 3$, it follows that

$$m - \deg v \le 3(n - 1) - 6$$

and so $m - \deg v = 3n - 6 - \deg v \le 3n - 9$. Thus, $\deg v \ge 3$. ■

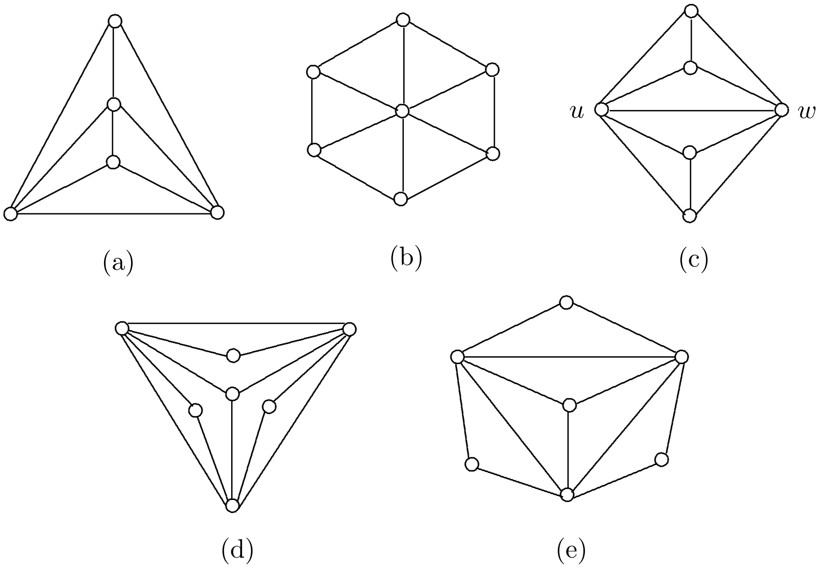


Figure 10.7: Nearly maximal planar graphs

Not only is the minimum degree of every maximal planar graph G of order 4 or more at least 3, the graph G is 3-connected (see Exercise 14).

In the early 20th century, Paul August Ludwig Wernicke studied under the supervision of Hermann Minkowski. (Dénes König, who, as we saw, wrote the first book on graph theory, also studied under Minkowski. Thus, Wernicke and König were "academic brothers".) By Corollary 10.5 and Theorem 10.10, every maximal planar graph of order 4 or more contains a vertex of degree 3, 4 or 5. In the very same year that he received his Ph.D., Wernicke [253] proved that every planar graph that doesn't have a vertex of a degree less than 5 must contain a vertex of degree 5 that is adjacent either to a vertex of degree 5 or to a vertex of degree 6. In the case of maximal planar graphs, Wernicke's result states the following.

Theorem 10.11 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3, (2) *a vertex of degree* 4, (3) *two adjacent vertices of degree* 5, (4) *two adjacent vertices, one of which has degree* 5 *and the other has degree* 6.

Proof. Assume, to the contrary, that there exists a maximal planar graph G of order $n \geq 4$ and size m containing none of (1)–(4). By Corollary 10.5, $\delta(G) = 5$. Let there be given a planar embedding of G, resulting in r regions. Then

$$n - m + r = 2.$$

Suppose that G has n_i vertices of degree i for $5 \leq i \leq \Delta(G) = \Delta$. Then

$$\sum_{i=5}^{\Delta} n_i = n \quad \text{and} \quad \sum_{i=5}^{\Delta} i n_i = 2m = 3r.$$

We now compute the number of regions whose boundary contains either a vertex of degree 5 or a vertex of degree 6. Since the boundary of every region is a triangle, it follows, by assumption, that no region has two vertices of degree 5 on its boundary or a vertex of degree 5 and a vertex of degree 6 on its boundary. On the other hand, the boundary of a region could contain two or perhaps three vertices of degree 6. Each vertex of degree 5 lies on the boundaries of five regions and every vertex of degree 6 lies on the boundaries of six regions. Furthermore, every region containing a vertex of degree 6 on its boundary can contain as many as three vertices of degree 6. Therefore, G has $5n_5$ regions whose boundary contains a vertex of degree 5 and at least $6n_6/3 = 2n_6$ regions whose boundary contains at least one vertex of degree 6. Thus,

$$\begin{aligned} r &\geq 5n_5 + 2n_6 \geq 5n_5 + 2n_6 - n_7 - 4n_8 - \cdots - (3\Delta - 20)n_\Delta \\ &= \sum_{i=5}^{\Delta}(20 - 3i)n_i = 20n - 3\sum_{i=5}^{\Delta} i n_i = 20(m - r + 2) - 3(2m) \\ &= (20m - 20r + 40) - 9r = (30r - 20r + 40) - 9r \\ &= r + 40, \end{aligned}$$

which is a contradiction. ■

The following result gives a relationship among the degrees of the vertices in a maximal planar graph of order at least 4.

Theorem 10.12 *Let G be a maximal planar graph of order $n \geq 4$ and size m containing n_i vertices of degree i for $3 \leq i \leq \Delta = \Delta(G)$. Then*

$$3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta.$$

Proof. Since $m = 3n - 6$, it follows that $2m = 6n - 12$. Therefore,

$$\sum_{i=3}^{\Delta} i n_i = \sum_{i=3}^{\Delta} 6n_i - 12$$

and so

$$\sum_{i=3}^{\Delta}(6 - i)n_i = 12. \tag{10.3}$$

Hence, $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta$. ■

Discharging

Heinrich Heesch introduced the idea of assigning what is called a "charge" to each vertex of a planar graph as well as **discharging rules** which indicate how charges can be redistributed among the vertices. In a maximal planar graph G, every vertex v of G is assigned a **charge** of $6 - \deg v$. In particular, every vertex of degree 5 receives a charge of $+1$, every vertex of degree 6 receives a charge of 0, and every vertex of degree 7 or more receives a negative charge. By appropriately redistributing positive charges, some useful results can often be obtained. According to equation (10.3) in the proof of Theorem 10.12, the sum of the charges of the vertices of a maximal planar graph of order 4 or more is 12. This is restated below.

Theorem 10.13 *If G is a maximal planar graph of order $n \geq 4$, size m and maximum degree $\Delta(G) = \Delta$ such that G has n_i vertices of degree i for $3 \leq i \leq \Delta$, then*

$$\sum_{i=3}^{\Delta}(6-i)n_i = 12.$$

According to Theorem 10.10, every vertex of a maximal planar graph of order 4 or more has degree 3 or more. Consequently, no vertex can contribute more than 3 to the sum in Theorem 10.13. Because this sum is 12, we have the following corollary, which is an extension of Corollary 10.5.

Corollary 10.14 *If G is a maximal planar graph of order at least 4, then G contains at least four vertices whose degrees are at most 5.*

We now use the discharging method to give an alternative proof of Theorem 10.11.

Theorem 10.15 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3, (2) *a vertex of degree* 4, (3) *two adjacent vertices of degree* 5, (4) *two adjacent vertices, one of which has degree* 5 *and the other has degree* 6.

Proof. Assume, to the contrary, that there exists a maximal planar graph G of order $n \geq 4$, where there are n_i vertices of degree i for $3 \leq i \leq \Delta = \Delta(G)$ such that G contains none of (1)–(4). Thus, $\delta(G) = 5$. To each vertex v of G assign the charge $6 - \deg v$. Hence, each vertex of degree 5 receives a charge of $+1$, each vertex of degree 6 receives no charge, and each vertex of degree 7 or more receives a negative charge. By Theorem 10.13, the sum of the charges of the vertices of G is

$$\sum_{i=3}^{\Delta}(6-i)n_i = 12.$$

Let there be given a planar embedding of G. For each vertex v of degree 5 in G, redistribute its charge of $+1$ by moving a charge of $\frac{1}{5}$ to each of its five neighbors, resulting in v now having a charge of 0. Hence, the sum of the charges of the vertices of G remains 12. Since G contains neither (3) nor (4), no vertex of degree 5 or 6 will have its charges increased. Consider a vertex u with $\deg u = k \geq 7$. Thus u received an initial charge of $6 - k$. Because no consecutive neighbors of u in the embedding can have degree 5, the vertex u can receive an added charge of $+\frac{1}{5}$ from at most $k/2$ of its neighbors. After the redistribution of charges, the new charge of u is at most

$$6 - k + \frac{k}{2} \cdot \frac{1}{5} = 6 - \frac{9k}{10} < 0.$$

Hence, no vertex of G now has a positive charge. This is impossible, however, since the sum of the charges of the vertices of G is 12. ■

Another result concerning maximal planar graphs that can be proved with the aid of the discharging method (see Exercise 19) is due to Philip Franklin [95].

Theorem 10.16 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3*,* (2) *a vertex of degree* 4*,* (3) *a vertex of degree* 5 *that is adjacent to two vertices, each of which has degree* 5 *or* 6*.*

10.3 Characterizations of Planar Graphs

In the preceding two sections, we discussed several characteristics of planar graphs. However, a fundamental question remains: For a given graph G, how does one determine whether G is planar or nonplanar? Of course, if G can be drawn in the plane without any of its edges crossing, then G is planar. On the other hand, if G cannot be drawn in the plane without edges crossing, then G is nonplanar. Nevertheless, it may very well be difficult to see how to draw a graph G in the plane without edges crossing or to know that such a drawing is impossible. We saw from Theorem 10.4 that if G has order $n \geq 3$ and size m where $m > 3n-6$, then G is nonplanar. Also, as a consequence of Theorem 10.4, we saw in Corollary 10.5 that if G contains no vertex of degree less than 6, then G is nonplanar. Of course, in that case, $m \geq 3n$.

Any graph that is a subgraph of a planar graph must surely be planar. Equivalently, every graph containing a nonplanar subgraph must itself be nonplanar. Thus, to show that a disconnected graph G is planar it suffices to show that each component of G is planar. Hence, when considering planarity, we may restrict our attention to connected graphs. Since a connected graph G is planar if and only if each block of G is planar (see Exercise 1), it is sufficient to concentrate on 2-connected graphs only.

According to Corollaries 10.8 and 10.9, the graphs K_5 and $K_{3,3}$ are nonplanar. Hence, if a graph G should contain K_5 or $K_{3,3}$ as a subgraph, then G

is nonplanar. For the maximal planar graph G of order 5 and size 9 shown in Figure 10.8 (that is, G is obtained by deleting one edge from K_5), we consider the graph $F = G \,\Box\, K_3$, shown in Figure 10.8. Thus, F consists of three copies of G, denoted by G_1, G_2 and G_3, where $u_1u_2 \notin E(G_1)$, $v_1v_2 \notin E(G_2)$ and $w_1w_2 \notin E(G_3)$. To make it easier to draw G, the nine edges of each graph G_i $(1 \le i \le 3)$ are not drawn. The graph F has order 15 and size $m = 42$. Since $m = 42 > 39 = 3n - 6$, it follows that F is nonplanar. Furthermore, it can be shown that no subgraph of F is isomorphic to either K_5 or $K_{3,3}$. Thus, despite the fact that F contains neither K_5 nor $K_{3,3}$ as a subgraph, the graph F is nonplanar. Consequently, there must exist some other explanation as to why this graph is nonplanar.

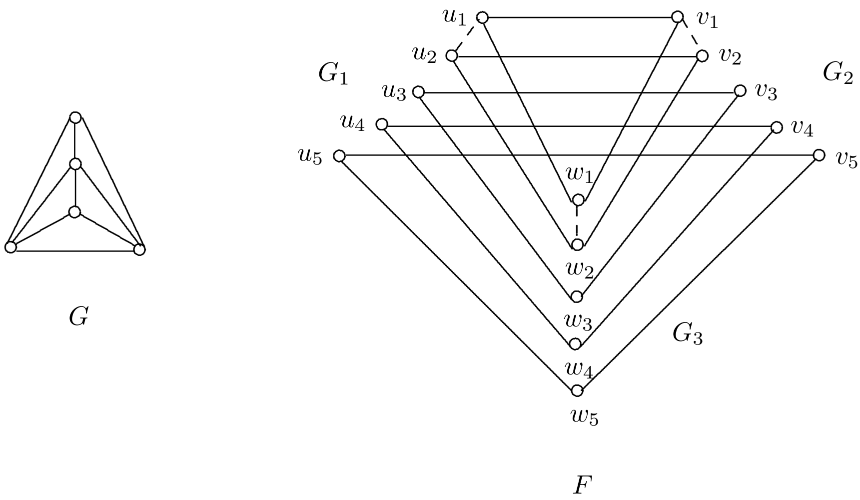


Figure 10.8: The graph $F = G \,\Box\, K_3$

Kuratowski's Theorem

A graph H is a **subdivision** of a graph G if either $H \cong G$ or H can be obtained from G by inserting vertices of degree 2 into some, all or none of the edges of G. Thus, for the graph G of Figure 10.9, all of the graphs H_1, H_2 and H_3 are subdivisions of G. Indeed, H_3 is also a subdivision of H_2.

Certainly, a subdivision H of a graph G is planar if and only if G is planar. Therefore, K_5 and $K_{3,3}$ are nonplanar as is any subdivision of K_5 or $K_{3,3}$. This provides a necessary condition for a graph to be planar.

Theorem 10.17 *A graph G is planar only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

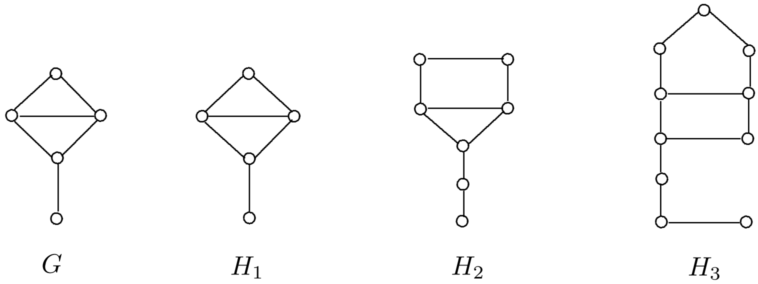


Figure 10.9: Subdivisions of a graph

The remarkable feature about this necessary condition for a graph to be planar is that the condition is also sufficient. The first published proof of this fact occurred in 1930. This theorem is due to the Polish topologist Kazimierz Kuratowski (1896–1980), who first announced this theorem in 1929. The title of Kuratowski's paper is "Sur le problème des courbes gauches en topologie" (On the problem of skew curves in topology), which suggests, and rightly so, that the setting of his theorem was in topology – not graph theory. Nonplanar graphs were sometimes called *skew graphs* during that period. The publication date of Kuratowski's paper was critical to having the theorem credited to him, for, as it turned out, later in 1930 the two American mathematicians Orrin Frink and Paul Althaus Smith submitted a paper containing a proof of this theorem as well but withdrew it after they became aware that Kuratowski's proof had preceded theirs, although just barely. They did publish a one-sentence announcement [96] of what they had accomplished in the *Bulletin of the American Mathematical Society* and, as the title "Irreducible non-planar graphs" of their note indicates, the setting for their proof was graph theoretical in nature.

It is believed by some that a proof of this theorem may have been discovered somewhat earlier by the Russian topologist Lev Semenovich Pontryagin, who was blind his entire adult life. Because the first proof of this theorem may have occurred in Pontryagin's unpublished notes, this result is sometimes referred to as the Pontryagin–Kuratowski theorem in Russia and elsewhere. However, since the possible proof of this theorem by Pontryagin did not satisfy the established practice of appearing in print in an accepted refereed journal, the theorem is generally credited to Kuratowski [151], and to Kuratowski alone.

Theorem 10.18 (**Kuratowski's Theorem**) *A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Proof. We have already noted the necessity of this condition for a graph to be planar. Hence, it remains to verify its sufficiency, namely that every graph containing no subgraph which is a subdivision of K_5 or $K_{3,3}$ is planar. Suppose that this statement is false. Then there is a nonplanar 2-connected graph G of minimum size containing no subgraph that is a subdivision of K_5 or $K_{3,3}$.

We claim, in fact, that G is 3-connected. Suppose that this is not the case. Then G contains a minimum vertex-cut consisting of two vertices x and y. Since G has no cut-vertices, it follows that each of x and y is adjacent to one or more vertices in each component of $G - \{x, y\}$. Let F_1 be one component of $G - \{x, y\}$ and let F_2 be the union of the remaining components of $G - \{x, y\}$. Furthermore, let

$$G_i = G[V(F_i) \cup \{x, y\}] \text{ for } i = 1, 2.$$

We consider two cases, depending on whether x and y are adjacent. Suppose first that x and y are adjacent. We claim that in this case at least one of G_1 and G_2 is nonplanar. If both G_1 and G_2 are planar, then there exist planar embeddings of these two graphs in which xy is on the boundary of the exterior region in each embedding. This, however, implies that G itself is planar, which is impossible. Thus G_1, say, is nonplanar. Since G_1 is a subgraph of G, it follows that G_1 contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. However, the size of G_1 is less than the size of G, which contradicts the defining property of G. Hence, x and y must be nonadjacent.

Let f be the edge obtained by joining x and y, and let $H_i = G_i + f$ for $i = 1, 2$. If H_1 and H_2 are both planar, then, as above, there is a planar embedding of $G + f$ and of G as well. Since this is impossible, at least one of H_1 and H_2 is nonplanar, say H_1 is nonplanar. Because the size of H_1 is less than the size of G, the graph H_1 contains a subgraph F that is a subdivision of K_5 or $K_{3,3}$. Since G_1 contains no such subgraph, it follows that $f \in E(F)$. Let P be an $x - y$ path in G_2. By replacing f in F by P, we obtain a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. This produces a contradiction. Hence, as claimed, G is 3-connected.

To summarize then, G is a nonplanar graph of minimum size containing no subgraph that is a subdivision of K_5 or $K_{3,3}$ and, as we just saw, G is 3-connected. Let $e = uv$ be an edge of G. Then $H = G - e$ is planar. Since G is 3-connected, H is 2-connected. By Theorem 3.4, there are cycles in H containing both u and v. Among all planar embeddings of H, choose one in which there is a cycle

$$C = (u = v_0, v_1, \ldots, v_\ell = v, \ldots, v_k = u)$$

containing u and v such that the number of regions interior to C is maximum.

It is convenient to define two subgraphs of H. By the *exterior subgraph* of H is meant the subgraph induced by those edges lying exterior to C and the *interior subgraph* of H is the subgraph induced by those edges lying interior to C. Both subgraphs exist, for otherwise the edge e could be added either to the exterior or interior subgraph of H so that the resulting graph (namely G) is planar.

No two distinct vertices of $\{v_0, v_1, \ldots, v_\ell\}$ or of $\{v_\ell, v_{\ell+1}, \ldots, v_k\}$ are connected by a path in the exterior subgraph of H, for otherwise there is a cycle in H containing u and v and having more regions interior to it than C has. Since G is nonplanar, there must be a $v_s - v_t$ path P in the exterior subgraph of H, where $0 < s < \ell < t < k$, such that v_s and v_t are the only vertices of P that belong to C. (See Figure 10.10.) In fact, the path P must be (v_s, v_t); for otherwise, if there is an interior vertex w on P, then, since G is 3-connected, there are three internally disjoint paths from w to C, creating a new cycle C' containing u and v such that C' has more regions interior to it than C has, which is a contradiction.

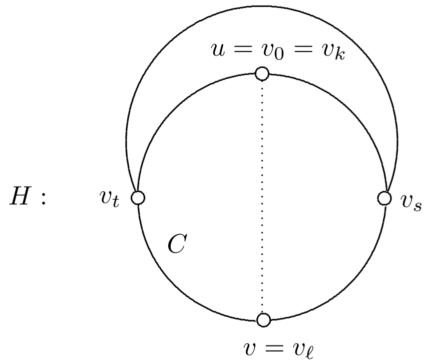


Figure 10.10: A step in the proof of Theorem 10.18

Let S be the set of vertices on C different from v_s and v_t, that is,

$$S = V(C) - \{v_s, v_t\},$$

and let H_1 be the component of $H - S$ that contains P. By the defining property of C, the subgraph H_1 cannot be moved to the interior of C in a plane manner. This fact together with the fact that $G = H + e$ is nonplanar implies that the interior subgraph of H must contain one of the following:

(1) A $v_a - v_b$ path with $0 < a < s$ and $\ell < b < t$ such that only v_a and v_b belong to C. (See Figure 10.11(a).)

(2) A vertex w not on C that is connected to C by three internally disjoint paths such that the terminal vertex of one such path P' is one of v_0, v_s, v_ℓ and v_t. If, for example, the terminal vertex of P' is v_0, then the terminal vertices of the other two paths are v_a and v_b, where $s \le a < \ell$ and $\ell < b \le t$ where not both $a = s$ and $b = t$ occur. (See Figure 10.11(b).) If the terminal vertex of P' is one of v_s, v_ℓ and v_t, then there are corresponding bounds for a and b for the terminal vertices of the other two paths.

(3) A vertex w not on C that is connected to C by three internally disjoint paths P_1, P_2 and P_3 such that the terminal vertices of these paths are three of the four vertices v_0, v_s, v_ℓ and v_t, say v_0, v_ℓ and v_s, respectively, together with a $v_c - v_t$ path P_4 ($v_c \ne v_0, v_\ell, w$), where v_c is on P_1 or P_2 and P_4 is disjoint from P_1, P_2 and C except for v_c and v_t. (See Figure 10.11(c).) The remaining choices for P_1, P_2 and P_3 produce three analogous cases.

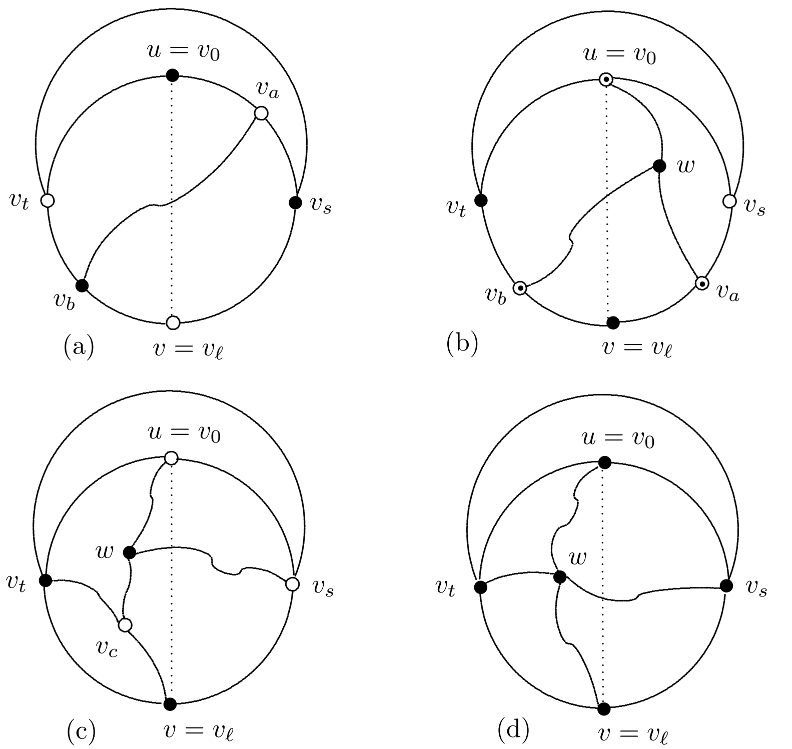


Figure 10.11: Situations (1)–(4) in the proof of Theorem 10.18

(4) A vertex w not on C that is connected to v_0, v_s, v_ℓ and v_t by four internally disjoint paths. (See Figure 10.11(d).)

In the first three cases, there is a subgraph of G that is a subdivision of $K_{3,3}$, while in the fourth case, there is a subgraph of G that is a subdivision of K_5. This is a contradiction. ■

While it is rarely easy to use Kuratowski's theorem to test a graph for planarity, a number of efficient algorithms have been developed that determine whether a graph is planar, including linear-time algorithms, the first of which was obtained by John Hopcroft and Robert Tarjan [131].

As a consequence of Kuratowski's theorem, the 4-regular graph G shown in Figure 10.12(a) is nonplanar since G contains the subgraph H in Figure 10.12(b) that is a subdivision of $K_{3,3}$.

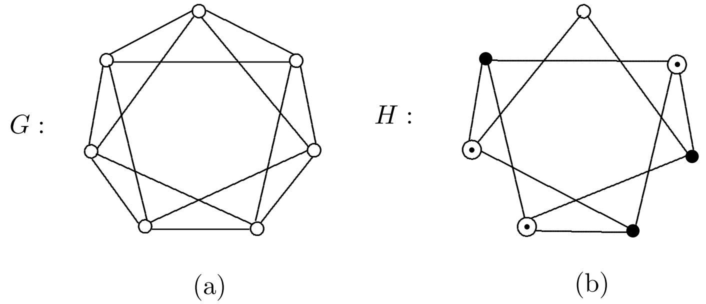


Figure 10.12: A nonplanar graph

Minors of Graphs

There is another characterization of planar graphs closely related to that given in Kuratowski's theorem. Before presenting this theorem, it is useful to introduce some additional terminology. If two adjacent vertices u and v in a graph G are identified, then we say that we have **contracted** the edge uv (denoting the resulting vertex by u or v). For the graph G of Figure 10.13, the graph G' is obtained by contracting the edge uv in G and where G'' is obtained by contracting the edge wy in G'.

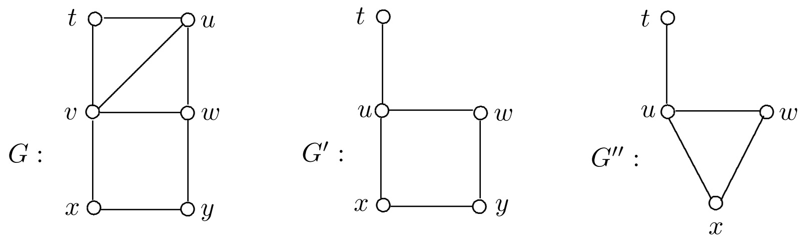


Figure 10.13: Contracting an edge

When dealing with edge contractions, it is often the case that we begin with a graph G, contract an edge in G to obtain a graph G', contract some edge in G' to obtain another graph G'', and so on, until finally arriving at a graph H. Any such graph H can be obtained in a different and perhaps simpler manner. In particular, H can be obtained from G by a succession of edge contractions if and only if the vertex set of H is the set of elements in a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ where each induced subgraph $G[V_i]$ is connected and V_i is adjacent to V_j $(i \neq j)$ if some vertex in V_i is adjacent to some vertex in V_j in G. For example, in the graph G of Figure 10.13, if we were to let

$$V_1 = \{t\},\ V_2 = \{u, v\},\ V_3 = \{x\} \text{ and } V_4 = \{w, y\},$$

then the resulting graph H is shown in Figure 10.14. This is the graph G'' of Figure 10.13 obtained by successively contracting the edge uv in G and then the edge wy in G'.

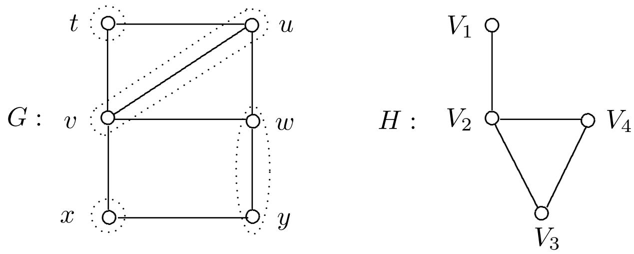


Figure 10.14: Edge contractions

A graph H is called a **minor** of a graph G if either $H \cong G$ or a graph isomorphic to H can be obtained from G by a succession of edge contractions, edge deletions and vertex deletions (in any order). Equivalently, H is a minor of G if $H \cong G$ or H can be obtained from a subgraph of G by a succession of edge contractions. Consequently, a graph G is a minor of itself. If H is a minor of G such that $H \ncong G$, then H is called a **proper minor** of G.

The planar graph H of Figure 10.14 is a minor of the planar graph G of that figure. In fact, every minor of a planar graph is planar.

Consider next the graph G_1 of Figure 10.15, where

$$V_1 = \{t_1, t_2\},\ V_2 = \{u_1, u_2, u_3, u_4\},$$
$$V_3 = \{v_1\},\ V_4 = \{w_1, w_2, w_3\},$$
$$V_5 = \{x_1, x_2\},\ V_6 = \{y_1\}, \text{ and } V_7 = \{z_1\}.$$

Then the graph H_1 of Figure 10.15 can be obtained from G_1 by successive edge contractions. Thus, H_1 is a minor of G_1. By deleting the edge V_2V_6 and the vertices V_6 and V_7 from H_1 (or equivalently, deleting V_6 and V_7 from H_1), we see that K_5 is also a minor of G_1.

The example in Figure 10.15 serves to illustrate the following observation.

Theorem 10.19 *If a graph G is a subdivision of a graph H, then H is a minor of G.*

The converse of Theorem 10.19 is not true, however (see Exercise 30). The following is an immediate consequence of Theorem 10.19.

Theorem 10.20 *If G is a nonplanar graph, then K_5 or $K_{3,3}$ is a minor of G.*

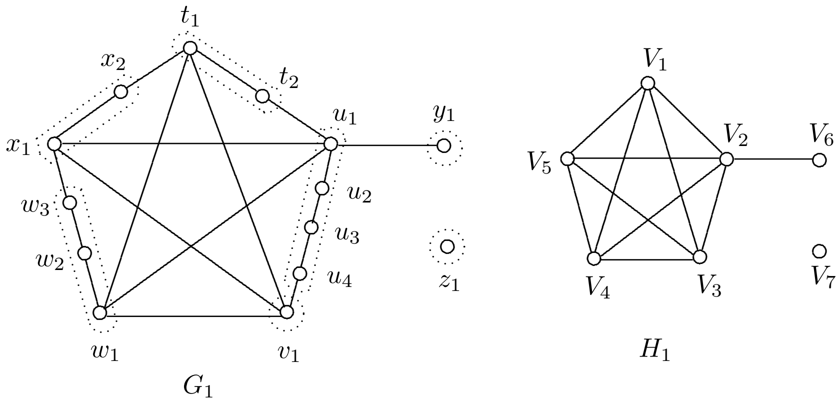


Figure 10.15: Minors of graphs

Wagner's Theorem

The German mathematician Klaus Wagner (1910–2000) showed that the converse of Theorem 10.20 is true [251] in 1937, only a year after obtaining his Ph.D. from Universität zu Köln (University of Cologne), thereby giving another characterization of planar graphs.

Theorem 10.21 (**Wagner's Theorem**) *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Proof. We have already mentioned (in Theorem 10.20) that if a graph G is nonplanar, then either K_5 or $K_{3,3}$ is a minor of G. For the converse, suppose that G is a graph having K_5 or $K_{3,3}$ as a minor. If G were planar, then every minor of G is planar, contradicting the assumption that K_5 or $K_{3,3}$ is a minor of G. Thus, G is nonplanar. ■

More specifically, Theorem 10.21 can be proved with the aid of the following result.

Theorem 10.22 *Let G be a graph.*

(*a*) *If G has $K_{3,3}$ as a minor, then G contains a subdivision of $K_{3,3}$.*

(*b*) *If G has K_5 as a minor, then G contains either a subdivision of K_5 or a subdivision of $K_{3,3}$.*

Proof. Suppose first that $H = K_{3,3}$ is a minor of G. The graph H can be obtained by first deleting edges and vertices of G (if necessary), obtaining a connected graph G', and then by a succession of edge contractions in G'. We show, in this case, that G' contains a subgraph that is a subdivision of $K_{3,3}$ and so G', and G as well, is nonplanar.

Denote the vertices of H by U_i and W_i $(1 \leq i \leq 3)$, where $\{U_1, U_2, U_3\}$ and $\{W_1, W_2, W_3\}$ are the partite sets of H. Since H is obtained from G' by a succession of edge contractions, the subgraphs

$$F_i = G'[U_i] \text{ and } H_i = G'[W_i],\ 1 \leq i \leq 3,$$

are connected. Since $U_iW_j \in E(H)$ for $1 \leq i, j \leq 3$, there is a vertex $u_{i,j} \in U_i$ that is adjacent in H to a vertex $w_{i,j} \in W_j$. Among the vertices $u_{i,1}, u_{i,2}, u_{i,3}$ in U_i $(1 \leq i \leq 3)$, two or possibly all three may represent the same vertex. If $u_{i,1} = u_{i,2} = u_{i,3}$, then denote this vertex by u_i; if two of $u_{i,1}, u_{i,2}, u_{i,3}$ are the same, say $u_{i,1} = u_{i,2}$, then denote this vertex by u_i; if $u_{i,1}, u_{i,2}$ and $u_{i,3}$ are distinct, then let u_i denote a vertex in U_i that is connected to $u_{i,1}, u_{i,2}$ and $u_{i,3}$ by internally disjoint paths in F_i (possibly $u_i = u_{i,j}$ for some j). We proceed in the same manner to obtain vertices $w_i \in W_i$ for $1 \leq i \leq 3$. The subgraph of G induced by the previously described nine edges joining $U_1 \cup U_2 \cup U_3$ and $W_1 \cup W_2 \cup W_3$ together with the edge sets of all of the previously mentioned paths in F_i and H_j $(1 \leq i, j \leq 3)$ is a subdivision of $K_{3,3}$.

Next, suppose that $H = K_5$ is a minor of G. Then H can be obtained by first deleting edges and vertices of G (if necessary), obtaining a connected graph G', and then by a succession of edge contractions in G'. We show in this case that either G' contains a subgraph that is a subdivision of K_5 or G' contains a subgraph that is a subdivision of $K_{3,3}$.

We may denote the vertices of H by V_i $(1 \leq i \leq 5)$, where $G_i = G'[V_i]$ is a connected subgraph of G' and each subgraph G_i contains a vertex that is adjacent to G_j for each pair i, j of distinct integers where $1 \leq i, j \leq 5$. For $1 \leq i \leq 5$, let $v_{i,j}$ be a vertex of G_i that is adjacent to a vertex of G_j, where $1 \leq j \leq 5$ and $j \neq i$.

For a fixed integer i with $1 \leq i \leq 5$, if the vertices $v_{i,j}$ $(i \neq j)$ represent the same vertex, then denote this vertex by v_i. If three of the four vertices $v_{i,j}$ are the same, then we also denote this vertex by v_i. If two of the vertices $v_{i,j}$ are the same, the other two are distinct, and there exist internally disjoint paths from the coinciding vertices to the other two vertices, then we denote the two coinciding vertices by v_i. If the vertices $v_{i,j}$ are distinct and G_i contains a vertex from which there are four internally disjoint paths (one of which may be trivial) to the vertices $v_{i,j}$, then denote this vertex by v_i. Hence, there are several instances in which we have defined a vertex v_i. Should v_i be defined for all i $(1 \leq i \leq 5)$, then G' (and therefore G as well) contains a subgraph that is a subdivision of K_5.

We may assume then that for one or more integers i $(1 \leq i \leq 5)$, the vertex v_i has not been defined. For each such i, there exist distinct vertices u_i and w_i, each of which is connected to two of the vertices $v_{i,j}$ by internally disjoint (possibly trivial) paths, while u_i and w_i are connected by a path none of whose internal vertices are the vertices $v_{i,j}$ and where every two of the five paths have only u_i or w_i in common. If two of the vertices $v_{i,j}$ coincide, then we denote this vertex by u_i. If the remaining two vertices $v_{i,j}$ should also coincide, then we denote this vertex by w_i. We may assume that $i = 1$, that u_1 is connected to $v_{1,2}$ and $v_{1,3}$ and that w_1 is connected to $v_{1,4}$ and $v_{1,5}$, as described above. Denote the edge set of these paths by E_1.

We now consider G_2. If $v_{2,1} = v_{2,4} = v_{2,5}$, then let w_2 be this vertex and set $E_2 = \emptyset$; otherwise, there is a vertex w_2 of G_2 (which may coincide with $v_{2,1}$, $v_{2,4}$ or $v_{2,5}$) connected by internally disjoint (possibly trivial) paths to the distinct vertices in $\{v_{2,1}, v_{2,4}, v_{2,5}\}$. We then let E_2 denote the edge set of these paths. Similarly, the vertices w_3, u_2, and u_3 and the sets E_3, E_4, and E_5 are defined with the aid of the sets $\{v_{3,1}, v_{3,4}, v_{3,5}\}$, $\{v_{4,1}, v_{4,4}, v_{4,5}\}$ and $\{v_{5,1}, v_{5,2}, v_{5,3}\}$, respectively. The subgraph of G' induced by the union of the sets E_i and the edges $v_{i,j}v_{j,i}$ contains a subdivision of $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$. ■

In the proof of Theorem 10.22, it was shown that if K_5 is a minor of a graph G, then G contains a subdivision of K_5 or a subdivision of $K_{3,3}$. In other words, we were unable to show that G necessarily contains a subdivision of K_5. There is good reason for this, which is illustrated in the next example.

The Petersen graph P has order $n = 10$ and size $m = 15$. Since $m < 3n - 6$, no conclusion can be drawn from Theorem 10.3 regarding the planarity or nonplanarity of P. Nevertheless, the Petersen graph is, in fact, nonplanar. Theorems 10.18 and 10.21 give two ways to establish this fact. Figures 10.16(a) and 10.16(b) show P drawn in two ways. Since $P - x$ (shown in Figure 10.16(c)) is a subdivision of $K_{3,3}$, the Petersen graph is nonplanar. The partition $\{V_1, V_2, \ldots, V_5\}$ of $V(P)$ shown in Figure 10.16(d), where $V_i = \{u_i, v_i\}$, $1 \le i \le 5$, shows that K_5 in Figure 10.16(d) is a minor of P and is therefore nonplanar. Since P is a cubic graph, there is no subgraph of P that is a subdivision of K_5, however.

Outerplanar Graphs

We now turn our attention to a special class of planar graphs. A graph G is **outerplanar** if there exists a planar embedding of G so that every vertex of G lies on the boundary of the exterior region. Actually, if there is a planar embedding of G so that every vertex of G lies on the boundary of the same region of G, then G is outerplanar. The following two results provide characterizations of outerplanar graphs.

Theorem 10.23 *A graph G is outerplanar if and only if the join $G \vee K_1$ is planar.*

Proof. Let G be an outerplanar graph and suppose that G is embedded in the plane such that every vertex of G lies on the boundary of the exterior region. Then a vertex v can be placed in the exterior region of G and joined to all vertices of G in such a way that a planar embedding of $G \vee K_1$ results. Thus, $G \vee K_1$ is planar.

For the converse, assume that G is a graph such that $G \vee K_1$ is planar. Hence $G \vee K_1$ contains a vertex v that is adjacent to every vertex of G. Let

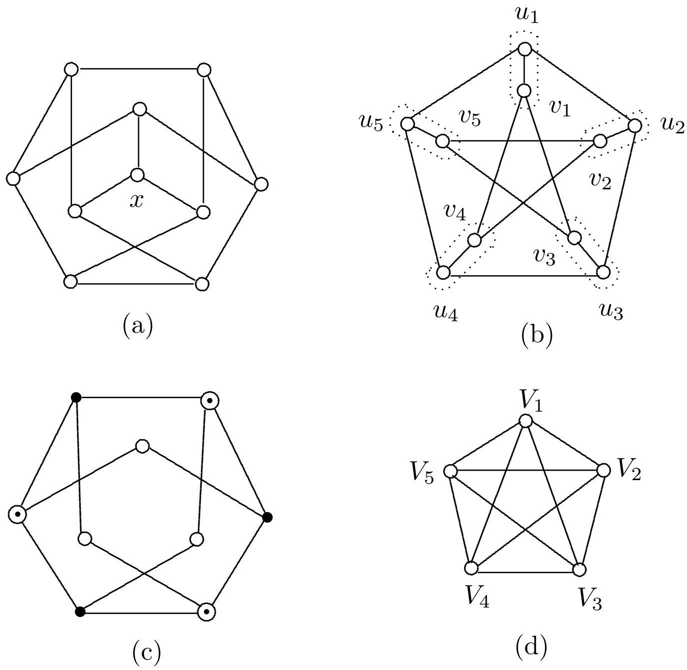


Figure 10.16: Showing that the Petersen graph is nonplanar

there be a planar embedding of $G \vee K_1$. Upon deleting the vertex v, we arrive at a planar embedding of G in which all vertices of G lie on the boundary of the same region. Thus, G is outerplanar. ■

The following characterization of outerplanar graphs is analogous to the characterization of planar graphs stated in Kuratowski's theorem.

Theorem 10.24 *A graph G is outerplanar if and only if G contains no subgraph that is a subdivision of K_4 or $K_{2,3}$.*

Proof. Suppose first that there exists some outerplanar graph G that contains a subgraph H that is a subdivision of K_4 or $K_{2,3}$. By Theorem 10.23, $G \vee K_1$ is planar. Since the subgraph $H \vee K_1$ of $G \vee K_1$ is a subdivision of K_5 or a subdivision of $K_{3,3}$, it follows that $G \vee K_1$ contains a subgraph that is a subdivision of K_5 or a subdivision of $K_{3,3}$ and so is nonplanar. This produces a contradiction.

For the converse, assume, to the contrary, that there exists a graph G that is not outerplanar but contains no subgraph that is a subdivision of K_4 or $K_{2,3}$. By Theorem 10.23, $G \vee K_1$ is not planar but contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. This contradicts Theorem 10.18. ■

An outerplanar graph G is **maximal outerplanar** if the addition to G

of any edge joining two nonadjacent vertices of G results in a graph that is not outerplanar. Necessarily then, there is a planar embedding of a maximal outerplanar graph G of order at least 3, where the boundary of whose exterior region is a Hamiltonian cycle of G and the boundary of every other region is a triangle. Every maximal outerplanar graph of order 3 or more is therefore nearly maximal planar. We now describe some other facts about outerplanar graphs.

Theorem 10.25 *Every nontrivial outerplanar graph contains at least two vertices of degree* 2 *or less.*

Proof. Let G be a nontrivial outerplanar graph. The result is obvious if the order of G is 4 or less, so we may assume that the order of G is at least 5. Add edges to G, if necessary, to obtain a maximal outerplanar graph. Thus, the boundary of the exterior region of G is a Hamiltonian cycle of G. Among the chords of C, let uv be one such that uv and a $u-v$ path on C produce a cycle containing a minimum number of interior regions. Necessarily, this minimum is 1. Then the degree of the remaining vertex y on the boundary of this region is 2. There is such a chord wx of C on the other $u-v$ path of C, producing another vertex z of degree 2. In G, the degrees of y and z are therefore 2 or less. ■

Theorem 10.26 *The size of every outerplanar graph of order* $n \geq 2$ *is at most* $2n-3$.

Proof. Let G be an outerplanar graph of order $n \geq 2$ and size m. By Theorem 10.23, $H = G \vee K_1$ is planar. Since H has order $n' = n+1$ and size $m' = m+n$, it follows that $m' \leq 3n'-6$ and so $m+n \leq 3(n+1)-6$. Thus, $m \leq 2n-3$. ■

In view of Theorem 10.26, the size of a maximal outerplanar graph of order $n \geq 2$ is $2n-3$.

10.4 Hamiltonian Planar Graphs

In Sections 10.1 and 10.2, we saw necessary conditions for a connected graph to be planar and necessary conditions for a graph of order at least 4 to be maximal planar. In this section, we will be introduced to a single result, namely a necessary condition for a planar graph to be Hamiltonian.

Grinberg's Theorem

Let G be a Hamiltonian planar graph of order n with Hamiltonian cycle C and let there be given a planar embedding of G. An edge of G not lying on C

is a **chord** of G. Every chord and every region of G then lies interior to C or exterior to C. For $i = 3, 4, \ldots, n$, let r_i denote the number of regions interior to C whose boundary contains exactly i edges and let r'_i denote the number of regions exterior to C whose boundary contains exactly i edges.

The plane graph G of Figure 10.17 of order 12 is Hamiltonian. The edges of the Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{12}, v_1)$ are drawn with bold lines. With respect to C, we have

$$r_3 = r'_3 = 1,\ r_4 = 3,\ r'_4 = 2,\ r_5 = r'_7 = 1,$$

while $r_i = 0$ for $6 \le i \le 12$ and $r'_i = 0$ for $i = 5, 6$ and $8 \le i \le 12$.

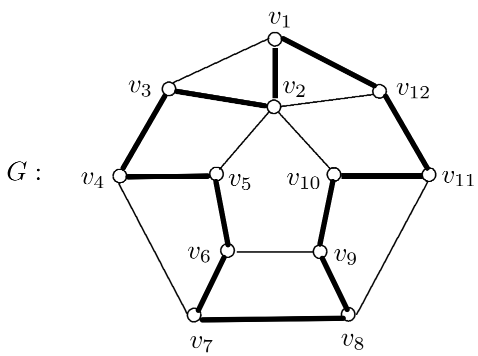


Figure 10.17: A Hamiltonian planar graph

In 1968 a necessary condition for a planar graph to be Hamiltonian was discovered by the Latvian mathematician Emanuels Ja. Grinberg [109].

Theorem 10.27 (**Grinberg's Theorem**) *For a plane graph G of order n with Hamiltonian cycle C,*

$$\sum_{i=3}^{n} (i-2)(r_i - r'_i) = 0.$$

Proof. Suppose that c chords of G lie interior to C. Then $c+1$ regions of G lie interior to C. Therefore,

$$\sum_{i=3}^{n} r_i = c + 1 \text{ and so } c = \sum_{i=3}^{n} r_i - 1.$$

Let N denote the result obtained by summing over all regions interior to C the number of edges on the boundary of each such region. Then each edge on C is counted once and each chord interior to C is counted twice, that is,

$$N = \sum_{i=3}^{n} i r_i = n + 2c.$$

Therefore,

$$\sum_{i=3}^{n} ir_i = n + 2c = n + 2\sum_{i=3}^{n} r_i - 2$$

and so

$$\sum_{i=3}^{n}(i-2)r_i = n-2.$$

Similarly,

$$\sum_{i=3}^{n}(i-2)r'_i = n-2.$$

Therefore, $\sum_{i=3}^{n}(i-2)(r_i - r'_i) = 0$. ■

The following observations are quite useful in applying Theorem 10.27. Let G be a plane graph with Hamiltonian cycle C. Furthermore, suppose that an edge e of G is on the boundary of two regions R_1 and R_2 of G. If e is an edge of C, then one of R_1 and R_2 is in the interior of C and the other is in the exterior of C. If, on the other hand, e is not an edge of C, then R_1 and R_2 are either both in the interior of C or both in the exterior of C.

The Tutte Graph

In 1880, the British mathematician Peter Guthrie Tait conjectured that every 3-connected cubic planar graph is Hamiltonian. This conjecture was disproved in 1946 by William T. Tutte [239], who produced the graph G in Figure 10.18 as a counterexample. In addition to disproving Tait's conjecture, Tutte [243] proved that every 4-connected planar graph is Hamiltonian.

Since Theorem 10.27 gives a necessary condition for a planar graph to be Hamiltonian, this theorem also provides a sufficient condition for a planar graph to be non-Hamiltonian. We now see how Grinberg's theorem can be used to show that the plane graph of Figure 10.18 is not Hamiltonian. This graph is called the **Tutte graph** and has a great deal of historical interest. We will encounter this graph again in Chapter 17.

Assume, to the contrary, that the Tutte graph G is Hamiltonian. Then G has a Hamiltonian cycle C. Necessarily, C must contain exactly two of the three edges e, f_1 and f_2, say f_1 and either e or f_2. Similarly, C must contain exactly two edges of the three edges e', f_2 and f_3. Since we may assume that C contains f_2, we may further assume that e is not on C. Consequently, R_1 and R_2 lie interior to C.

Let G_1 denote the component of $G - \{e, f_1, f_2\}$ containing w. Thus, G_1 contains a Hamiltonian $v_1 - v_2$ path P'. Therefore, $G_2 = G_1 + v_1v_2$ is Hamiltonian and contains a Hamiltonian cycle C' consisting of P' and v_1v_2. Applying Grinberg's theorem to G_2 with respect to C', we obtain

$$1(r_3 - r'_3) + 2(r_4 - r'_4) + 3(r_5 - r'_5) + 6(r_8 - r'_8) = 0. \qquad (10.4)$$

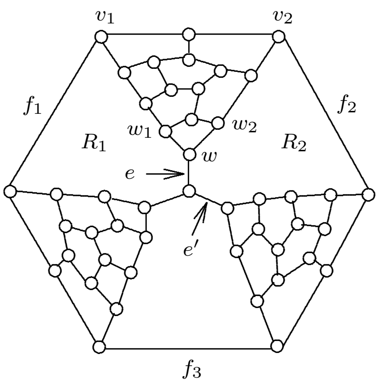


Figure 10.18: The Tutte graph

Since v_1v_2 is on C' and the exterior region of G_2 lies exterior to C', it follows that

$$r_3 - r_3' = 1 - 0 = 1 \quad \text{and} \quad r_8 - r_8' = 0 - 1 = -1.$$

Therefore, from (10.4), we have

$$2(r_4 - r_4') + 3(r_5 - r_5') = 5.$$

Necessarily, both ww_1 and ww_2 are edges of C' and so $r_4 \geq 1$, implying that either

$$r_4 - r_4' = 1 - 1 = 0 \text{ or } r_4 - r_4' = 2 - 0 = 2.$$

If $r_4 - r_4' = 0$, then $3(r_5 - r_5') = 5$, which is impossible. On the other hand, if $r_4 - r_4' = 2$, then $3(r_5 - r_5') = 1$, which is also impossible. Hence, G is not Hamiltonian.

For many years, Tutte's graph was the only known example of a 3-connected cubic planar graph that was not Hamiltonian. Much later, however, other such graphs have been found; for example, Grinberg himself found another counterexample to Tait's conjecture (see Exercise 39).

Exercises for Chapter 10

Section 10.1: The Euler Identity

1. Prove that a graph is planar if and only if each of its blocks is planar.

2. Prove Corollary 10.2: *If G is a plane graph with n vertices, m edges and r regions, then $n - m + r = 1 + k(G)$.*

3. Give an example of a graph G of order 8 such that G and $\overline{G}$ are planar.

4. Prove that if G is a planar graph of order 11, then $\overline{G}$ is nonplanar.

5. (a) Prove that the order of every 3-regular planar graph containing no triangle or 4-cycle is at least 20.
   (b) Show that the Petersen graph is nonplanar.

6. (a) Show that if G is a planar graph containing no vertex of degree less than 5, then G contains at least 12 vertices of degree 5.
   (b) Give an example of a planar graph that contains no vertex of degree less than 5.

7. Show that every graph G of order $n \geq 6$ that contains three spanning trees T_1, T_2 and T_3 such that every edge of G belongs to exactly one of these three trees is nonplanar.

8. If the boundary of every interior region of a plane graph G of order n and size m is a triangle and the boundary of the exterior region is a k-cycle ($k \geq 3$), express m in terms of n and k.

9. Determine all connected regular planar graphs G such that the number of regions in a planar embedding of G equals its order.

10. A cubic polyhedron P has only 5-sided, 6-sided and 7-sided faces. Determine a formula for the number of 5-sided faces in P.

Section 10.2: Maximal Planar Graphs

11. Give an example of two non-isomorphic maximal planar graphs of the same order.

12. Prove that there exists only one 4-regular maximal planar graph.

13. Prove that a planar graph of order $n \geq 3$ and size m is maximal planar if and only if $m = 3n - 6$.

14. Prove that every maximal planar graph of order 4 or more is 3-connected.

15. In a planar embedding of a maximal planar graph G of order 6, a vertex is placed in each interior region of G and joined to the vertices on its boundary, producing a graph H. Prove or disprove: H is Hamiltonian.

16. Determine all maximal planar graphs G of order 3 or more such that the number of regions in a planar embedding of G equals its order.

17. Determine whether the graph G shown in Figure 10.19 is nearly maximal planar.

G :

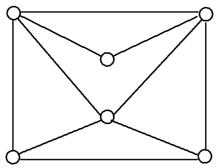


Figure 10.19: The graph G in Exercise 17

18. A nontrivial tree T of order n has the property that $\overline{T}$ is a maximal planar graph.

    (a) What is n?

    (b) Give an example of a tree T with this property.

19. Use a discharging method to prove Theorem 10.16: *If G is a maximal planar graph of order* 4 *or more, then G contains at least one of the following:* (1) *a vertex of degree* 3*,* (2) *a vertex of degree* 4*,* (3) *a vertex of degree* 5 *that is adjacent to two vertices, each of which has degree* 5 *or* 6*.*

20. If every vertex v of a nontrivial tree T is given a charge of $2 - \deg v$, then what is the sum of the charges of the vertices of T?

Section 10.3: Characterizations of Planar Graphs

21. Let U be a subset of the vertex set of a graph G, and suppose that $H = G[U]$ is an induced subgraph of G of order k. Let H' be the graph obtained by inserting a vertex of degree 2 into every edge of H. Let G' be the graph obtained by inserting a vertex of degree 2 into every edge of G.

    (a) Prove or disprove: If G is planar, then H' is planar.

    (b) What is $G'[U]$?

22. Let T be a tree of order at least 4, and let $e_1, e_2, e_3 \in E(\overline{T})$. Prove that $T + e_1 + e_2 + e_3$ is planar.

23. Let T be a tree of order at least 5, and let $e_1, e_2, \cdots, e_5 \in E(\overline{T})$. Let $G = T + \{e_1, e_2, \cdots, e_5\}$. Prove that if G does not contain a subdivision of $K_{3,3}$, then G is planar.

24. (a) Determine the order n, the size m and the number $3n - 6$ for the graph $K_4 \square K_2$.
    (b) What does the information in (a) say about the planarity of $K_4 \square K_2$?
    (c) Is $K_4 \square K_2$ planar or nonplanar?

25. Determine all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that for each edge e of G, the graph $G - e$ is planar.

26. Let $S_{a,b}$ denote the double star in which the degrees of the two vertices that are not end-vertices are a and b. Determine all pairs a, b of integers such that $\overline{S}_{a,b}$ is planar.

27. A nonplanar graph G of order 7 has the property that $G - v$ is planar for every vertex v of G.
    (a) Show that G does not contain $K_{3,3}$ as a subgraph.
    (b) Give an example of a graph with this property.

28. Determine all integers $n \geq 3$ such that $\overline{C}_n$ is planar.

29. Determine all integers $n \geq 3$ such that C_n^2 is nonplanar.

30. Show that the converse of Theorem 10.19 is not, in general, true.

31. It has been observed that if a graph H is a minor of a planar graph, then H is planar. Prove or disprove: If a minor H of a graph G is planar, then G is planar.

32. Let G be the graph shown in Figure 10.20.
    (a) Show that G contains $K_{3,3}$ as a subgraph.
    (b) Show that G does not contain a subdivision of K_5 as a subgraph.
    (c) Show that K_5 is a minor of G.

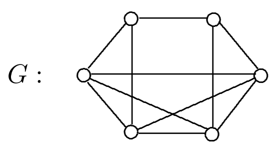


Figure 10.20: The graph G in Exercise 32

33. (a) Let G be a 4-regular graph of order 10 and size m. What can be deduced about the planarity of G by comparing the numbers m and $3n - 6$?

(b) Prove or disprove: There is a planar 2-connected 4-regular graph of order 10.

(c) Show that the 2-connected 4-regular graph H of order 10 shown in Figure 10.21 does not contain K_5 as a subgraph but does contain K_5 as a minor. What can you conclude from this?

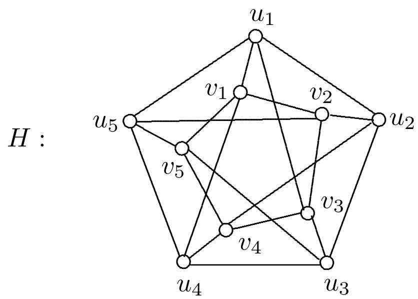


Figure 10.21: A 2-connected 4-regular graph of order 10

(d) Prove that if G is a graph of order $n \geq 5$ and size $m \geq 3n - 5$, then G need not contain K_5 as a subgraph but must contain a subgraph with minimum degree 4.

34. (a) What is the minimum possible order of a graph G containing only vertices of degree 3 and degree 4 and an equal number of each such that G contains a subdivision of K_5?

(b) Does the graph H of Figure 10.22 contain a subdivision of K_5 or a subdivision of $K_{3,3}$?

(c) Does the graph H of Figure 10.22 contain K_5 or $K_{3,3}$ as a minor?

(d) Is the graph H of Figure 10.22 planar or nonplanar?

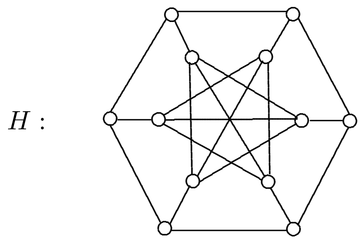


Figure 10.22: The graph H in Exercise 34

35. Determine all connected graphs G of order $n \geq 4$ such that $G \vee K_1$ is outerplanar.

36. For a positive integer k, a graph G of order $n > k$ and size m is said to have property π_k if (1) $m = kn - \binom{k+1}{2}$ and (2) for every induced subgraph H of order p and size q in G, where $k \leq p < n$, it follows that $q \leq kp - \binom{k+1}{2}$.

    (a) Show that $\delta(G) \geq k$.
    (b) Show that $\omega(G) \leq k + 1$.
    (c) What familiar class of graphs has property π_2? Show that there is a graph having property π_2 that does not belong to this class.
    (d) What familiar class of graphs has property π_3? Show that there is a graph having property π_3 that does not belong to this class.

Section 10.4: Hamiltonian Planar Graphs

37. Show, by applying Theorem 10.27, that $K_{2,3}$ is not Hamiltonian.

38. Show, by applying Theorem 10.27, that each of the graphs in Figure 10.23 is not Hamiltonian.

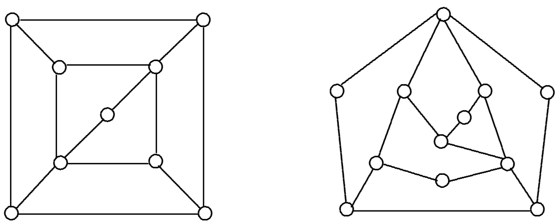


Figure 10.23: Graphs in Exercise 38

39. Show, by applying Theorem 10.27, that the **Grinberg graph** in Figure 10.24 is not Hamiltonian.

40. Show, by applying Theorem 10.27, that the **Herschel graph** in Figure 10.25 is not Hamiltonian.

41. Show, by applying Theorem 10.27, that no Hamiltonian cycle in the graph of Figure 10.26 contains both the edges e and f.

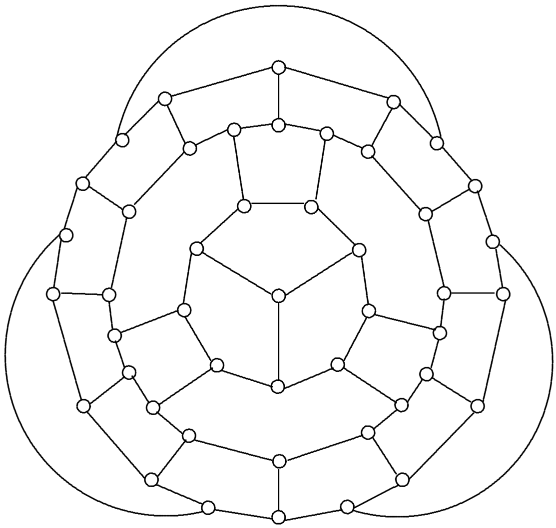


Figure 10.24: The Grinberg graph in Exercise 39

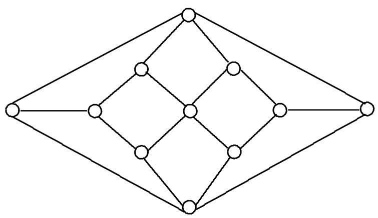


Figure 10.25: The Herschel graph in Exercise 40

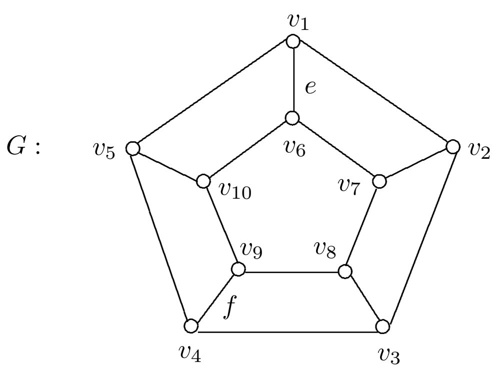


Figure 10.26: A Hamiltonian planar graph in Exercise 41

Chapter 11

Nonplanar Graphs

We saw in Chapter 10 that if G is a maximal planar graph containing two nonadjacent vertices u and v, then the graph $G + uv$ obtained by adding the edge uv is nonplanar. Although nonplanar, the graph $G + uv$ is very close to being planar. We now look at various ways of measuring how close nonplanar graphs are to being planar.

11.1 The Crossing Number of a Graph

Nonplanar graphs cannot, of course, be embedded in the plane. Hence, whenever one attempts to draw a nonplanar graph in the plane, some of its edges must cross. This rather simple observation leads to a concept.

The **crossing number** $\text{cr}(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane. Before proceeding further, we comment on assumptions we are making regarding all drawings under consideration. In particular, we assume that

- adjacent edges never cross
- two nonadjacent edges cross at most once
- no edge crosses itself
- no more than two edges cross at a point of the plane
- the (open) arc in the plane corresponding to an edge of the graph contains no vertex of the graph.

A few observations will prove useful. If $G \subseteq H$, then $\text{cr}(G) \leq \text{cr}(H)$; while if H is a subdivision of G, then $\text{cr}(G) = \text{cr}(H)$. A graph G is planar if and only if $\text{cr}(G) = 0$. In particular, if G is a maximal planar graph of order $n \geq 3$ and size m, then $m = 3n - 6$ and $\text{cr}(G) = 0 = m - 3n + 6$. If $m > 3n - 6$ and so $m - 3n + 6 > 0$, then G is nonplanar and so $\text{cr}(G) \geq 1$. In fact, the number

$m - 3n + 6$ provides a lower bound for the crossing number of a graph of order $n \geq 3$ and size m.

Theorem 11.1 *If G is a graph of order $n \geq 3$ and size m, then*

$$\mathrm{cr}(G) \geq m - 3n + 6.$$

Proof. Let there be given a drawing of G in the plane with $\mathrm{cr}(G) = c$ crossings. At each crossing, a new vertex is introduced, producing a plane graph H of order $n + c$ and size $m + 2c$. Since H is planar, it follows by Theorem 10.3 that

$$m + 2c \leq 3(n + c) - 6.$$

Thus, $\mathrm{cr}(G) = c \geq m - 3n + 6$. ■

While the lower bound for the crossing number of a graph can be useful in determining $\mathrm{cr}(G)$ for certain graphs G, this bound can differ significantly from $\mathrm{cr}(G)$. For example, for large integers s and t, let $H = P_s \,\square\, P_t$. Then this planar graph H can be embedded in the plane where one region has a $(2s + 2t - 4)$-cycle for its boundary, while the boundary of all other regions are 4-cycles. Edges can be added to H to produce a maximal planar graph G. By appropriately selecting nonadjacent vertices x and y of G, it follows that $\mathrm{cr}(G + xy) \geq 1$ by Theorem 11.1 but $G + xy$ can have a large crossing number.

Crossing Numbers of Complete Graphs

One class of graphs whose crossing number has been a subject of study is complete graphs. By Theorem 11.1, it follows for $n \geq 3$ that

$$\mathrm{cr}(K_n) \geq \binom{n}{2} - 3n + 6 = \frac{(n-3)(n-4)}{2}. \qquad (11.1)$$

Richard K. Guy [112] discovered an even better lower bound for $\mathrm{cr}(K_n)$.

Theorem 11.2 *For $n \geq 5$, $\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$.*

Proof. We proceed by induction on n. For $n = 5$, we have $\frac{1}{5}\binom{n}{4} = \frac{1}{5}\binom{5}{4} = 1$. Since K_5 is nonplanar, $\mathrm{cr}(K_5) \geq 1$. In fact, the drawing of K_5 in Figure 11.1 with one crossing shows that $\mathrm{cr}(K_5) = 1$.

Assume that $\mathrm{cr}(K_{n-1}) \geq \frac{1}{5}\binom{n-1}{4}$ for an integer $n \geq 6$. Let there be a drawing of K_n in the plane with $\mathrm{cr}(K_n)$ crossings. When a vertex of K_n is deleted, a drawing of K_{n-1} is obtained, where the number of crossings in this copy of K_{n-1} is at least $\mathrm{cr}(K_{n-1})$. Hence, the n vertex-deleted subgraphs of K_n produce n graphs K_{n-1} having a total of at least $n\,\mathrm{cr}(K_{n-1})$ crossings.

A crossing in K_n involves two nonadjacent edges, say uv and xy. This crossing occurs in every vertex-deleted subgraph K_{n-1} of K_n except when u, v, x

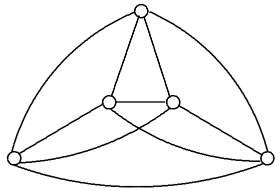


Figure 11.1: A drawing of K_5 with one crossing

or y is deleted; that is, this crossing occurs in $n-4$ subgraphs K_{n-1} of K_n. Thus, the total number of crossings in these n drawings of vertex-deleted subgraphs K_{n-1} of K_n is $(n-4)\,\mathrm{cr}(K_n)$. Hence,

$$(n-4)\,\mathrm{cr}(K_n) \ge n\,\mathrm{cr}(K_{n-1}) \ge \frac{n}{5}\binom{n-1}{4}.$$

Therefore,

$$\mathrm{cr}(K_n) \ge \frac{n}{5(n-4)}\binom{n-1}{4} = \frac{1}{5}\binom{n}{4},$$

as desired. ■

The lower bounds for $\mathrm{cr}(K_n)$ given in (11.1) and in Theorem 11.2 are the same when $n = 5$ and $n = 6$, while

$$\frac{1}{5}\binom{n}{4} > \frac{(n-3)(n-4)}{2}$$

when $n \ge 7$. Thus the lower bound $\mathrm{cr}(K_n) \ge \frac{1}{5}\binom{n}{4}$ is an improvement over that given in (11.1). When $n = 6$, these bounds state that $\mathrm{cr}(K_6) \ge 3$. The drawing of K_6 with three crossings in Figure 11.2 shows that $\mathrm{cr}(K_6) = 3$.

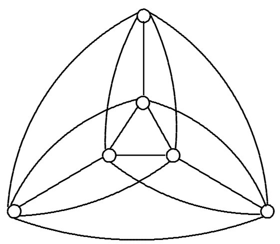


Figure 11.2: A drawing of K_6 with three crossings

It has been shown by Jaroslav Blažek and Milan Koman [30] and Richard K. Guy [111], among others, that for complete graphs,

$$\mathrm{cr}(K_n) \le \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{11.2}$$

Guy conjectured, in fact, that the upper bound in (11.2) is, in fact, the crossing number of K_n for all positive integers n. As far as known results are concerned, the best obtained is the following.

Theorem 11.3 *For* $1 \leq n \leq 12$*,*

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \qquad (11.3)$$

Guy [112] established Theorem 11.3 for $1 \leq n \leq 10$, while Shengjun Pan and R. Bruce Richter [182] verified Theorem 11.3 for $n = 11$ and $n = 12$.

Turán's Brick-Factory Problem

We now turn to the crossing number of complete bipartite graphs. The problem of determining $\mathrm{cr}(K_{s,t})$ has a memorable history. This problem is sometimes referred to as **Turán's Brick-Factory Problem**, named for the Hungarian mathematician Paul Turán (1910–1976).

Born in Budapest, Hungary, Paul Turán displayed remarkable mathematical ability at a very early age. Turán was one of many young students in Budapest who studied graph theory under Dénes König. Turán met Paul Erdős in September 1930 and became and remained friends with him. Turán received a Ph.D. in 1935 from the University of Budapest, with a dissertation in number theory. By the end of 1935, Turán had seven papers in print. Despite his outstanding record at such an early age, Turán had great difficulty securing a faculty position because of his Jewish heritage. He could only make a living as a private mathematics tutor although he continued his research.

While Turán finally secured a position in 1938 (as a school teacher), his personal situation had grown worse. After the German invasion of Poland which began World War II, Hungary was not involved in the war at first but was nevertheless greatly influenced by Nazi policies. In 1940 Turán was sent to a labor camp. Indeed, Turán was in and out of several labor camps during the war. Turán himself [238] wrote:

> *We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was precious to all of us. We were all sweating and cursing at such*

occasions, I too; but nolens volens the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with s kilns and t storage yards seemed to be very difficult ... the problem occurred to me again ... at my first visit to Poland where I met Zarankiewicz. I mentioned to him my 'brick-factory'–problem ... and Zarankiewicz thought to have solved (it). But Ringel found a gap in his published proof, which nobody has been able to fill so far – in spite of much effort. This problem has also become a notoriously difficult unsolved problem ....

Crossing Numbers of Complete Bipartite Graphs

Kazimierz Zarankiewicz [262] thought he had proved that

$$\mathrm{cr}(K_{s,t}) = \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor \tag{11.4}$$

but, in actuality, he had only verified that the right-hand expression of (11.4) is an upper bound for $\mathrm{cr}(K_{s,t})$. As it turned out, both Paul C. Kainen and Gerhard Ringel found flaws in Zarankiewicz's argument. Hence, (11.4) remains only a conjecture. The best general result on crossing numbers of complete bipartite graphs is the following, due to the combined work of Daniel Kleitman [144] and Douglas Woodall [260].

Theorem 11.4 *If s and t are positive integers with $s \le t$ and either* (1) $s \le 6$ *or* (2) $s = 7$ *and $t \le 10$, then*

$$\mathrm{cr}(K_{s,t}) = \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor .$$

It follows, therefore, from Theorem 11.4 that

$$\mathrm{cr}(K_{3,t}) = \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor, \qquad \mathrm{cr}(K_{4,t}) = 2\left\lfloor \tfrac{t}{2} \right\rfloor \left\lfloor \tfrac{t-1}{2} \right\rfloor,$$

$$\mathrm{cr}(K_{5,t}) = 4\left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor \quad \text{and} \quad \mathrm{cr}(K_{6,t}) = 6\left\lfloor \tfrac{t}{2} \right\rfloor \left\lfloor \tfrac{t-1}{2} \right\rfloor$$

for all t. For example, $\mathrm{cr}(K_{3,3}) = 1$, $\mathrm{cr}(K_{4,4}) = 4$, $\mathrm{cr}(K_{5,5}) = 16$, $\mathrm{cr}(K_{6,6}) = 36$ and $\mathrm{cr}(K_{7,7}) = 81$. A drawing of $K_{4,4}$ with four crossings is shown in Figure 11.3.

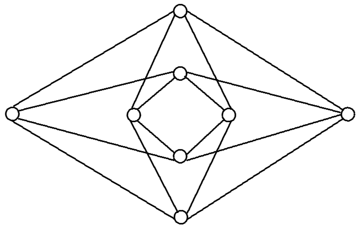


Figure 11.3: A drawing of $K_{4,4}$ with four crossings

As would be expected, the situation regarding crossing numbers of complete k-partite graphs, $k \geq 3$, is even more complicated. For the most part, only bounds and highly specific results have been obtained in these cases. On the other hand, some of the proof techniques employed have been enlightening. As an example, the following result of Arthur T. White [254, p. 67] establishes the crossing number of $K_{2,2,3}$.

Theorem 11.5 *The crossing number of* $K_{2,2,3}$ *is* 2.

Proof. The graph $K_{2,2,3}$ has order 7 and size 16. Suppose that $\text{cr}(K_{2,2,3}) = c$. Since $K_{3,3}$ is nonplanar and $K_{3,3} \subseteq K_{2,2,3}$, it follows that $K_{2,2,3}$ is nonplanar and so $c \geq 1$. Let there be given a drawing of $K_{2,2,3}$ in the plane with c crossings. At each crossing we introduce a new vertex, producing a connected plane graph G of order $n = 7 + c$ and size $m = 16 + 2c$. By Theorem 10.3, $m \leq 3n - 6$.

Let u_1u_2 and v_1v_2 be two (nonadjacent) edges of $K_{2,2,3}$ that cross in the given drawing, giving rise to a new vertex. If G is a maximal planar graph, then $C = (u_1, v_1, u_2, v_2, u_1)$ is a cycle of G, implying that the subgraph induced by $\{u_1, u_2, v_1, v_2\}$ in $K_{2,2,3}$ is K_4. However, $K_{2,2,3}$ contains no such subgraph; thus, G is not a maximal planar graph and so $m < 3n - 6$. Therefore,

$$16 + 2c < 3(7 + c) - 6,$$

from which it follows that $c \geq 2$. The inequality $c \leq 2$ follows from the fact that there exists a drawing of $K_{2,2,3}$ with two crossings (see Figure 11.4). ■

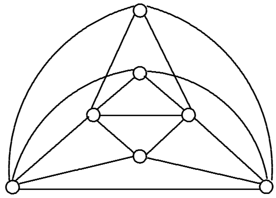


Figure 11.4: A drawing of $K_{2,2,3}$ with two crossings

Crossing Numbers of Cartesian Products of Graphs

Other graphs whose crossing numbers have been investigated with little success are the n-cubes Q_n. Since Q_n is planar for $n = 1, 2, 3$, $\text{cr}(Q_n) = 0$ for each such n. Roger Eggleton and Richard Guy [76] have shown that $\text{cr}(Q_4) = 8$ but $\text{cr}(Q_n)$ is unknown for $n \geq 5$. One might observe that

$$Q_4 = K_2 \square K_2 \square K_2 \square K_2 = C_4 \square C_4$$

so that $\text{cr}(C_4 \square C_4) = 8$. This raises the question of determining $\text{cr}(C_s \square C_t)$ for $s, t \geq 3$. For the case $s = t = 3$, Frank Harary, Paul Kainen and Allen Schwenk [120] showed that $\text{cr}(C_3 \square C_3) = 3$. Their proof consisted of the following three steps:

Step 1. Exhibiting a drawing of $C_3 \square C_3$ with three crossings so that $\text{cr}(C_3 \square C_3) \leq 3$.

Step 2. Showing that $C_3 \square C_3 - e$ is nonplanar for every edge e of $C_3 \square C_3$ so that $\text{cr}(C_3 \square C_3) \geq 2$.

Step 3. Showing, by case exhaustion, that it is impossible to have a drawing of $C_3 \square C_3$ with exactly two crossings so that $\text{cr}(C_3 \square C_3) \geq 3$ (see Exercise 6).

Richard D. Ringeisen and Lowell W. Beineke [200] then significantly extended this result by determining $\text{cr}(C_3 \square C_t)$ for all integers $t \geq 3$.

Theorem 11.6 *For each integer $t \geq 3$,*

$$\text{cr}(C_3 \square C_t) = t.$$

Proof. We label the vertices of $C_3 \square C_t$ by the $3t$ ordered pairs $(0, j), (1, j)$ and $(2, j)$, where $j = 0, 1, \ldots, t-1$, and, for convenience, we let

$$u_j = (0, j),\ v_j = (1, j) \text{ and } w_j = (2, j).$$

First, we note that $\text{cr}(C_3 \square C_t) \leq t$. This observation follows from the fact that there exists a drawing of $C_3 \square C_t$ with t crossings. A drawing of $C_3 \square C_4$ with four crossings is shown in Figure 11.5. Drawings of $C_3 \square C_t$ with t crossings for other values of t can be given similarly.

To complete the proof, we show that $\text{cr}(C_3 \square C_t) \geq t$. We verify this by induction on $t \geq 3$. For $t = 3$, we recall the previously mentioned result $\text{cr}(C_3 \square C_3) = 3$.

Assume that $\text{cr}(C_3 \square C_k) \geq k$ for some integer $k \geq 3$, and consider the graph $C_3 \square C_{k+1}$. We show that $\text{cr}(C_3 \square C_{k+1}) \geq k + 1$. Let there be given a drawing of $C_3 \square C_{k+1}$ with $\text{cr}(C_3 \square C_{k+1})$ crossings. We consider two cases.

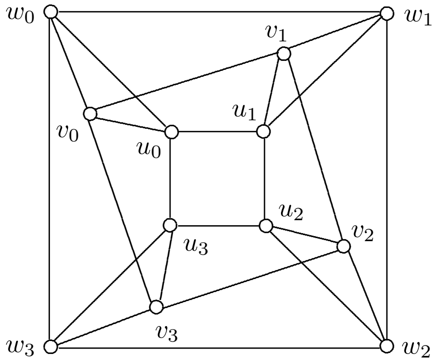


Figure 11.5: A drawing of $C_3 \,\Box\, C_4$ with four crossings

Case 1. *No edge of any triangle* $T_j = G[\{u_j, v_j, w_j\}]$, $j = 0, 1, \ldots, k$, *is crossed.* For $j = 0, 1, \ldots, k$, define

$$H_j = G[\{u_j, v_j, w_j, u_{j+1}, v_{j+1}, w_{j+1}\}],$$

where the subscripts are expressed modulo $k+1$. We show that for each $j = 0, 1, \ldots, k$, the number of times edges of H_j are crossed totals at least 2. Since, by assumption, no triangle T_j has an edge crossed and since every edge not in any triangle T_j belongs to exactly one subgraph H_j, it will follow that there are at least $k+1$ crossings in the drawing because then every crossing of an edge in H_j involves either two edges of H_j or an edge of H_j and an edge of H_i for some $i \neq j$.

If two of the edges $u_j u_{j+1}$, $v_j v_{j+1}$ and $w_j w_{j+1}$ cross each other, then two edges of H_j are crossed. Assume then that no two edges of H_j cross each other. Thus, H_j is a plane subgraph in the drawing of $C_3 \,\Box\, C_{k+1}$ (see Figure 11.6). The triangle T_{j+2} must lie within some region of H_j. If T_{j+2} lies in a region of H_j bounded by a triangle, say T_j, then at least one edge of the cycle $(u_0, u_1, \ldots, u_k, u_0)$, for example, must cross an edge of T_j, contradicting our assumption. Thus, T_{j+2} must lie in a region of H_j bounded by a 4-cycle, say $(u_j, u_{j+1}, w_{j+1}, w_j, u_j)$. However then, edges of the cycle $(v_0, v_1, \ldots, v_k, v_0)$ must cross edges of the cycle $(u_j, u_{j+1}, w_{j+1}, w_j, u_j)$ at least twice and, hence, edges of H_j at least twice, as asserted.

Case 2. *Some triangle, say* T_0, *has at least one of its edges crossed.* Suppose that $\mathrm{cr}(C_3 \,\Box\, C_{k+1}) < k+1$. Then the graph $C_3 \,\Box\, C_{k+1} - E(T_0)$, which is a subdivision of $C_3 \,\Box\, C_k$, is drawn with fewer than k crossings, contradicting the inductive hypothesis. ■

The only other result giving the crossing number of graphs $C_s \,\Box\, C_t$ is the following theorem of Beineke and Ringeisen [19].

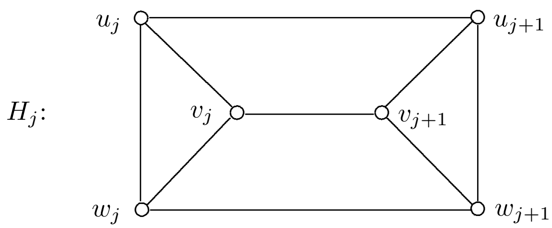


Figure 11.6: The subgraph H_j in the proof of Theorem 11.6

Theorem 11.7 *For all* $t \geq 4$,

$$\text{cr}(C_4 \,\square\, C_t) = 2t.$$

Beineke and Ringeisen [19] also determined $\text{cr}(K_4 \,\square\, C_t)$ for $t \geq 4$.

Theorem 11.8 *For all* $t \geq 4$,

$$\text{cr}(K_4 \,\square\, C_t) = 3t.$$

Fáry's Theorem

In a planar embedding of a graph G, an edge of G can be any curve, including a straight-line segment. The **rectilinear crossing number** $\overline{\text{cr}}(G)$ of a graph G is the minimum number of crossings among all those drawings of G in the plane in which each edge is a straight-line segment. Since the crossing number $\text{cr}(G)$ considers all drawings of G in the plane (not just those for which edges are straight-line segments), we have the obvious inequality

$$\text{cr}(G) \leq \overline{\text{cr}}(G). \tag{11.5}$$

Clearly, $\overline{\text{cr}}(G) \geq 0$ for every planar graph G. However, an interesting feature of planar graphs is that they can be embedded in the plane so that every edge is a straight-line segment. Such an embedding is referred to as a **straight-line embedding**. This result is known as Fáry's theorem but was proved independently not only by István Fáry [89] but by Sherman K. Stein [225] and Klaus Wagner [250] as well.

Theorem 11.9 (**Fáry's Theorem**) *If G is a planar graph, then*

$$\overline{\text{cr}}(G) = 0.$$

Proof. If the rectilinear crossing number of every maximal planar graph is 0, then the rectilinear crossing number of every planar graph is 0. Hence, it suffices to prove the theorem for maximal planar graphs. This result is obvious for K_1 and K_2.

We prove by induction on $n \geq 3$ that for every maximal plane graph G of order n, the boundary of whose exterior region contains the vertices u, v and w, there exists a straight-line embedding of G, each region of which has the same boundary as the given planar embedding of G, and whose exterior region has boundary vertices u, v and w. The result is certainly true for $n = 3$ and $n = 4$. Assume that the statement is true for all maximal plane graphs of order k for some integer $k \geq 4$. Let G be a maximal plane graph of order $k + 1$ whose exterior region has boundary vertices u, v and w.

By Corollary 10.5, G contains a vertex $x \notin \{u, v, w\}$ such that $\deg x = r$ and $3 \leq r \leq 5$. Let $N_G(x) = \{x_1, x_2, \ldots, x_r\}$. Remove x from G and let R be the region of $G - x$ whose boundary vertices are $N_G(x)$. Add $r - 3$ edges to the region R of $G - x$ so that a maximal plane graph G' results. Since G' is a maximal plane graph of order k, it follows by the induction hypothesis that there is a straight-line embedding of G' resulting in a graph G'' each region of which has the same boundary and whose exterior region has boundary vertices u, v and w.

Now remove the $r-3$ edges that were added to $G-x$ to produce a straight-line embedding G^* of $G - x$ such that the boundary of the region R^* with boundary vertices $N_G(x)$ is an r-gon, where $3 \leq r \leq 5$. If the r-gon is convex, then the vertex x can be added anywhere in R^* and joined to the vertices of $N_G(x)$ by straight-line segments, producing a straight-line embedding of G.

Suppose that the r-gon P is not convex and so $r = 4$ or $r = 5$. We then triangulate P by adding $r-3$ straight-line segments. If $r = 4$, then one straight-line segment is added. Place a new vertex x on this segment and remove this segment. Straight-line segments can then be drawn from x to each of the vertices x_i $(1 \leq i \leq 4)$, producing a straight-line embedding of G. See Figure 11.7.

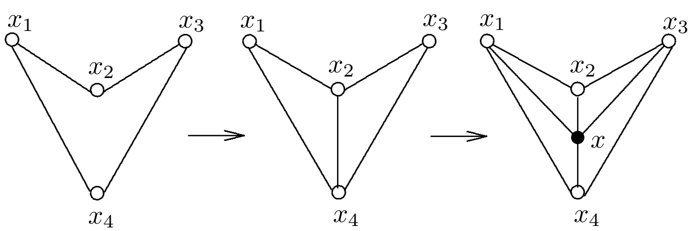


Figure 11.7: A step $(r = 4)$ in the proof of Theorem 11.9

If P is a pentagon, then two straight-line segments can be added to triangulate P, producing three triangles. One of the five vertices of P, say x_1, lies on each of these three triangles. One of these triangles, say T, is a neighbor of the other two triangles. Since a triangle is convex, a straight-line segment can be drawn in T and so in P to each of x_2, x_3, x_4 and x_5 (see Figure 11.8), again producing a straight-line embedding of G. ■

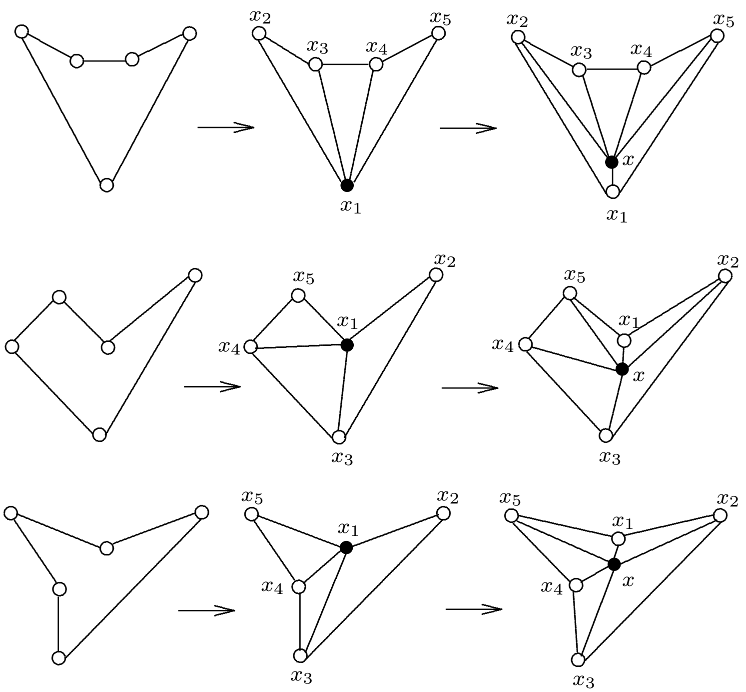


Figure 11.8: A step ($r = 5$) in the proof of Theorem 11.9

The Art Gallery Problem

By Fáry's Theorem, there is a straight-line embedding of every planar graph. In the proof of this theorem, we used the fact that within the interior of each triangle, quadrilateral and pentagon, a point can be placed that can be joined by straight-line segments to the vertices of the polygon P so that they lie within the interior of P. Actually this is a special case of a result from geometry:

The Art Gallery Problem

> *Suppose that a certain art gallery consists of a single large room with n walls on which paintings are hung. What is the minimum number of security guards that must be hired and stationed in the gallery to guarantee that for every painting hung on a wall there is a guard who has straight line vision of the artwork?*

This problem was posed in 1973 by the geometer Victor Klee after a discussion with Vašek Chvátal. It was shown by Chvátal [56], with a simpler proof by Steve Fisk [91], that no more than $\lfloor n/3 \rfloor$ guards are needed and that examples exist where $\lfloor n/3 \rfloor$ guards are required. In the case of an art gallery with three, four or five walls, only one guard need be hired, which is the fact used in the proof of Theorem 11.9.

It has been conjectured that $\text{cr}(K_{s,t}) = \overline{\text{cr}}(K_{s,t})$ for all positive integers s and t. In the case of complete graphs, not only is $\overline{\text{cr}}(K_n) = \text{cr}(K_n)$ for $1 \le n \le 4$ (when K_n is planar), it is also known that $\overline{\text{cr}}(K_n) = \text{cr}(K_n)$ for $5 \le n \le 7$ and $n = 9$. However, $\text{cr}(K_8) = 18$ and $\overline{\text{cr}}(K_8) = 19$ (see Guy [112]), so strict inequality in (11.5) is indeed a possibility. Furthermore, Alex Brodsky, Stephane Durocher and Ellen Gethner [37] showed that $\overline{\text{cr}}(K_{10}) = 62$, while $\text{cr}(K_{10}) = 60$. The members Oswin Aichholzer, Franz Aurenhammer and Hannes Krasser of the *Rectilinear Crossing Number Project* determined $\overline{\text{cr}}(K_n)$ for all n with $11 \le n \le 21$ except $n = 20$. In particular

- $\text{cr}(K_{11}) = 100$ and $\overline{\text{cr}}(K_{11}) = 102$ and
- $\text{cr}(K_{12}) = 150$ and $\overline{\text{cr}}(K_{12}) = 153$.

In a paper by Bernardo M. Ábrego, Silvia Fernández-Merchant and Gelasio Salazar [1], the values of $\overline{\text{cr}}(K_n)$ are given for $5 \le n \le 30$, including $\overline{\text{cr}}(K_{30}) = 9726$.

11.2 The Genus of a Graph

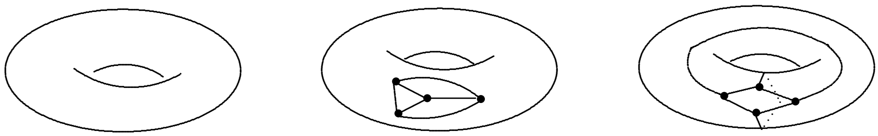
While only planar graphs can be embedded in the plane, there is a host of common and increasingly complex surfaces on which a nonplanar graph G might possibly be embedded. Determining the simplest of these surfaces on which G can be embedded gives an indication of how close G is to being planar. We will see that the embeddings of interest are those called 2-cell embeddings and discuss the possible surfaces on which a given graph has such an embedding.

We have seen that a graph G is planar if G can be drawn in the plane in such a way that no two edges cross and that such a drawing is called an embedding of G in the plane or a planar embedding. Furthermore, a graph G can be embedded in the plane if and only if G can be embedded on (the surface of) a sphere.

Of course, not all graphs are planar. Indeed, Kuratowski's theorem (Theorem 10.18) and Wagner's theorem (Theorem 10.21) describe conditions (involving the two nonplanar graphs K_5 and $K_{3,3}$) under which G can be embedded in the plane. Graphs that are not embeddable in the plane (or on a sphere) may be embeddable on other surfaces, however.

Embeddings on a Torus

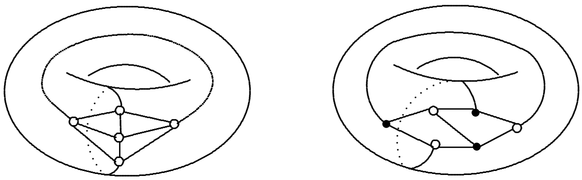
A common surface on which a graph may be embedded is the **torus**, a doughnut-shaped surface (see Figure 11.9(a)). Two different embeddings of the (planar) graph K_4 on a torus are shown in Figures 11.9(b) and 11.9(c).



(a) (b) (c)

Figure 11.9: Embedding K_4 on a torus

While it is easy to see that every planar graph can be embedded on a torus, some nonplanar graphs can be embedded on a torus as well. For example, embeddings of K_5 and $K_{3,3}$ on a torus are shown in Figures 11.10(a) and 11.10(b).



(a) (b)

Figure 11.10: Embedding K_5 and $K_{3,3}$ on a torus

Another way to represent a torus and to visualize an embedding of a graph on a torus is to begin with a rectangular piece of (flexible) material as in Figure 11.11 and first make a cylinder from it by identifying sides a and c, which are the same after the identification occurs. Sides b and d are then circles. These circles are then identified to produce a torus.

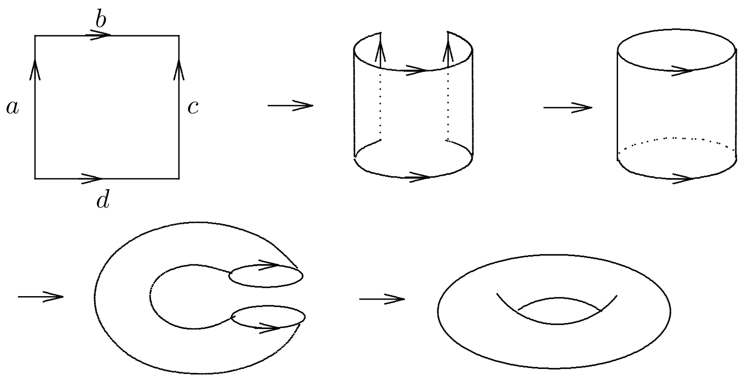
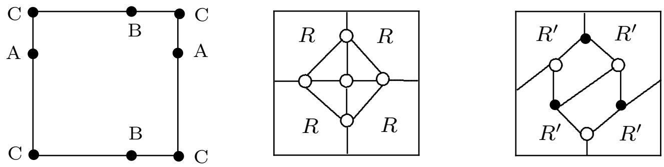


Figure 11.11: Constructing a torus

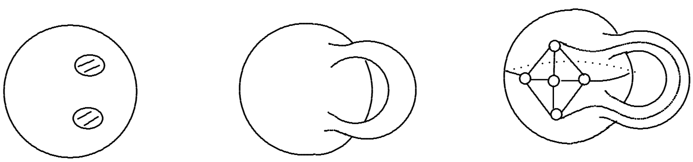
After seeing how a torus can be constructed from a rectangle, it follows that the points labeled A in the rectangle in Figure 11.12(a) represent the same point on the torus. This is also true of the points labeled B and the points labeled C. Figures 11.12(b) and 11.12(c) show embeddings of K_5 and $K_{3,3}$ on the torus. There are five regions in the embedding of K_5 on the torus shown in Figure 11.12(b) and R is a single region in this embedding. Moreover, there are three regions in the embedding of $K_{3,3}$ on the torus shown in Figure 11.12(c) and R' is a single region.



(a) (b) (c)

Figure 11.12: Embedding K_5 and $K_{3,3}$ on a torus

Another way to represent a torus and an embedding of a graph on a torus is to begin with a sphere, insert two holes in its surface (as in Figure 11.13(a)) and attach a handle on the sphere, where the ends of the handle are placed over the two holes (as in Figure 11.13(b)). An embedding of K_5 on the torus constructed in this manner is shown in Figure 11.13(c).



(a) (b) (c)

Figure 11.13: Embedding K_5 on a torus

While a torus is a sphere with one handle, a sphere with k handles, $k \geq 0$, is called a **surface of genus** k and is denoted by S_k. Thus, S_0 is a sphere and S_1 is a torus. The surfaces S_k are the **orientable surfaces**.

Let G be a nonplanar graph. When drawing G on a sphere, some edges of G will cross. The graph G can always be drawn so that only two edges cross at any point of intersection. At each such point of intersection, a handle can be suitably placed on the sphere so that one of these two edges passes over the handle and the intersection of the two edges has been avoided. Consequently, every graph can be embedded on some orientable surface. The smallest nonnegative integer k such that a graph G can be embedded on S_k is called the **genus** of G and is denoted by $\gamma(G)$. Therefore, $\gamma(G) = 0$ if and only if G is planar; while

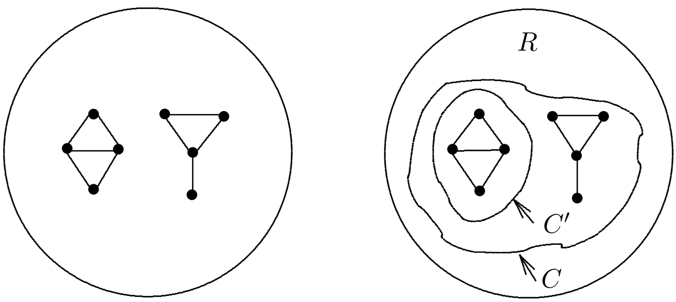
$\gamma(G) = 1$ if and only if G is nonplanar but G can be embedded on the torus. An embedding of a graph G on the torus is called a **toroidal embedding** of G. In particular,

$$\gamma(K_5) = 1 \text{ and } \gamma(K_{3,3}) = 1.$$

Figure 11.14(a) shows an embedding of a disconnected graph H on a sphere. In this case, $n = 8$, $m = 9$ and $r = 4$. Thus $n - m + r = 8 - 9 + 4 = 3$. That $n - m + r \neq 2$ is not particularly surprising as the Euler Identity (Theorem 10.1) requires that H be a *connected* plane graph. Although this is a major reason why we will restrict our attention to connected graphs here, it is not the only reason. There is a desirable property possessed by every embedding of a connected planar graph on a sphere that is possessed by no embedding of a disconnected planar graph on a sphere.

2-Cell Embeddings of Graphs

Suppose that G is a graph embedded on a surface S_k, $k \geq 0$. A region of this embedding is a **2-cell** if every closed curve in that region can be continuously deformed in that region to a single point. (Topologically, a region is a 2-cell if it is homeomorphic to a disk.) While the closed curve C in R in the embedding of the graph on a sphere shown in Figure 11.14(b) can in fact be continuously deformed in R to a single point, the curve C' cannot. Hence R is not a 2-cell in this embedding.



(a) (b)

Figure 11.14: An embedding on a sphere that is not a 2-cell embedding

An embedding of a graph G on some surface is a **2-cell embedding** if every region in the embedding is a 2-cell. Consequently, the embedding of the graph shown in Figure 11.14(a) is not a 2-cell embedding. It turns out, however, that every embedding of a connected graph on a sphere is necessarily a 2-cell embedding. Of course, such a graph is necessarily planar. If a connected graph is embedded on a surface S_k where $k > 0$, then the embedding may or may not be a 2-cell embedding, however. For example, the embedding of K_4 in Figure 11.9(b) is not a 2-cell embedding. The curves C and C' shown in Figures 11.15(a) and 11.15(b) cannot be continuously deformed to a single point in the region in which these curves are drawn. On the other hand, the embedding of K_4 shown in Figure 11.9(c) and shown again in Figure 11.15(c) is a 2-cell embedding.

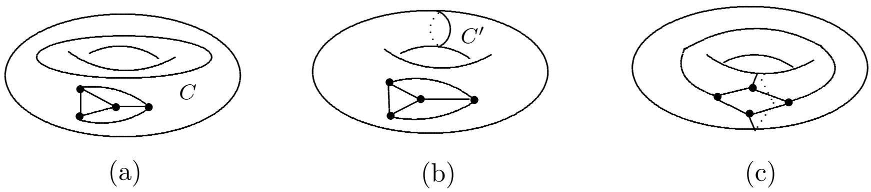


Figure 11.15: Non-2-cell and 2-cell embeddings of K_4 on the torus

The Generalized Euler Identity

The embeddings of K_4, K_5 and $K_{3,3}$ on a torus given in Figures 11.9(c), 11.10(a) and 11.10(b), respectively, are all 2-cell embeddings. Furthermore, in each case, $n - m + r = 0$. As it turns out, if G is a connected graph of order n and size m that is 2-cell embedded on a torus resulting in r regions, then it is always the case that $n - m + r = 0$. This fact together with the Euler Identity (Theorem 10.1) are special cases of a more general result. The mathematician Simon Antoine Jean Lhuilier spent much of his life working on problems related to the Euler Identity. Lhuilier, like Euler, was from Switzerland and was taught mathematics by one of Euler's former students (Louis Bertrand). Lhuilier saw that the Euler Identity did *not* hold for graphs embedded on spheres containing handles. In fact, he proved a more general form of this identity [156].

Theorem 11.10 (**The Generalized Euler Identity**) *If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus $k \geq 0$, resulting in r regions, then*

$$n - m + r = 2 - 2k.$$

Proof. We proceed by induction on k. If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus 0, then G is a plane graph. By the Euler Identity, $n - m + r = 2 = 2 - 2 \cdot 0$. Thus the basis step of the induction holds.

Assume, for every connected graph G' of order n' and size m' that is 2-cell embedded on a surface S_k for some nonnegative integer k, resulting in r' regions, that

$$n' - m' + r' = 2 - 2k.$$

Let G be a connected graph of order n and size m that is 2-cell embedded on S_{k+1}, resulting in r regions. We may assume, without loss of generality, that no vertex of G lies on any handle of S_{k+1} and that the edges of G are drawn on the handles so that a closed curve can be drawn around each handle that intersects no edge of G more than once.

Let H be one of the $k+1$ handles of S_{k+1}. There are necessarily edges of G on H; for otherwise, the handle belongs to a region R in which case any closed curve around H cannot be continuously deformed in R to a single point, contradicting the assumption that R is a 2-cell. We now draw a closed curve C around H, which intersects some edges of G on H but intersects no edge more than once. Suppose that there are $t \geq 1$ points of intersection of C and the edges on H. At each point of intersection, a new vertex is introduced. Each of the t edges then becomes two edges. Also, the segments of C between vertices become edges. We add two vertices of degree 2 along C to produce two additional edges. (This guarantees that the resulting structure will be a graph, not a multigraph.)

Let G_1 be the graph just constructed, where G_1 has order n_1, size m_1 and r_1 regions. Then

$$n_1 = n + t + 2 \text{ and } m_1 = m + 2t + 2.$$

Since each portion of C that became an edge of G_1 is in a region of G, the addition of such an edge divides that region into two regions, each of which is a 2-cell. Since there are t such edges,

$$r_1 = r + t.$$

We now cut the handle H along C and "patch" the two resulting holes, producing two duplicate copies of the vertices and edges along C (see Figure 11.16). Denote the resulting graph by G_2, which is now 2-cell embedded on a surface S_k.

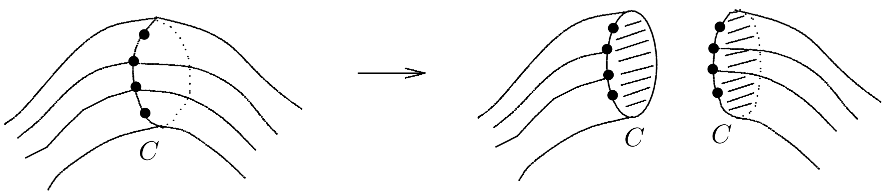


Figure 11.16: Converting a 2-cell embedding of G_1 on S_{k+1} into a 2-cell embedding of G_2 on S_k

Let G_2 have order n_2, size m_2 and r_2 regions, all of which are 2-cells. Then

$$n_2 = n_1 + t + 2,\ m_2 = m_1 + t + 2 \text{ and } r_2 = r_1 + 2.$$

Furthermore, $n_2 = n + 2t + 4$, $m_2 = m + 3t + 4$ and $r_2 = r + t + 2$. By the induction hypothesis, $n_2 - m_2 + r_2 = 2 - 2k$. Therefore,

$$\begin{aligned} n_2 - m_2 + r_2 &= (n + 2t + 4) - (m + 3t + 4) + (r + t + 2) \\ &= n - m + r + 2 = 2 - 2k. \end{aligned}$$

Therefore, $n - m + r = 2 - 2(k + 1)$. ∎

We noted that every embedding of a connected planar graph G in the plane is always a 2-cell embedding of G. This fact is a special case of a useful result obtained by J. W. T. (Ted) Youngs [261].

Theorem 11.11 *Every embedding of a connected graph G of genus k on S_k, where k is a nonnegative integer, is a 2-cell embedding of G on S_k.*

Lower Bounds for the Genus of a Graph

With the aid of Theorems 11.10 and 11.11, we have the following.

Corollary 11.12 *If G is a connected graph of order n and size m that is embedded on a surface of genus $\gamma(G)$, resulting in r regions, then*

$$n - m + r = 2 - 2\gamma(G).$$

The following result is a consequence of Corollary 11.12.

Theorem 11.13 *If G is a connected graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1.$$

Proof. Since the result is obviously true for $n = 3$, we may assume that $n \geq 4$. Suppose that G is embedded on a surface of genus $\gamma(G)$, resulting in r regions. By Corollary 11.12, $n - m + r = 2 - 2\gamma(G)$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and let m_i be the number of edges on the boundary of R_i $(1 \leq i \leq r)$. Thus, $m_i \geq 3$ for $1 \leq i \leq r$. Since every edge is on the boundary of either one or two regions, it follows that

$$3r \leq \sum_{i=1}^{r} m_i \leq 2m$$

and so $3r \leq 2m$. Therefore,

$$6 - 6\gamma(G) = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m. \tag{11.6}$$

Solving (11.6) for $\gamma(G)$, we have $\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1$. ∎

Theorem 11.13 is a generalization of Theorem 10.3, for when G is planar (and so $\gamma(G) = 0$) Theorem 11.13 becomes Theorem 10.3. The lower bound for $\gamma(G)$ presented in Theorem 11.13 can be improved when more information on cycle lengths in G is available (see Exercise 13).

Theorem 11.14 *If G is a connected graph of order n, size m and girth k, then*

$$\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1.$$

The following result is a consequence of Theorem 11.14 that includes bipartite graphs as a special case. Recall that a graph is called triangle-free if it contains no triangles.

Corollary 11.15 *If G is a connected, triangle-free graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{4} - \frac{n}{2} + 1.$$

While there is no general formula for the genus of an arbitrary graph, the following result by Joseph Battle, Frank Harary and Yukihiro Kodama [13] implies that, as far as genus formulas are concerned, only 2-connected graphs need be investigated.

Theorem 11.16 *If G is a graph having blocks B_1, B_2, …, B_k, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(B_i).$$

The following corollary is a consequence of the preceding result (see Exercise 14).

Corollary 11.17 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(G_i).$$

The Genus of Some Well-Known Graphs

As is often the case when no general formula exists for the value of a parameter for an arbitrary graph, formulas (or partial formulas) are established for certain families of graphs. Ordinarily the first classes to be considered are the complete graphs, the complete bipartite graphs and the n-cubes. The genus offers no exception to this statement.

According to Theorem 11.13,

$$\gamma(K_5) \geq \tfrac{1}{6},\ \gamma(K_6) \geq \tfrac{1}{2} \ \text{ and } \ \gamma(K_7) \geq 1.$$

This says that all three graphs K_5, K_6 and K_7 are nonplanar. Of course, we already knew this by Corollary 10.8. Because K_5 is nonplanar, so too are K_6 and K_7. We have seen that $\gamma(K_5) = 1$. Actually, $\gamma(K_7) = 1$ as well.

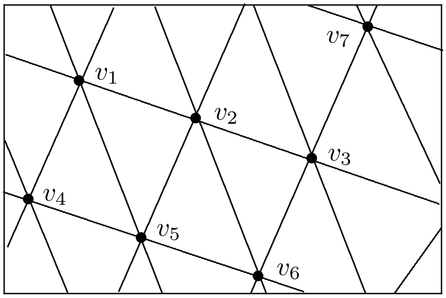


Figure 11.17: An embedding of K_7 on the torus

Figure 11.17 shows an embedding of K_7 on the torus where the vertex set of K_7 is $\{v_1, v_2, \ldots, v_7\}$. Because K_6 is nonplanar and K_6 is a subgraph of a graph that can be embedded on a torus, $\gamma(K_6) = 1$.

Applying Theorem 11.13 to the complete graph K_n, $n \geq 3$, we have

$$\gamma(K_n) \geq \frac{\binom{n}{2}}{6} - \frac{n}{2} + 1 = \frac{(n-3)(n-4)}{12}.$$

Since $\gamma(K_n)$ is an integer,

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil. \tag{11.7}$$

Born in India, Ted Youngs (1910–1970) received his Ph.D. in 1934. In 1964 he was appointed one of the first faculty members of the University of California Santa Cruz. Born in Austria, Gerhard Ringel (1919–2008) was one of the pioneers of graph theory. He received his Ph.D. from the University of Bonn in Germany in 1951. In addition to his mathematical skills, Ringel was well known for his interest in collecting and breeding butterflies. Ringel left Germany in 1970 to join Youngs at the University of California Santa Cruz in order to complete a proof of a famous theorem in graph theory, which will be discussed in Chapter 16. However, the truth of this theorem required showing that the lower bound for $\gamma(K_n)$ in (11.7) is, in fact, the value of $\gamma(K_n)$, which Ringel and Youngs were successful in accomplishing [204].

Theorem 11.18 *For every integer $n \geq 3$,*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Ringel [203] was also successful in obtaining a formula for the genus of every complete bipartite graph.

Theorem 11.19 *For every two integers $s, t \geq 2$,*

$$\gamma(K_{s,t}) = \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil.$$

In particular, Theorem 11.19 implies that a complete bipartite graph G can be embedded on a torus if and only if G is planar or is a subgraph of $K_{4,4}$ or $K_{3,6}$.

A formula for the genus of the n-cube was found by Ringel [201] and by Beineke and Harary [18]. We prove this result to illustrate some of the techniques involved.

Theorem 11.20 *For $n \geq 2$, the genus of the n-cube is given by*

$$\gamma(Q_n) = (n-4) \cdot 2^{n-3} + 1.$$

Proof. Since the n-cube is a triangle-free graph of order 2^n and size $n \cdot 2^{n-1}$, it follows by Corollary 11.15 that

$$\gamma(Q_n) \geq (n-4) \cdot 2^{n-3} + 1.$$

To verify the reverse inequality, we employ induction on n. In fact, we show for every integer $n \geq 2$, that there is an embedding of Q_n on the surface of genus $(n-4) \cdot 2^{n-3} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{n-2} regions with pairwise disjoint boundaries. Since Q_2 and Q_3 are planar and $(n-4) \cdot 2^{n-3} + 1 = 0$ for $n = 2$ and $n = 3$, this is certainly true for $n = 2$ and $n = 3$.

Assume for an integer $k \geq 4$ that there is an embedding of Q_{k-1} on the surface S of genus $(k-5) \cdot 2^{k-4} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{k-3} regions with pairwise disjoint boundaries. Since the order of Q_{k-1} is 2^{k-1}, each vertex of Q_{k-1} belongs to the boundary of precisely one of the aforementioned 2^{k-3} regions. Furthermore, let Q_{k-1} be embedded on another copy S' of the surface of genus $(k-5) \cdot 2^{k-4} + 1$ such that the embedding of Q_{k-1} on S' is a "mirror image" of the embedding of Q_{k-1} on S (that is, if v_1, v_2, v_3, v_4 are the vertices of the boundary of a region of Q_{k-1} on S, where the vertices are listed clockwise about the 4-cycle, then there is a region on S' with the vertices v_1, v_2, v_3, v_4 on its boundary listed counterclockwise).

We now consider the 2^{k-3} distinguished regions of S together with the corresponding regions of S' and join each pair of associated regions by a handle. The addition of the first handle produces the surface of genus $2[(k-5) \cdot 2^{k-4} + 1]$ while the addition of each of the other $2^{k-3} - 1$ handles results in an increase of 1 to the genus. Thus, the surface just constructed has genus $(k-4) \cdot 2^{k-3} + 1$. Now each set of four vertices on the boundary of a distinguished region can be joined to the corresponding four vertices on the boundary of the associated region so that the four edges are embedded on the handle joining the regions. It is now immediate that the resulting graph is isomorphic to Q_k and that every region is bounded by a 4-cycle. Furthermore, each added handle gives rise to four regions, "opposite" ones of which have disjoint boundaries, so there exist 2^{k-2} regions of Q_k that are pairwise disjoint. ■

The Möbius Strip and Projective Plane

There are other surfaces on which graphs can be embedded. The **Möbius strip** (or **Möbius band**) is a one-sided surface that can be constructed from a rectangular piece of material by giving one side of the rectangle a half-twist (or a rotation through 180^o) and then identifying opposite sides of the rectangle (see Figure 11.18). Thus, A represents the same point on the Möbius strip. The Möbius strip is named for the German mathematician August Ferdinand Möbius who discovered it in 1858 (even though the mathematician Johann Benedict Listing discovered it shortly before Möbius).

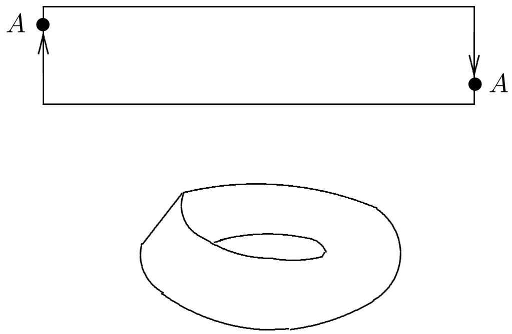


Figure 11.18: The Möbius strip

Certainly, every planar graph can be embedded on the Möbius strip. Figure 11.19 shows that $K_{3,3}$ can also be embedded on the Möbius strip.

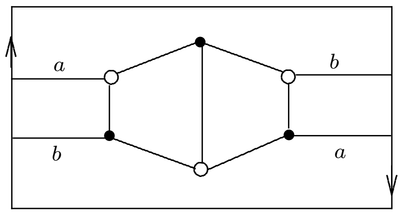


Figure 11.19: An embedding of $K_{3,3}$ on the Möbius strip

Of more interest are the nonorientable surfaces (or the nonorientable 2-dimensional manifolds), the simplest example of which is the projective plane. The **projective plane** can be represented by identifying opposite sides of a rectangle in the manner shown in Figure 11.20(a). Note that A represents the same point in the projective plane, as does B. Figure 11.20(b) shows an embedding of K_5 on the projective plane.

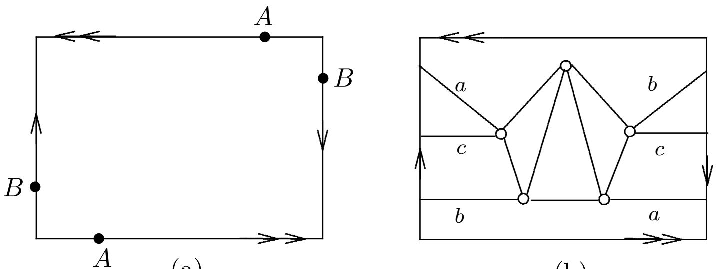


Figure 11.20: An embedding of K_5 on the projective plane

The projective plane can also be represented by a circle where antipodal pairs of points on the circumference are the same point. Using this representation, we can give an embedding of K_6 on the projective plane shown in Figure 11.21.

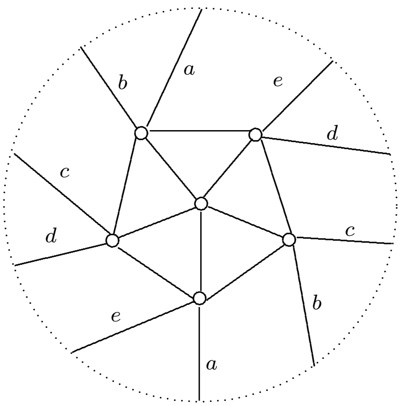


Figure 11.21: An embedding of K_6 on the projective plane

For the embedding of K_5 on the projective plane shown in Figure 11.20(b), $n = 5$, $m = 10$ and $r = 6$; while for the embedding of K_6 shown in Figure 11.21, $n = 6$, $m = 15$ and $r = 10$. In both cases, $n - m + r = 1$. In fact, for any connected graph of order n and size m that is 2-cell embedded on the projective plane, resulting in r regions,

$$n - m + r = 1.$$

Recall that S_k denotes the surface of genus k. Thus, S_0 represents the sphere (or plane), S_1 represents the torus and S_2 represents the **double torus** (or sphere with two handles). We have already mentioned that the torus can be represented as a square with opposite sides identified. More generally, the surface S_k $(k > 0)$ can be represented as a regular $4k$-gon whose $4k$ sides can be listed in clockwise order as

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}, \tag{11.8}$$

where, for example, a_1 is a side directed clockwise and a_1^{-1} is a side also labeled a_1 but directed counterclockwise. These two sides are then identified in a manner consistent with their directions. Thus, the double torus can be represented by a regular octagon, as shown in Figure 11.22. The "two" points labeled X are actually the same point on S_2 while the "eight" points labeled Y are, in fact, a single point.

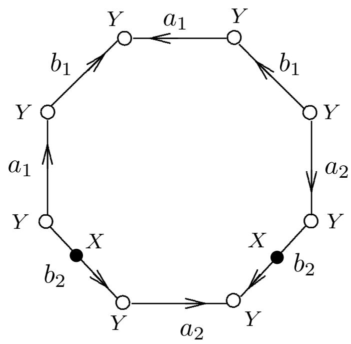


Figure 11.22: A representation of the double torus

Although it is probably obvious that there exist numerous graphs that can be embedded on the surface S_k of a given nonnegative integer k, it may not be entirely obvious that there always exist graphs for which a 2-cell embedding on S_k exists.

Theorem 11.21 *For every nonnegative integer k, there exists a connected graph that has a 2-cell embedding on S_k.*

Proof. For $k = 0$, every connected planar graph has the desired property; thus, we assume that $k > 0$.

We represent S_k as a regular $4k$-gon whose $4k$ sides are described and identified as in (11.8). First, we define a pseudograph H as follows. At each vertex of the $4k$-gon, let there be a vertex of H. Actually, the identification process associated with the $4k$-gon implies that there is only one vertex of H. Let each side of the $4k$-gon represent an edge of H. The identification produces $2k$ distinct edges, each of which is a loop. This completes the construction of H. Hence, the pseudograph H has order 1 and size $2k$. Furthermore, there is only one region, namely the interior of the $4k$-gon; this region is clearly a 2-cell. Therefore, there exists a 2-cell embedding of H on S_k.

To convert the pseudograph H into a graph, we subdivide each loop twice, producing a graph G having order $4k + 1$, size $6k$ and again a single 2-cell region. ∎

Figure 11.23 illustrates the construction given in the proof of Theorem 11.21 in the case of the torus S_1. The graph G so constructed is shown in Figure 11.23(a). In Figure 11.23(b)-(e), we see a variety of ways of visualizing the embedding. In Figure 11.23(b), a 3-dimensional embedding is described. In Figures 11.23(c) and (d), the torus is represented as a rectangle with opposite sides identified. (Figure 11.23(c) is the actual drawing described in the proof of the theorem.) In Figure 11.23(e), a portion of G is drawn in the plane. Then two circular holes are made in the plane and a tube (or handle) is placed over the plane joining the two holes. The edge uv is then drawn over the handle, completing the 2-cell embedding.

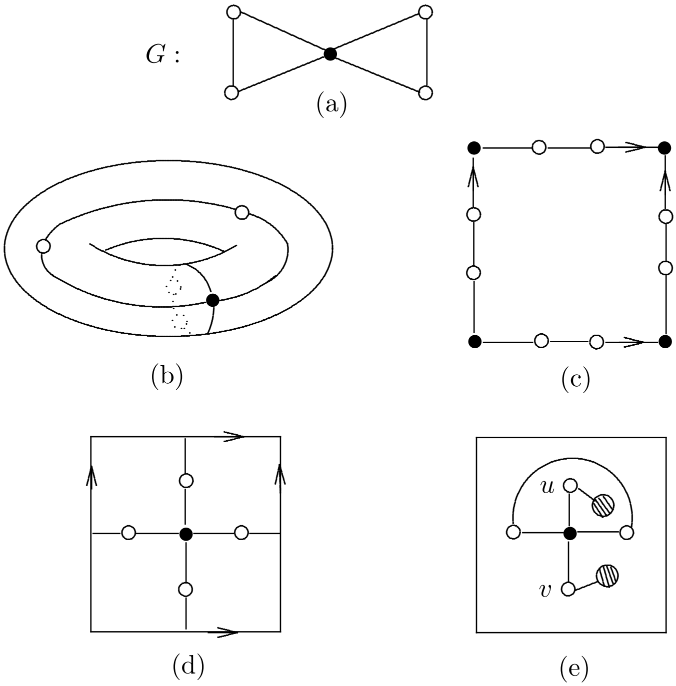


Figure 11.23: A graph 2-cell embedded on the torus

The graphs G constructed in the proof of Theorem 11.21 are planar. Hence, for every nonnegative integer k, there exist planar graphs that can be 2-cell embedded on S_k. It is also true that for every planar graph G and every positive integer k, there exists an embedding of G on S_k that is *not* a 2-cell embedding. In general, for a given graph G and *positive* integer k with $k > \gamma(G)$, there always exists an embedding of G on S_k that is not a 2-cell embedding, which can be obtained from an embedding of G on $S_{\gamma(G)}$ by adding $k - \gamma(G)$ handles to the interior of some region of G. If $k = \gamma(G)$ and G is connected, then by Theorem 11.11, every embedding of G on S_k is a 2-cell embedding. Of course, if $k < \gamma(G)$, there is no embedding whatsoever of G on S_k.

11.3 The Graph Minor Theorem

We have seen by Wagner's theorem (Theorem 10.21) that a graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. That is, Wagner's theorem is a **forbidden minor** characterization of planar graphs – in this case, two forbidden minors: K_5 and $K_{3,3}$. A natural question to ask is whether a forbidden minor characterization may exist for graphs that can be embedded on other surfaces. It was shown by Daniel Archdeacon and Philip Huneke [10] that there are exactly 35 forbidden minors for graphs that can be embedded on the projective plane. More general results involving minors have been obtained. Klaus Wagner conjectured that in every infinite collection of graphs, there are always two graphs where one is isomorphic to a minor of the other. In what must be considered one of the major theorems of graph theory, Neil Robertson and Paul Seymour [206] verified this conjecture. Its lengthy proof is a consequence of a sequence of several papers that required years to complete.

Theorem 11.22 (**The Graph Minor Theorem**) *In every infinite set of graphs, there are two graphs where one is* (*isomorphic to*) *a minor of the other.*

One consequence of Theorem 11.22 is another major theorem in graph theory, also due to Robertson and Seymour [206]. A set S of graphs is said to be **minor-closed** if for every graph G in S, every minor of G also belongs to S. For example, the set S of planar graphs is minor-closed because every minor of a planar graph is planar; that is, if $G \in S$ and H is a minor of G, then $H \in S$ as well.

Theorem 11.23 *Let S be a minor-closed set of graphs. Then there exists a finite set M of graphs such that $G \in S$ if and only if no graph in M is a minor of G.*

Proof. Let $\overline{S}$ be the set of graphs not belonging to S and let M be the set of all graphs F in $\overline{S}$ such that every proper minor of F belongs to S. We claim that this set M has the required properties. First, we show that $G \in S$ if and only if no graph in M is a minor of G.

First, suppose that there is some graph $G \in S$ such that there is a graph F that is a minor of G and for which $F \in M$. Since $G \in S$ and S is minor-closed, it follows that $F \in S$. However, since $F \in M$, we have $F \in \overline{S}$, which is a contradiction.

For the converse, assume to the contrary that there is a graph $G \in \overline{S}$ such that no graph in M is a minor of G. We consider two cases.

Case 1. *All of the proper minors of G are in S.* Then by the defining property of M, it follows that $G \in M$. Since $G \in M$ and G is a minor of itself, this contradicts our assumption that no graph in M is a minor of G.

Case 2. *Some proper minor of G, say G', is not in S.* Thus, $G' \in \overline{S}$. Then G' either satisfies the condition of Case 1 or of Case 2. We proceed in this manner as long as we remain in Case 2, producing a sequence $G = G^{(0)}, G^{(1)}, G^{(2)}, \ldots$ of proper minors.

If this process terminates, we have a finite sequence

$$G = G^{(0)}, G' = G^{(1)}, \ldots, G^{(p)},$$

where each graph in the sequence is a proper minor of all those graphs that precede it. Then $G^{(p)} \in M$, which returns us to Case 1. Otherwise, the sequence $G = G^{(0)}, G' = G^{(1)}, G^{(2)}, \ldots$ is infinite and where each graph $G^{(i)}$, $i \geq 1$, is a proper minor of all those graphs that precede it. This, however, is impossible since for each $i \geq 0$, either the order of $G^{(i+1)}$ is less than that of $G^{(i)}$ or the orders are the same and the size of $G^{(i+1)}$ is less than that of $G^{(i)}$.

It therefore remains only to show that M is finite. Assume, to the contrary, that M is infinite. By the Graph Minor Theorem, M contains two graphs, H_1 and H_2 say, such that one is a minor of the other. Suppose that H_2 is a minor (necessarily a proper minor) of H_1. Since every proper minor of H_1 belongs to S, it follows that $H_2 \in S$. However, since $H_2 \in M$, it follows that $H_2 \in \overline{S}$, producing a contradiction. ■

We now return to the question concerning the existence of a forbidden minor characterization for graphs embeddable on a surface S_k of genus $k \geq 0$. Certainly, if G is a sufficiently small graph (in terms of its order and/or size), then G can be embedded on S_k. Hence, if we begin with a graph F that cannot be embedded on S_k and perform successive edge contractions, edge deletions and vertex deletions, then eventually we arrive at a graph F' that also cannot be embedded on S_k but such that any additional edge contraction, edge deletion or vertex deletion of F' produces a graph that *can* be embedded on S_k. Such a graph F' is said to be **minimally nonembeddable on** S_k. Consequently, a graph F' is minimally nonembeddable on S_k if F' cannot be embedded on S_k but every proper minor F' can be embedded on S_k. Thus, the set of graphs embeddable on S_k is minor-closed. As a consequence of the Graph Minor Theorem, we have the following.

Theorem 11.24 *For each integer $k \geq 0$, the set of minimally nonembeddable graphs on S_k is finite.*

Consequently, for each nonnegative integer k, there is a finite set $\mathcal{M}_k$ of graphs such that a graph G is embeddable on S_k if and only if no graph in $\mathcal{M}_k$ is a minor of G. Of course, the set of minimally nonembeddable graphs on the sphere is $\mathcal{M}_0 = \{K_5, K_{3,3}\}$. Although the number of minimally nonembeddable graphs on the torus is finite, the actual value of this number is not known. However, it is known that this number exceeds 800 and so $|\mathcal{M}_1| > 800$.

Exercises for Chapter 11

Section 11.1: The Crossing Number of a Graph

1. Draw K_7 in the plane with nine crossings.

2. Determine $\text{cr}(K_{3,3})$ without using Theorem 11.4.

3. Show that $\text{cr}(K_{5,5}) \leq 16$.

4. Determine $\text{cr}(K_{2,2,2})$.

5. Determine $\text{cr}(K_{1,2,3})$.

6. Show that $2 \leq \text{cr}(C_3 \square C_3) \leq 3$ without using Theorem 11.6.

7. (a) It is known that $\text{cr}(W_4 \square K_2) = 2$, where W_4 is the wheel $C_4 \vee K_1$ of order 5. Draw $W_4 \square K_2$ in the plane with two crossings.

   (b) Prove or disprove: If G is a nonplanar graph containing an edge e such that $G - e$ is planar, then $\text{cr}(G) = 1$.

8. Prove that $\overline{\text{cr}}(C_3 \square C_t) = t$ for $t \geq 3$.

9. Give an example of a straight-line embedding of a maximal planar graph of order exceeding 4 containing exactly four vertices of degree 5 or less.

10. Let G be a connected planar graph. Prove or disprove: If $\text{cr}(G \square K_2) = 0$, then G is outerplanar.

Section 11.2: The Genus of a Graph

11. Determine $\gamma = \gamma(K_{4,4})$ without using Theorem 11.19 and label the regions in a 2-cell embedding of $K_{4,4}$ on the surface of genus γ.

12. (a) Show that $\gamma(G) \leq \text{cr}(G)$ for every graph G.

    (b) Prove that for every positive integer k, there exists a graph G such that $\gamma(G) = 1$ and $\text{cr}(G) = k$.

13. Prove Theorem 11.14: *If G is a connected graph of order n and size m whose smallest cycle has length k, then $\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1$.*

14. Use Theorem 11.16 to prove Corollary 11.17: *If G is a graph with components $G_1, G_2, \ldots, G_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i)$.*

15. Show for every two integers $s, t \geq 2$ that

$$\gamma(K_{s,t}) \geq \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil.$$

16. (a) Find a lower bound for $\gamma(K_{3,3} \vee \overline{K}_n)$.

    (b) Determine $\gamma(K_{3,3} \vee \overline{K}_n)$ exactly for $n = 1, 2, 3$.

17. Determine $\gamma(K_2 \square C_4 \square C_6)$.

18. Prove, for every positive integer γ, that there exists a connected graph G of genus γ.

19. Prove, for each positive integer k, that there exists a connected planar graph G such that $\gamma(G \square K_2) \geq k$.

20. Show, in a manner similar to the embedding of $K_{3,3}$ shown in Figure 11.19, that K_5 can be embedded on the Möbius strip.

21. By Theorem 11.18, $\gamma(K_7) = 1$. Let there be an embedding of K_7 on the torus and let R_1 and R_2 be two neighboring regions. Let G be the graph obtained by adding a new vertex v in R_1 and joining v to the vertices on the boundaries of both R_1 and R_2. What is $\gamma(G)$?

22. The graph H is a certain 6-regular graph of order 12. It is known that $G = H \square K_2$ can be embedded on S_3. What is $\gamma(G)$?

23. Does there exist a graph G containing two nonadjacent vertices u and v such that $\gamma(G) = \gamma(G + uv)$ but $\text{cr}(G) \neq \text{cr}(G + uv)$?

24. For an r-regular graph H of order n where $r \equiv 5 \pmod 6$, the graph $G = H \square K_2$ can be embedded on the surface S_k where $k = \frac{(r-5)n}{6} + 1$. Show that $\gamma(G) = k$.

Section 11.3: The Graph Minor Theorem

25. (a) Show that the set $\mathcal{F}$ of forests is a minor-closed family of graphs.

    (b) What are the forbidden minors of $\mathcal{F}$?

26. Is the set B of bipartite graphs a minor-closed family of graphs? If so, what are the forbidden minors of B?

27. We have seen that the set of planar graphs is a minor-closed family of graphs. Is the set O of outerplanar graphs a minor-closed family of graphs? If so, what are the forbidden minors of O?

28. Use the Graph Minor Theorem (Theorem 11.22) to show that for any infinite set $S = \{G_1, G_2, G_3, \ldots\}$ of graphs, there exist infinitely many pairwise disjoint 2-element sets $\{i, j\}$ of integers such that one of G_i and G_j is a minor of the other.

Chapter 12

Matchings, Independence and Domination

Of the numerous problems concerning sets of edges or sets of vertices in graphs, many of these deal with the idea of independence (in which every two elements in the set are nonadjacent). Such a set of edges is a matching in a graph. The fact that two adjacent vertices u and v result in the edge uv gives rise to two other fundamental concepts in graph theory, namely covers and domination. These two concepts are also discussed in this chapter.

12.1 Matchings

A set of edges in a graph G is **independent** if no two edges in the set are adjacent in G. The edges in an independent set of edges of G are called a **matching** in G. If $M = \{e_1, e_2, \ldots, e_k\}$ is a matching in a graph G where $e_i = u_iv_i$ for $1 \leq i \leq k$, then the edges of M match (or pair off) the vertices $u_1, u_2, \ldots, u_k$ to the vertices $v_1, v_2, \ldots, v_k$. A matching of maximum size in G is a **maximum matching** in G. The **edge independence number** $\alpha'(G)$ of G is the number of edges in a maximum matching of G. In fact, $\alpha'(G)$ is sometimes referred to as the **matching number** of G. (We will see that the edge independence number has a connection with other parameters in Section 12.3.) In the graph G of Figure 12.1, the set $M_1 = \{e_1, e_4\}$ is a matching that is not a maximum matching, while $M_2 = \{e_1, e_3, e_5\}$ and $M_3 = \{e_1, e_3, e_6\}$ are maximum matchings in G. For the graph G of Figure 12.1 then, $\alpha'(G) = 3$.

If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a **perfect matching** in G. Clearly, if G has a perfect matching M, then G has even order and the edge-induced subgraph $G[M]$ is a 1-regular spanning subgraph of G. Thus, the graph G of Figure 12.1 cannot have a perfect matching.

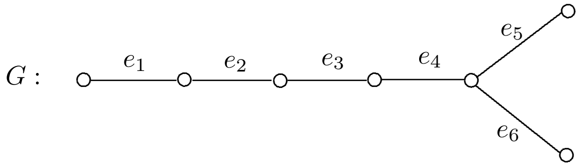


Figure 12.1: Matchings and maximum matchings

If M is a matching in a graph G, then every vertex of G is incident with at most one edge of M. A vertex that is incident with no edges of M is referred to as an **M-unmatched vertex** or simply an **unmatched vertex** if the matching M is clear. The following theorem will prove to be useful.

Theorem 12.1 *Let M_1 and M_2 be matchings in a graph G. Then each component of the spanning subgraph H of G with $E(H) = (M_1 - M_2) \cup (M_2 - M_1)$ is one of the following:*

(*i*) *an isolated vertex,*

(*ii*) *an even cycle whose edges are alternately in M_1 and in M_2,*

(*iii*) *a nontrivial path whose edges are alternately in M_1 and in M_2 and such that each end-vertex of the path is either M_1-unmatched or M_2-unmatched but not both.*

Proof. First, we note that $\Delta(H) \leq 2$, for if H contains a vertex v such that $\deg_H v \geq 3$, then v is incident with at least two edges in the same matching. Since $\Delta(H) \leq 2$, every component of H is a path (possibly trivial) or a cycle. Since no two edges in a matching are adjacent, the edges of each cycle and path in H are alternately in M_1 and in M_2. Thus each cycle in H is even.

Suppose that $e = uv$ is an edge of H and u is the end-vertex of a path P that is a component of H. The proof will be complete once we have shown that u is M_1-unmatched or M_2-unmatched but not both. Since $e \in E(H)$, it follows that $e \in M_1 - M_2$ or $e \in M_2 - M_1$. If $e \in M_1 - M_2$, then u is not M_1-unmatched. We show that u is M_2-unmatched. If this is not the case, then there is an edge f in M_2 (thus $f \neq e$) such that f is incident to u. Since e and f are adjacent, $f \notin M_1$. Thus, $f \in M_2 - M_1 \subseteq E(H)$. This, however, is impossible since u is the end-vertex of P. Therefore, u is M_2-unmatched. Similarly, if $e \in M_2 - M_1$, then u is M_1-unmatched. ■

Augmenting Paths and Maximum Matchings

In order to present a characterization of maximum matchings in a graph, we introduce two new terms. Let M be a matching in a graph G. An **M-alternating path** of G is a path whose edges are alternately in M and not in M. An M**-augmenting path** is an M-alternating path P both of whose end-vertices are unmatched, that is, the first and last edges of P do not belong to M. Necessarily, every M-augmenting path has odd length.

Figure 12.2 shows a graph G of order 13 and a matching

$$M = \{v_2v_4, v_3v_6, v_5v_8, v_7v_{10}, v_{11}v_{12}\}$$

of size 5, whose edges are indicated by bold lines. The path $P =(v_1, v_2, v_4, v_7, v_{10}, v_9)$ is then an M-augmenting $v_1 - v_9$ path in G.

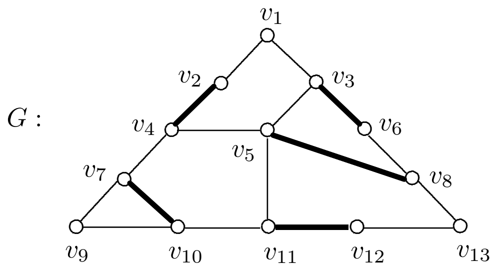


Figure 12.2: A matching M in a graph G and an M-augmenting path $P = (v_1, v_2, v_4, v_7, v_{10}, v_9)$

The following result is due to Claude Berge [21].

Theorem 12.2 *A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.*

Proof. Assume, to the contrary, that a graph G contains an M-augmenting path P, where M is a maximum matching in G. Let M' denote the edges of P belonging to M, and let $M'' = E(P) - M'$. Since $|M''| = |M'| + 1$, the set $(M - M') \cup M''$ is a matching having cardinality exceeding that of M, producing a contradiction.

For the converse, suppose that M_1 is a matching of a graph G that is not a maximum matching in G. Let M_2 be a maximum matching in G and let H be the spanning subgraph of G with

$$E(H) = (M_1 - M_2) \cup (M_2 - M_1).$$

Since $|M_2| > |M_1|$, it follows by Theorem 12.1 that there is a component P of H that is a path whose edges are alternately in M_1 and in M_2 such that P contains more edges belonging to M_2 than to M_1. Necessarily, each end-vertex of P is M_1-unmatched, which implies that P is an M_1-augmenting path. ■

Since the path $P = (v_1, v_2, v_4, v_7, v_{10}, v_9)$ in the graph G of Figure 12.2 is an M-augmenting path for the matching

$$M = \{v_2v_4, v_3v_6, v_5v_8, v_7v_{10}, v_{11}v_{12}\},$$

it follows by Theorem 12.2 that M is not a maximum matching in G. Following the proof of Theorem 12.2, we let $M' = E(P) \cap M$ and $M'' = E(P) - M'$. Then $(M - M') \cup M''$ is a matching whose size exceeds that of M. In fact, $(M - M') \cup M''$ is a maximum matching in the graph G of Figure 12.2 (see Figure 12.3).

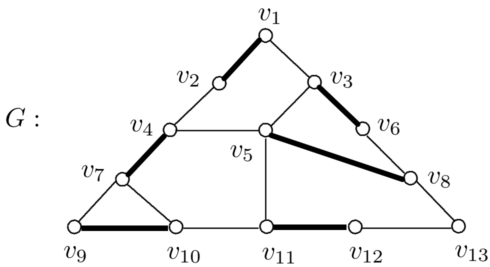


Figure 12.3: A maximum matching in a graph G

Although matchings are of interest in all graphs, they are of particular interest in bipartite graphs. Let G be a bipartite graph with partite sets U and W, where $|U| \leq |W|$. Necessarily, for any matching of k edges, $k \leq |U|$. If $|U| = k$, then U is said to be **matched** to a subset of W.

Hall's Theorem

For a bipartite graph G with partite sets U and W and for $S \subseteq U$, let $N(S)$ be the set of all vertices in W having a neighbor in S. The condition that

$$|N(S)| \geq |S| \text{ for all } S \subseteq U$$

is referred to as **Hall's condition**. This condition is named for Philip Hall (1904–1982). The following 1935 theorem of Hall [118] shows that this condition is necessary and sufficient for one partite set of a bipartite graph to be matched to a subset of the other.

Theorem 12.3 (**Hall's Theorem**) *Let G be a bipartite graph with partite sets U and W. Then U can be matched to a subset of W if and only if Hall's condition is satisfied.*

Proof. Suppose the U can be matched to a subset of W under a matching M^*. Then every nonempty subset S of U can be matched under M^* to some subset of W, implying that $|N(S)| \geq |S|$; so, U satisfies Hall's condition.

To verify the converse, suppose that there exists a bipartite graph G for which Hall's condition is satisfied but U cannot be matched to a subset of W. Let M be a maximum matching in G. By assumption, there is a vertex u in U that is unmatched. Let X be the set of all vertices of G that are connected to u by an M-alternating path. Since M is a maximum matching, it follows by Theorem 12.2 that u is the only unmatched vertex in X.

Let $U_1 = X \cap U$ and let $W_1 = X \cap W$. Since no vertex of $X - \{u\}$ is unmatched, it follows that $U_1 - \{u\}$ is matched under M to W_1. Therefore, $|W_1| = |U_1| - 1$ and $W_1 \subseteq N(U_1)$. Furthermore, for every $w \in N(U_1)$, the graph G contains an M-alternating $u - w$ path and so $N(U_1) \subseteq W_1$. Thus, $N(U_1) = W_1$ and

$$|N(U_1)| = |W_1| = |U_1| - 1 < |U_1|.$$

This, however, contradicts the fact that Hall's condition is satisfied. ∎

If G is a bipartite graph with partite sets U and W where $|U| \leq |W|$ and Hall's condition is satisfied, then by Theorem 12.3, G contains a matching of size $|U|$, which is a maximum matching. If $|U| = |W|$, then such a matching is a perfect matching in G. The following result is another consequence of Theorem 12.3.

Theorem 12.4 *Every r-regular bipartite graph $(r \geq 1)$ has a perfect matching.*

Proof. Let G be an r-regular bipartite graph with partite sets U and W. Then $|U| = |W|$. Let S be a nonempty subset of U. Suppose that a total of k edges join the vertices of S and the vertices of $N(S)$. Thus, $k = r|S|$. Since there are $r|N(S)|$ edges incident with the vertices of $N(S)$, it follows that $k \leq r|N(S)|$ or $r|S| \leq r|N(S)|$. Therefore, $|S| \leq |N(S)|$ and Hall's condition is satisfied in G. By Theorem 12.3, G has a perfect matching. ∎

A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets is said to have a **system of distinct representatives** if there exists a set $\{s_1, s_2, \ldots, s_n\}$ of distinct elements such that $s_i \in S_i$ for $1 \leq i \leq n$.

Theorem 12.5 *A collection $\{S_1, S_2, \ldots, S_n\}$ of finite nonempty sets has a system of distinct representatives if and only if for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.*

Proof. Assume first that $\{S_1, S_2, \ldots, S_n\}$ has a system of distinct representatives. Then, for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.

For the converse, suppose that $\{S_1, S_2, \ldots, S_n\}$ is a collection of n sets such that for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements. We now consider the bipartite graph G with partite sets

$$U = \{S_1, S_2, \ldots, S_n\} \text{ and } W = S_1 \cup S_2 \cup \cdots \cup S_n$$

such that a vertex S_i $(1 \leq i \leq n)$ in U is adjacent to a vertex w in W if $w \in S_i$. Let X be any subset of U with $|X| = k$, where $1 \leq k \leq n$. Since the union of any k sets in U contains at least k elements, it follows that $|N(X)| \geq |X|$. Thus, G satisfies Hall's condition. By Theorem 12.3, G contains a matching from U to a subset of W. This matching pairs off the sets $S_1, S_2, \ldots, S_n$ with n distinct elements in $S_1 \cup S_2 \cup \ldots \cup S_n$, producing a system of distinct representatives for $\{S_1, S_2, \ldots, S_n\}$. ∎

Theorem 12.5 is also due to Philip Hall. Indeed, it was through systems of distinct representative that Hall proved the theorem that bears his name. Born in England, Hall's interest in mathematics was greatly influenced by the mathematics teachers he had. In 1927 Hall obtained an important result in group theory, generalizing the Sylow theorems for finite solvable groups, which is now often called Hall's theorem. He then published a paper in 1932 on groups of prime power order, perhaps his best known work. In 1933 he was then appointed a Lecturer at Cambridge. In 1935 his theorem on matchings was published, although it was not stated in terms of graph theory. Except for a period during World War II when he worked for a Foreign Office at Bletchley Park, he remained at Cambridge from 1933 to 1967. Hall spent much of his life making important contributions to algebra and is considered one of the great mathematicians of the 20th century.

The preceding discussion is directly related to a well-known combinatorial problem called the **Marriage Problem**:

> *Given a set of boys and a set of girls where each girl knows some of the boys, under what conditions can all girls get married, each to a boy she knows?*

In this context, Theorem 12.5 may be reformulated to produce what is often referred to as **Hall's Marriage Theorem**:

> *If there are n girls, then the Marriage Problem has a solution if and only if every subset of k girls $(1 \le k \le n)$ collectively know at least k boys.*

12.2 1-Factors

We have already noted that if M is a perfect matching in a graph G, then $G[M]$ is a 1-regular spanning subgraph of G. Any spanning subgraph of a graph G is referred to as a **factor** of G. A k-regular factor is called a **k-factor**. Thus, a 1-factor of a graph G is the subgraph induced by a perfect matching in G. Therefore, all theorems stated earlier that concern perfect matchings can be restated in terms of 1-factors. For example, the theorem below is a restatement of Theorem 12.4.

Theorem 12.6 *Every r-regular bipartite graph $(r \ge 1)$ has a 1-factor.*

The determination of whether a given graph contains a 1-factor is a problem that has received much attention in the literature. Of course, if a graph G has a 1-factor, then G has even order. It turns out that for a graph G to contain a 1-factor, this depends not only on the order of G being even but on the orders of components of certain subgraphs of G. We refer to a component of a graph as **odd** or **even** according to whether its order is odd or even.

While the graph G of Figure 12.4 has even order, it does not contain a 1-factor. Let $S = \{s_1, s_2, s_3\}$. Since each of the components $G_1, G_2, \ldots, G_5$ of $G - S$ is odd, it follows that if G has a 1-factor F, then some edge of F must join a vertex of S and a vertex of G_i for each i ($1 \le i \le 5$). However, since there are more odd components in $G - S$ than vertices in S, this is impossible.

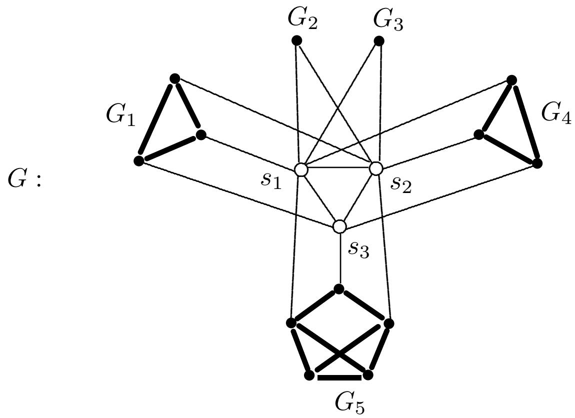


Figure 12.4: A graph that contains no 1-factor

Tutte's Theorem on 1-Factors

The explanation as to why the graph G in Figure 12.4 does not have a 1-factor is, in fact, the key reason as to why any graph does not have a 1-factor. The theorem referred to here is due to William Tutte, who characterized graphs containing a 1-factor while he was a graduate student. The proof of Tutte's theorem [242] we present is due to Ian Anderson [8]. The number of odd components in a graph G is denoted by $k_o(G)$.

Theorem 12.7 (**Tutte's Theorem**) *A nontrivial graph G contains a 1-factor if and only if $k_o(G - S) \le |S|$ for every proper subset S of $V(G)$.*

Proof. First, suppose that G contains a 1-factor F. Let S be a proper subset of $V(G)$. If $G - S$ has no odd components, then $k_o(G - S) = 0$ and $k_o(G - S) \le |S|$. Thus, we may assume that $k_o(G - S) = k \ge 1$. Let $G_1, G_2, \ldots, G_k$ be the odd components of $G - S$. (There may be some even components of $G - S$ as well.) For each odd component G_i of $G - S$, there is at least one edge of F joining a vertex of G_i and a vertex of S. Thus, $k_o(G - S) \le |S|$.

We now verify the converse. Let G be a graph such that $k_o(G - S) \le |S|$ for every proper subset S of $V(G)$. In particular, $k_o(G - \emptyset) \le |\emptyset| = 0$, implying that every component of G is even and so G itself has even order. We now show that G has a 1-factor by employing induction on the (even) order of G. Since K_2 is the only graph of order 2 having no odd components and K_2 has a 1-factor, the base case of the induction is verified.

For a given even integer $n \geq 4$, assume that all graphs H of even order less than n and satisfying $k_o(H - S) \leq |S|$ for every proper subset S of $V(H)$ contain a 1-factor. Now, let G be a graph of order n satisfying $k_o(G - S) \leq |S|$ for every proper subset S of $V(G)$. As we saw above, every component of G has even order. We show that G has a 1-factor.

For a vertex v of G that is not a cut-vertex and $R = \{v\}$, it follows that $k_o(G - R) = |R| = 1$. Hence, there are nonempty proper subsets T of $V(G)$ for which $k_o(G - T) = |T|$. Among all such sets T, let S be one of maximum cardinality. Suppose that $k_o(G - S) = |S| = k \geq 1$ and let $G_1, G_2, \ldots, G_k$ be the odd components of $G - S$.

We claim that $k(G-S) = k$, that is, $G_1, G_2, \ldots, G_k$ are the *only* components of $G - S$. Assume, to the contrary, that $G - S$ has an even component G_0. Let v_0 be a vertex of G_0 that is not a cut-vertex of G_0. Let $S_0 = S \cup \{v_0\}$. Since G_0 has even order,

$$k_o(G - S_0) \geq k_o(G - S) + 1 = k + 1.$$

On the other hand, $k_o(G - S_0) \leq |S_0| = k + 1$. Therefore,

$$k_o(G - S_0) = |S_0| = k + 1,$$

which is impossible. Thus, as claimed, $G_1, G_2, \ldots, G_k$ are the only components of $G - S$.

For each integer i $(1 \leq i \leq k)$, let S_i denote the set of those vertices in S adjacent to at least one vertex of G_i. Since G has only even components, each set S_i is nonempty.

We claim, for each integer ℓ with $1 \leq \ell \leq k$, that the union of any ℓ of the sets $S_1, S_2, \ldots, S_k$ contains at least ℓ vertices. Assume, to the contrary, that this is not the case. Then there is an integer j such that the union S' of j of the sets $S_1, S_2, \ldots, S_k$ has fewer than j elements. Suppose that $S_1, S_2, \ldots, S_j$ have this property. Thus,

$$S' = S_1 \cup S_2 \cup \cdots \cup S_j \text{ and } |S'| < j.$$

Then $G_1, G_2, \ldots, G_j$ are at least some of the components of $G - S'$ and so

$$k_o(G - S') \geq j > |S'|,$$

which contradicts the hypothesis. Thus, as claimed, for each integer ℓ with $1 \leq \ell \leq k$, the union of any ℓ of the sets $S_1, S_2, \ldots, S_k$ contains at least ℓ vertices.

By Theorem 12.5, there is a set $\{v_1, v_2, \ldots, v_k\}$ of k distinct vertices of S such that $v_i \in S_i$ for $1 \leq i \leq k$. Since every component G_i of $G - S$ contains a vertex u_i such that u_iv_i is an edge of G, it follows that $\{u_iv_i : 1 \leq i \leq k\}$ is a matching of G.

We now show that for each nontrivial component G_i of $G - S$ $(1 \leq i \leq k)$, the graph $G_i - u_i$ contains a 1-factor. Let W be a proper subset of $V(G_i - u_i)$. We claim that

$$k_o(G_i - u_i - W) \leq |W|.$$

Assume, to the contrary, that $k_o(G_i - u_i - W) > |W|$. Since G_i has odd order, $G_i - u_i$ has even order and so $k_o(G_i - u_i - W)$ and $|W|$ are both even or both odd. Hence,

$$k_o(G_i - u_i - W) \geq |W| + 2.$$

Let $X = S \cup W \cup \{u_i\}$. Then

$$\begin{aligned} |X| &= |S| + |W| + 1 = |S| + (|W| + 2) - 1 \\ &\leq k_o(G - S) + k_o(G_i - u_i - W) - 1 \\ &= k_o(G - X) \leq |X|, \end{aligned}$$

which implies that $k_o(G - X) = |X|$ and contradicts the defining property of S. Thus, as claimed, $k_o(G_i - u_i - W) \leq |W|$.

Therefore, by the induction hypothesis, for each nontrivial component G_i of $G - S$ $(1 \leq i \leq k)$, the graph $G_i - u_i$ has a 1-factor. The collection of 1-factors of $G_i - u_i$ for all nontrivial graphs G_i of $G - S$ together with the edges in $\{u_iv_i : 1 \leq i \leq k\}$ produce a 1-factor of G. ■

Petersen's Theorem

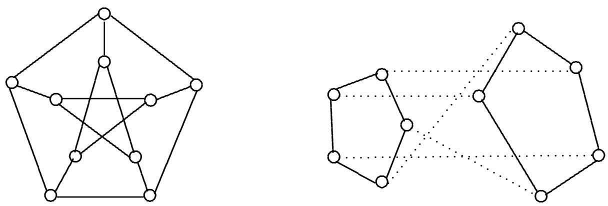
Clearly, every 1-regular graph contains a 1-factor, while the only 2-regular graphs containing a 1-factor are those that are the union of even cycles. Determining which cubic graphs contain a 1-factor is considerably more complex. One of the best known theorems in this connection is due to Julius Petersen [186].

Peter Christian Julius Petersen (1839–1910) was born in Sorø, Denmark. Petersen received his Ph.D. in 1871 from Copenhagen University. On the occasion of receiving his doctorate, Petersen wrote (translated from Danish):

> *Mathematics had, from the time I started to learn it, taken my complete interest, and most of my work consisted of solving problems of my own and of my friends, and in seeking the trisection of the angle, a problem that has had a great influence on my whole development.*

Although Petersen worked in and made contributions to many areas of mathematics, it is only graph theory for which he is known. Indeed, in his day he enjoyed an international reputation. Petersen's contributions to graph theory were primarily contained within a single paper he wrote, published in 1891 and titled "Die Theorie der regulären Graphen". Prior to 1891, the important results on graph theory (including Leonhard Euler's work on Eulerian graphs and Gustav Kirchhoff's work on spanning trees) were not results in graph *theory* as there really was no graph theory at that time. In the case of Petersen's paper, however, an argument could be made that for the first time a paper had been written containing fundamental results on the theory of graphs. Among the important results occurring in this paper were two theorems we will see shortly, namely, Theorem 12.8 below, referred to as Petersen's theorem (*Every* 3*-regular bridgeless graph contains a* 1*-factor*), and a theorem we will visit in Chapter 13.

Even though Petersen's major contribution to graph theory was his 1891 paper, that is not what he is known for. His primary claim to fame lies not for a single paper he wrote but for a single graph that appeared in one of his papers: the Petersen graph. We have encountered this graph several times already and we will continue to encounter it. Petersen first mentioned this graph, not in his 28-page classic 1891 paper, but in his 3-page 1898 note [187] "Sur le théorème de Tait" in which he presented this graph as a counterexample to a statement we will encounter in the next section. His graph did not appear in the aesthetic way in which it is commonly drawn, as shown in Figure 12.5(a), but in the less appealing way shown in Figure 12.5(b).



(a) (b)

Figure 12.5: The Petersen graph

The relatively short proof of Petersen's theorem that we present is not that of Petersen but makes use of Tutte's theorem, which appeared decades after Petersen's paper.

Theorem 12.8 (**Petersen's Theorem**) *Every bridgeless cubic graph contains a* 1*-factor.*

Proof. Let G be a bridgeless cubic graph and let S be a proper subset of $V(G)$ with $|S| = k$. We show that $k_o(G - S) \leq |S|$. This is true if $G - S$ has no odd components; so we assume that $G - S$ has $\ell \geq 1$ odd components, say $G_1, G_2, \ldots, G_\ell$.

Let E_i $(1 \leq i \leq \ell)$ denote the set of edges joining the vertices of G_i and the vertices of S. Since G is cubic, every vertex of G_i has degree 3 in G. Because the sum of the degrees in G of the vertices of G_i is odd and the sum of the degrees in G_i of the vertices of G_i is even, it follows that $|E_i|$ is odd. Because G is bridgeless, $|E_i| \neq 1$ and so $|E_i| \geq 3$ for $1 \leq i \leq \ell$. This implies that there are at least 3ℓ edges joining the vertices of $G - S$ and the vertices of S. Since $|S| = k$, at most $3k$ edges join the vertices of $G-S$ and the vertices of S. Thus,

$$3k_o(G - S) = 3\ell \leq 3k = 3|S|$$

and so $k_o(G-S) \leq |S|$. By Tutte's theorem (Theorem 12.7), G has a 1-factor. ■

Indeed, Petersen showed that a cubic graph with at most two bridges contains a 1-factor (see Exercise 14). This result cannot be extended further, however, since the graph G of Figure 12.6 is cubic and contains three bridges but no 1-factor since $k_o(G - v) = 3 > 1 = |\{v\}|$.

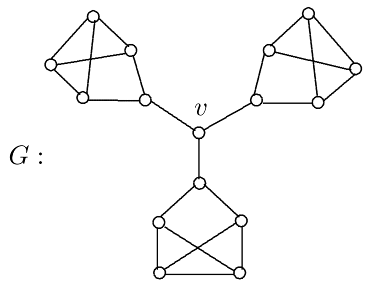


Figure 12.6: A cubic graph with no 1-factor

Errera's Theorem

If a connected cubic graph G contains two bridges, then, as noted above, G contains a 1-factor. Furthermore, these two edges lie on a single path of G. In fact, if all the bridges of a cubic graph, regardless of how many there might be, lie on a single path, then Alfred Errera [85] showed that G must have a 1-factor.

Theorem 12.9 (**Errera's Theorem**) *If all the bridges of a connected cubic graph G lie on a single path of G, then G has a 1-factor.*

Proof. We proceed by induction on the number of bridges in a connected cubic graph, all of whose bridges lie on a single path. We have already mentioned that if any such a graph G contains one bridge or two bridges, then G has a 1-factor. Assume that every connected cubic graph with k (≥ 2) bridges, all of which lie on a single path, contains a 1-factor.

Let G be a connected cubic graph with $k + 1$ bridges $e_1, e_2, \ldots, e_{k+1}$, all of which lie on a single path of G. Thus, G contains $k+2$ blocks $B_1, B_2, \ldots, B_{k+2}$ such that $e_i = u_i w_i$ ($1 \leq i \leq k + 1$), where $u_i \in V(B_i)$ and $w_i \in V(B_{i+1})$. (See

Figure 12.7.) Thus, B_i has odd order when $i = 1$ or $i = k + 2$; while B_i has even order if $2 \leq i \leq k + 1$. We now construct two connected graphs H and H', each containing k bridges, all of which lie on a single path. The graph H is obtained from G by deleting $V(B_2)$ and joining u_1 and w_2 by the edge e. The graph H' is obtained from G by deleting $V(B_{k+1})$ and joining u_k and w_{k+1} by the edge e'.

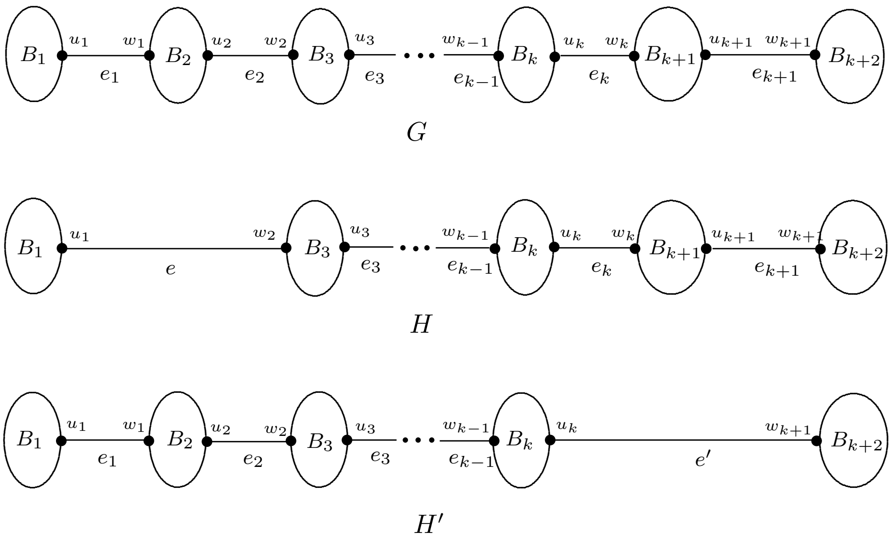


Figure 12.7: The graphs G, H and H' in the proof of Theorem 12.9

By the induction hypothesis, H contains a 1-factor F and H' contains a 1-factor F'. Let $M_1 = E(F) \cap E(B_1)$. Then u_1 is the only M_1-unmatched vertex in B_1. Since F is a 1-factor in H, it follows that $e \in E(F)$. This in turn implies that neither w_2 nor u_3 is incident with an edge of F lying in B_3. Thus, $u_3w_3 \in E(F)$. Continuing in this manner, we see that every bridge of H belongs to F. In a similar manner, every bridge of H' belongs to F'. This says that in G, it follows that $B_1 - u_1$ and $B_{k+2} - w_{k+1}$ contain 1-factors and that $B_i - w_{i-1} - u_i$ contains a 1-factor for $2 \leq i \leq k + 1$. These 1-factors together with the $k + 1$ bridges of G produce a 1-factor in G. ■

The three bridges of the cubic graph G of Figure 12.6 do not lie on a single path of G and G does not have a 1-factor. It is a direct consequence of Errera's theorem that if the bridges of a cubic graph G of order n lie on a single path, then $\alpha'(G) = n/2$.

12.3 Independence and Covers

We now explore not only the edge independence number in more detail but the vertex independence number, as well as two related parameters. There is, in fact, a formula for the edge independence number suggested by Theorem 12.7.

The Tutte–Berge Formula

By Tutte's theorem, it follows, of course, that if G is a graph of order $n \geq 2$ such that for each proper subset S of $V(G)$, the number of odd components of $G-S$ does not exceed $|S|$, then n is even and $\alpha'(G) = n/2$. Berge [23] observed that Tutte's theorem (Theorem 12.7) gives rise to a formula for the size of a maximum matching in a graph, that is, a formula for its edge independence number.

Theorem 12.10 (**The Tutte–Berge Formula**) *If G is a graph of order n, then*

$$\alpha'(G) = \min_{S \subset V(G)} \left\{ \frac{n + |S| - k_o(G-S)}{2} \right\}, \qquad (12.1)$$

where the minimum is taken over all proper subsets S of $V(G)$.

Proof. Let S be a proper subset of $V(G)$ and let M be a maximum matching in G. Furthermore, let M_1 be the subset of M consisting of those edges incident with at least one vertex of S and let $M_2 = M - M_1$. Then

$$\begin{aligned} \alpha'(G) &= |M| = |M_1| + |M_2| \leq |S| + \alpha'(G-S) \\ &\leq |S| + \frac{(n-|S|) - k_o(G-S)}{2} = \frac{n + |S| - k_o(G-S)}{2}. \end{aligned}$$

Thus,

$$\alpha'(G) \leq \min_{S \subset V(G)} \left\{ \frac{n + |S| - k_o(G-S)}{2} \right\}.$$

Next, let

$$p = \max_{S \subset V(G)} \{k_o(G-S) - |S|\}, \qquad (12.2)$$

where the maximum is taken over all proper subsets S of $V(G)$. If $S = \emptyset$, then $k_o(G-S) - |S| = k_o(G) \geq 0$ and so $p \geq 0$. If $p = 0$, then $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$ and so, by Tutte's theorem (Theorem 12.7), G has a 1-factor. Therefore, the formula (12.1) holds. Hence, we may assume that $p \geq 1$. Let $G' = G \vee K_p$. Then the order of G' is $n' = n + p$. Observe that G' is connected and that n and $k_o(G-S) - |S|$ are of the same parity for each proper subset S of $V(G)$. Thus, n' is even.

We now show that $k_o(G'-S') \leq |S'|$ for every proper subset S' of $V(G')$. First, the inequality holds for $S' = \emptyset$. So, we may assume that $S' \neq \emptyset$. Suppose first that S' does not contain all p vertices of K_p. Then $G' - S'$ is connected and so $k_o(G' - S') \le |S'|$. Hence, we may assume that S' contains all p vertices of K_p. Let $S = S' - V(K_p)$. So, $|S'| = |S| + p$ and $G' - S' = G - S$. Consequently, by (12.2),

$$k_o(G' - S') = k_o(G - S) \le |S| + p = |S'|.$$

Therefore, by Tutte's theorem, G' has a 1-factor and thus a perfect matching. Deleting the p vertices of K_p from G' produces a matching in G of size at least

$$\begin{aligned} \frac{n'}{2} - p &= \frac{n+p}{2} - p = \frac{n-p}{2} \\ &= \frac{1}{2}\left(n - \max_{S \subset V(G)} \{k_o(G-S) - |S|\}\right) \\ &= \min_{S \subset V(G)} \left\{\frac{n + |S| - k_o(G-S)}{2}\right\} \end{aligned}$$

and so

$$\alpha'(G) \ge \min_{S \subset V(G)} \left\{\frac{n + |S| - k_o(G-S)}{2}\right\}.$$

Therefore, (12.1) holds. ■

Vertex Covers and Edge Covers

A vertex and an edge are said to **cover each other** in a graph G if they are incident in G. A **vertex cover** in G is a set of vertices that covers all the edges of G. An **edge cover** in a graph G without isolated vertices is a set of edges that covers all the vertices of G. The minimum cardinality of a vertex cover in a graph G is called the **vertex covering number** of G and is denoted by $\beta(G)$. As expected, the **edge covering number** $\beta'(G)$ of a graph G (without isolated vertices) is the minimum cardinality of an edge cover in G.

For $s \le t$, we have $\beta(K_{s,t}) = s$ and $\beta'(K_{s,t}) = t$. Recall that the (vertex) independence number $\alpha(G)$ of G is the maximum cardinality among the independent sets of vertices in G. We have already seen that $\alpha(K_{s,t}) = t$. An independent set S of vertices in a graph G is **maximal independent** if S is not a proper subset of any other independent set of vertices in G. Therefore, there are two maximal independent sets of vertices in $K_{s,t}$, one with s vertices and the other with t vertices, namely the two partite sets of $K_{s,t}$. Furthermore, $\alpha'(K_{s,t}) = s$. As another illustration of these four parameters, we note that for $n \ge 2$, $\alpha(K_n) = 1$, $\alpha'(K_n) = \lfloor n/2 \rfloor$, $\beta(K_n) = n - 1$ and $\beta'(K_n) = \lceil n/2 \rceil$. The independence and covering concepts are summarized below.

$\alpha(G)$	vertex independence number	maximum number of vertices, no two of which are adjacent
$\alpha'(G)$	edge independence number	maximum number of edges, no two of which are adjacent
$\beta(G)$	vertex covering number	minimum number of vertices that cover the edges of G
$\beta'(G)$	edge covering number	minimum number of edges that cover the vertices of G

Gallai's Theorem

Observe that for the two graphs G of order n considered above, namely $K_{s,t}$ with $n = s + t$ and K_n, we have

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) = n.$$

These two examples illustrate the next theorem [99], due to the Hungarian mathematician Tibor Gallai (1912–1992).

Gallai was a winner of national mathematics competitions, along with Paul Erdős and Paul Turán, and became lifelong friends of both. As a consequence of his accomplishments, Gallai was admitted to Pázmány University in Budapest. He was one of a group of enthusiastic students in the 1930s in Budapest that included Erdős, Turán, George Szekeres and Esther Klein. Some of these students attended the graph theory course given by Dénes König, who was a professor at the Technical University of Budapest. This was to have a profound effect on Gallai's mathematical interests. Gallai, who was König's only doctoral student, helped König with his graph theory book and König mentioned some of Gallai's results in the book and used other ideas of Gallai. Many of Gallai's contributions to graph theory were to prove fundamental to the subject and aided in the rapid development of graph theory and combinatorics. For example, he was among the first to recognize the importance of so-called min-max theorems (such as Menger's theorem).

Gallai was an exceptionally modest individual and rarely made public appearances or attended conferences. In fact, much of his work became known only because of the efforts of his students. While Gallai was quick to praise the work of others, he often underestimated the merits of his own contributions, even though he had important results in many areas of graph theory. Consequently, he was notoriously slow to publish his own results. Several of his results went unpublished, only later to be independently rediscovered (and published) by others.

Theorem 12.11 (**Gallai's Theorem**) *If G is a graph of order n having no isolated vertices, then*

$$\alpha(G) + \beta(G) = n \tag{12.3}$$

and

$$\alpha'(G) + \beta'(G) = n. \tag{12.4}$$

Proof. We begin with (12.3). Let U be an independent set of vertices of G with $|U| = \alpha(G)$. Then the set $V(G) - U$ is a vertex cover in G. Therefore, $\beta(G) \leq n - \alpha(G)$. If, however, W is a set of $\beta(G)$ vertices that covers all edges of G, then $V(G) - W$ is independent; thus $\alpha(G) \geq n - \beta(G)$. This proves (12.3).

Next, we verify (12.4). Let E_1 be a maximum independent set of edges of G and so $|E_1| = \alpha'(G)$. Then E_1 covers $2\alpha'(G)$ vertices of G. The remaining $n - 2\alpha'(G)$ vertices can be covered by $n - 2\alpha'(G)$ edges not in E_1. Thus, $\beta'(G) \leq \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G)$ and so $\alpha'(G) + \beta'(G) \leq n$.

Next, let E' be an edge cover in G with $|E'| = \beta'(G)$. The minimality of E' implies that each component of $G[E']$ is a tree. Select from each component of $G[E']$ one edge, denoting the resulting set of edges by E''. Then E'' is independent and so $|E''| \leq \alpha'(G)$. If $G[E']$ is a forest with k components, then, by Corollary 3.16, the size of $G[E']$ is $n - k$. Thus,

$$\alpha'(G) + \beta'(G) \geq |E''| + |E'| = k + (n - k) = n,$$

completing the proof of (12.4) and the theorem. ■

An elementary relationship involving the independence and covering numbers is stated next.

Theorem 12.12 *If G is a graph having no isolated vertices, then*

$$\beta(G) \geq \alpha'(G) \quad \textit{and} \quad \beta'(G) \geq \alpha(G).$$

Proof. Let S be a vertex cover in G and let X be an independent set of edges with $|X| = \alpha'(G)$. For each edge e of X, there is a vertex v_e in S that is incident with e. Furthermore, for every two distinct edges e and f of X, the vertices v_e and v_f are distinct. Thus, $|S| \geq |X| = \alpha'(G)$, which implies that $\beta(G) \geq \alpha'(G)$.

For a graph G of order n, it therefore follows by Gallai's theorem (Theorem 12.11) that

$$n - \alpha(G) = \beta(G) \geq \alpha'(G) = n - \beta'(G)$$

and so $\beta'(G) \geq \alpha(G)$. ■

The König–Egerváry Theorem

While $\beta(G) \geq \alpha'(G)$ for every graph G, equality does not hold in general. If, however, G is bipartite, then $\beta(G) = \alpha'(G)$. This result was discovered independently by König [147] and Eugene Egerváry [75].

Theorem 12.13 (**The König–Egerváry Theorem**) *If G is a bipartite graph, then*

$$\beta(G) = \alpha'(G).$$

Proof. Since $\beta(G) \geq \alpha'(G)$, it suffices to show that $\beta(G) \leq \alpha'(G)$. Let U and W be the partite sets of G and let M be a maximum matching in G. Then $\alpha'(G) = |M|$. Denote by A the set of all M-unmatched vertices in U. (If $A = \emptyset$, then the proof is complete.) Observe that $|M| = |U| - |A|$. Let S be the set of all vertices of G that are connected to some vertex in A by an M-alternating path. Define $U' = S \cap U$ and $W' = S \cap W$.

As in the proof of Hall's theorem (Theorem 12.3), we have that $U' - A$ is matched to W' and that $N(U') = W'$. Since $U' - A$ is matched to W', it follows that $|U'| - |W'| = |A|$.

Observe that $C = (U - U') \cup W'$ is a vertex cover in G; for otherwise, there is an edge vw in G such that $v \in U'$ and $w \notin W'$. Furthermore,

$$|C| = |U| - |U'| + |W'| = |U| - |A| = |M|.$$

Therefore, $\beta(G) \leq |C| = |M| = \alpha'(G)$ and the proof is complete. ■

As a consequence of the König–Egerváry theorem (Theorem 12.13) and Gallai's theorem (Theorem 12.11), we have the following result of König and Richard Rado [192].

Theorem 12.14 (**The König–Rado Theorem**) *If G is a bipartite graph without isolated vertices, then*

$$\alpha(G) = \beta'(G).$$

Proof. By Theorems 12.11 and 12.13,

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) \text{ and } \beta(G) = \alpha'(G).$$

Thus $\alpha(G) = \beta'(G)$. ■

12.4 Domination

An area of graph theory that has received increased attention during recent decades is that of domination in graphs. A vertex v in a graph G is said to **dominate** itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices of G is a **dominating set** of G if every vertex of G is dominated by at least one vertex of S. Equivalently, a set S of vertices of G is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the **domination number** of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as a **minimum dominating set**. Although $\gamma(G)$ is the same notation that is used for the genus of a graph G, the notation in both instances is common and we will never use the terms domination number and the genus of a graph in the same discussion.

The sets $S_1 = \{v_1, v_2, y_1, y_2\}$ and $S_2 = \{w_1, w_2, x\}$ are both dominating sets for the graph G of Figure 12.8, indicated by solid circles. Since the dominating set S_2 consists of three vertices, $\gamma(G) \leq 3$. A vertex of degree 4 dominates five vertices. Because $\Delta(G) = 4$ and G has order 11, two vertices of G can dominate at most ten vertices of G and so $\gamma(G) \geq 3$. Thus, $\gamma(G) = 3$.

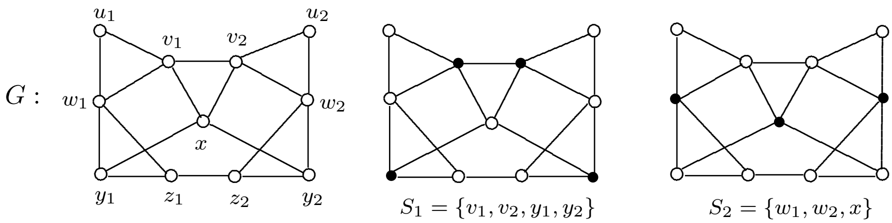


Figure 12.8: Dominating sets

Dominating sets appear to have their origins in the game of chess, where the goal is to cover or dominate the squares of a chessboard by certain chess pieces. In 1862 Carl Friedrich de Jaenisch [65] considered the problem of determining the minimum number of queens (which can move either horizontally, vertically or diagonally over any number of unoccupied squares) that can be placed on a chessboard such that every square is either occupied by a queen or can be occupied by one of the queens in a single move. The minimum number of such queens is 5 and one possible placement of five such queens is shown in Figure 12.9.

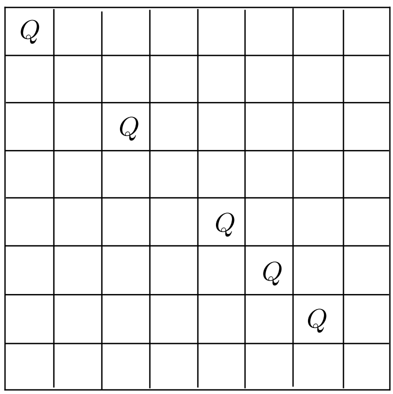


Figure 12.9: The minimum number of queens dominating the squares of a chessboard

Two queens on a chessboard are **attacking** if the square occupied by one of the queens can be reached by the other queen in a single move; otherwise, they are **nonattacking queens**. Clearly, every pair of queens on the chessboard of Figure 12.9 are attacking. The minimum number of nonattacking queens such that every square of the chessboard can be reached by one of the queens is also 5. A possible placement of five nonattacking queens is shown in Figure 12.10.

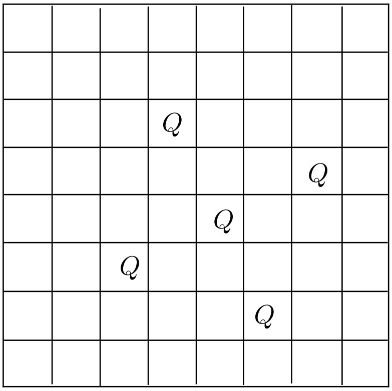


Figure 12.10: The minimum number of nonattacking queens dominating the squares of a chessboard

The connection between the chessboard problem described above and dominating sets in graphs is immediate. The 64 squares of a chessboard are the vertices of a graph G and two vertices (squares) are adjacent in G if each of the two squares can be reached by a queen on the other square in a single move. The graph G is referred to as the **queen's graph**. Then the minimum number of queens that dominate all the squares of a chessboard is $\gamma(G)$. The minimum number of nonattacking queens that dominate all the squares of a chessboard is the minimum cardinality of a dominating set in G that is also independent.

Domination as a theoretical area in graph theory was formalized by Claude Berge in 1958 [22, p. 40] and Oystein Ore [180, Chapter 13] in 1962. Since 1977, when Ernest Cockayne and Stephen Hedetniemi [60] presented a survey of domination results, domination theory has received considerable attention.

Minimal Dominating Sets

A **minimal dominating set** in a graph G is a dominating set that contains no dominating set as a proper subset. A minimal dominating set of minimum cardinality is, of course, a minimum dominating set and consists of $\gamma(G)$ vertices. For the graph G of Figure 12.8, the set $S_1 = \{v_1, v_2, y_1, y_2\}$ is a minimal dominating set that is not a minimum dominating set. Minimal dominating sets were characterized by Ore [180, p. 206].

Theorem 12.15 *A dominating set S of a graph G is a minimal dominating set of G if and only if every vertex v in S satisfies at least one of the following two properties:*

(i) there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$;

(ii) v is adjacent to no vertex of S.

Proof. First, observe that if each vertex v in S has at least one of the properties (i) and (ii), then $S - \{v\}$ is not a dominating set of G. Consequently, S is a minimal dominating set of G.

Conversely, assume that S is a minimal dominating set of G. Then certainly for each $v \in S$, the set $S - \{v\}$ is not a dominating set of G. Hence, there is a vertex w in $V(G) - (S - \{v\})$ that is adjacent to no vertex of $S - \{v\}$. If $w = v$, then v is adjacent to no vertex of S and (ii) holds. Suppose then that $w \neq v$. Since S is a dominating set of G and $w \notin S$, the vertex w is adjacent to at least one vertex of S. However, w is adjacent to no vertex of $S - \{v\}$. Consequently, $N(w) \cap S = \{v\}$ and so (i) holds. ■

Theorem 12.15 can be reworded as follows: A dominating set S of a graph G is a minimal dominating set of G if and only if for every vertex v in S either (1) v dominates some vertex of $V(G) - S$ that is not dominated by any other vertex of S or (2) no other vertex of S dominates v.

The following result of Ore [180, p. 207] gives a property of the complementary set of a minimal dominating set in a graph without isolated vertices.

Theorem 12.16 *If S is a minimal dominating set of a graph G without isolated vertices, then $V(G) - S$ is a dominating set of G.*

Proof. Let $v \in S$. Then v has at least one of the two properties (i) and (ii) described in the statement of Theorem 12.15. Suppose first that there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$. Hence, v is adjacent to some vertex in $V(G) - S$. Suppose next that v is adjacent to no vertex in S. Then v is an isolated vertex of the subgraph $G[S]$. Since v is not isolated in G, the vertex v is adjacent to some vertex of $V(G) - S$. Thus, $V(G) - S$ is a dominating set of G. ■

For graphs G without isolated vertices, we now have an upper bound for $\gamma(G)$ in terms of the order of G.

Corollary 12.17 *If G is a graph of order n without isolated vertices, then*

$$\gamma(G) \leq n/2.$$

Proof. Let S be a minimal dominating set of G. By Theorem 12.16, $V(G)-S$ is a dominating set of G. Thus, $\gamma(G) \leq \min\{|S|, |V(G) - S|\} \leq n/2$. ■

Nearly all connected graphs attaining the bound in Corollary 12.17 can be produced with the aid of the following operation. The **corona** $cor(H)$ of a graph H is that graph obtained from H by adding a pendant edge to each vertex of H. Let $G = cor(H)$, where G has order n. Then G has no isolated vertices and $\gamma(G) = n/2$. Indeed, Charles Payan and Nguyen Huy Xuong [184] showed that every component of a graph G of order n without isolated vertices having $\gamma(G) = n/2$ is either C_4 or the corona of some (connected) graph.

Vladimir Arnautov [7] and, independently, Payan [183] obtained an upper bound for the domination number that is an improvement to that given in Corollary 12.17 when $\delta(G) \geq 5$. We will see this again in Chapter 21.

Theorem 12.18 *If G is a graph of order n and $\delta(G) \geq 2$, then*

$$\gamma(G) \leq \frac{n + n\ln(\delta(G)+1)}{\delta(G)+1}.$$

Béla Bollobás and Ernest Cockayne [31] showed that every graph without isolated vertices contains a minimum dominating set in which every vertex satisfies property (i) described in the statement of Theorem 12.15.

Theorem 12.19 *Every graph G without isolated vertices contains a minimum dominating set S such that for every vertex v of S, there exists a vertex w of $G - S$ such that $N(w) \cap S = \{v\}$.*

Proof. Among all minimum dominating sets of G, let S be one such that $G[S]$ has maximum size. Suppose, to the contrary, that S contains a vertex v that does not have the desired property. Then by Theorem 12.15, v is an isolated vertex in $G[S]$. Moreover, every vertex of $V(G) - S$ that is adjacent to v is adjacent to some other vertex of S as well. Since G contains no isolated vertices, v is adjacent to a vertex w in $V(G) - S$. Consequently, $(S - \{v\}) \cup \{w\}$ is a minimum dominating set of G whose induced subgraph contains at least one edge incident with w and hence has a greater size than $G[S]$. This produces a contradiction. ■

The domination number of a graph without isolated vertices is also bounded above by all of the independence and covering numbers.

Theorem 12.20 *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \min\{\alpha(G), \alpha'(G), \beta(G), \beta'(G)\}.$$

Proof. Since every vertex cover of a graph without isolated vertices is a dominating set, as is every maximum independent set of vertices, $\gamma(G) \leq \beta(G)$ and $\gamma(G) \leq \alpha(G)$. Let X be an edge cover of cardinality $\beta'(G)$. Then every vertex of G is incident with at least one edge in X.

Let S be a set of vertices, obtained by selecting an incident vertex with each edge in X. Then S is a dominating set of vertices and $\gamma(G) \leq |S| \leq |X| = \beta'(G)$.

Next, let M be a maximum matching in G. We construct a set S of vertices consisting of one vertex incident with an edge of M for each edge of M. Let

$uv \in M$. The vertices u and v cannot be adjacent to distinct unmatched vertices x and y, respectively; for otherwise, (x, u, v, y) is an M-augmenting path in G, contradicting Theorem 12.2. If u is adjacent to a unmatched vertex, place u in S; otherwise, place v in S. This is done for each edge of M. Thus, S is a dominating set of G and $\gamma(G) \leq |S| = |M| = \alpha'(G)$. ∎

Vadim Vizing conjectured that the domination number of the Cartesian product of two graphs is always at least as large as the product of the domination numbers of these two graphs.

Vizing's Conjecture *For every two graphs G and H,*

$$\gamma(G \,\square\, H) \geq \gamma(G) \cdot \gamma(H).$$

Independent Dominating Sets

Recall that an independent set S of vertices of a graph G is a maximal independent set if S is not a proper subset of any independent set in G. Thus, every maximum independent set is maximal. The minimum cardinality of a maximal independent set of vertices of G is denoted by $i(G)$. Thus, for $K_{s,t}$, where $s < t$, there are only two maximal independent sets of vertices, namely, the partite sets of $K_{s,t}$. Hence, $\alpha(K_{s,t}) = t$ while $i(K_{s,t}) = s$.

Since every maximal independent set of vertices in a graph G is a dominating set of G, it follows that $\gamma(G) \leq i(G)$. Not every dominating set is independent, however. Indeed, not every minimum dominating set is independent. For example, in the graph G of Figure 12.11, the set $S_1 = \{u_1, u_2, v_1, v_2, w_1, w_2\}$ is a maximal independent set (and consequently a dominating set) of G; while $S_2 = \{x, y, z\}$ is a minimum dominating set of G and certainly S_2 is not independent. (These sets are indicated in Figure 12.11 by solid circles.) However, G does contain a minimum dominating set of G that is independent, namely, $S_3 = \{u, v, w\}$.

A set S of vertices in a graph G is an **independent dominating set** of G if S is both an independent and a dominating set of G. Thus, the sets S_1 and S_3 in Figure 12.11 are independent dominating sets while S_2 is not. The **independent domination number** $i(G)$ of G is the minimum cardinality among all independent dominating sets of G. That this is precisely the notation used for the minimum cardinality of a maximal independent set of vertices in a graph is justified by the following observation of Berge [25].

Theorem 12.21 *A set S of vertices in a graph is an independent dominating set if and only if S is a maximal independent set of vertices.*

Proof. We have already noted that every maximal independent set of vertices is a dominating set. Conversely, suppose that S is an independent dominating

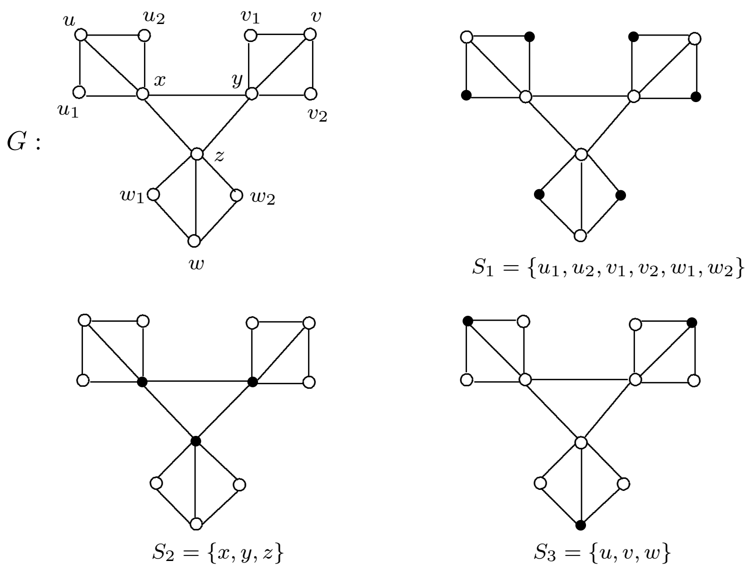


Figure 12.11: Dominating sets and maximal independent sets

set. Then S is independent and every vertex not in S is adjacent to a vertex of S and so S is a maximal independent set of vertices. ■

Another observation now follows.

Corollary 12.22 *Every maximal independent set of vertices in a graph is a minimal dominating set.*

Proof. Let S be a maximal independent set of vertices in a graph G. By Theorem 12.21, S is a dominating set. Since S is independent, certainly every vertex of S is adjacent to no vertex of S. Thus, every vertex of S satisfies property (ii) of Theorem 12.15. So, by Theorem 12.15, S is a minimal dominating set. ■

We have seen that $\gamma(G) \leq i(G)$ for every graph G. That the difference between the independent domination number and domination number of a graph can be arbitrarily large can be seen in the double star T containing two vertices of degree $k \geq 2$, where $i(T) = k$ and $\gamma(T) = 2$. Graphs G also exist for which $\gamma(G) = i(G)$. For example, $\gamma(G) = i(G) = 5$ for the queen's graph G, as Figures 12.9 and 12.10 show. The following theorem provides a large class of graphs for which the domination number and the independent domination number are equal.

Theorem 12.23 *If G is a claw-free graph, then $\gamma(G) = i(G)$.*

Proof. Since $\gamma(G) \leq i(G)$, it remains only to show that $i(G) \leq \gamma(G)$. Let S be a minimum dominating set of G. Thus, $|S| = \gamma(G)$. Let S' be a maximal independent subset of S and let $T = V(G) - N(S')$. That is, T is the set of vertices of G that are not dominated by any vertex of S'. In addition, let T' be a maximal independent subset of T. Since no vertex of T is adjacent to a vertex of S', no vertex of T' is adjacent to a vertex of S'; that is, $S' \cup T'$ is an independent set in G. Because S' is a maximal independent subset of S, every vertex of S is dominated by a vertex of S'. Similarly, every vertex of T is dominated by a vertex of T'. Consequently, $S' \cup T'$ is an independent dominating set of G and so $i(G) \leq |S' \cup T'|$.

Next, we show that $|S' \cup T'| \leq |S| = \gamma(G)$. Since (i) S' is a maximal independent subset of S, (ii) S is a dominating set of G, (iii) $S' \cup T'$ is an independent set of G and (iv) the graph G is claw-free, it follows that (a) every vertex of T' is adjacent to at least one vertex in $S - S'$ and (b) every vertex of $S - S'$ is adjacent to at most one vertex in T'. These facts imply that T' can be matched to a subset of $S - S'$ and so $|T'| \leq |S - S'|$. Then $i(G) \leq |S' \cup T'| = |S'| + |T'| \leq |S| = \gamma(G)$. Therefore, $i(G) = \gamma(G)$. ■

Exercises for Chapter 12

Section 12.1. Matchings

1. Show that every tree has at most one perfect matching.

2. Determine the maximum size m of a graph of order n having a maximum matching of k edges if (a) $n = 2k$, (b) $n = 2k + 2$.

3. Use Menger's theorem (Theorem 4.10) to prove the following implication in Hall's theorem (Theorem 12.3): *Let G be a bipartite graph with partite sets U and W. If Hall's condition is satisfied, then U can be matched to a subset of W.*

4. Let G be a bipartite graph with partite sets U and W such that $|U| = |W| = r \geq 1$. Suppose that $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_r\}$. Two vertices u_i and w_j are adjacent in G if and only if $i + j \geq r + 1$. Use Hall's theorem (Theorem 12.3) to show that G has a perfect matching.

5. Let G be a bipartite graph of size m with partite sets U and W where $|U| = |W| = k$.

   (a) Prove that if $m \geq k^2 - k + 1$, then G has a perfect matching.

   (b) Show that for $m = k^2 - k$, there exists a bipartite graph H with partite sets U and W such that $|U| = |W| = k$ but H does not have a perfect matching.

6. For which connected graphs G does the subdivision graph $S(G)$ (where every edge of G is subdivided exactly once) contain a perfect matching?

7. Let G be a bipartite graph with partite sets U and W where $|U| \leq |W|$. The **deficiency** $\operatorname{def}(S)$ of a set $S \subseteq U$ is defined as $\max\{|A| - |N(A)|\}$, where the maximum is taken over all nonempty subsets A of S. Show that
$$\alpha'(G) = \min\{|U|, |U| - \operatorname{def}(U)\}.$$

8. For a tree T, let $S(T)$ denote the subdivision graph of T.

   (a) Prove or disprove: There exists a tree T such that $S(T)$ has a perfect matching.

   (b) Give examples of three trees T_1, T_2, T_3 such that $\alpha'(S(T_1)) < 2\alpha'(T_1)$, $\alpha'(S(T_2)) = 2\alpha'(T_2)$ and $\alpha'(S(T_3)) > 2\alpha'(T_3)$.

9. The **matching graph** $M(G)$ of a nonempty graph G has the maximum matchings of G as its vertices, and two vertices M_1 and M_2 of $M(G)$ are adjacent if M_1 and M_2 differ in only one edge. Show that each cycle C_n, $n = 3, 4, 5, 6$, is the matching graph of some graph.

10. (a) Show that there is only one regular maximal planar graph G whose order $n \in \{5, 6, \ldots, 11\}$.
    (b) For the graph G in (a), show that $\overline{G}$ has a perfect matching M. Determine the genus of the graph $G + M$.
    (c) Prove that if G is a maximal planar graph G of order $n > 4$ whose complement contains a perfect matching M, then the genus
$$\gamma(G + M) \geq \frac{n}{12}.$$

11. A matching M' in a graph G is a **maximal matching** if there exists no matching M of G such that M' is a proper subset of M. Let G be a nonempty graph. Prove or disprove the following.
    (a) For every two nonadjacent edges e and f, there is a maximal matching of G containing e and f.
    (b) For every two adjacent edges e and f, there is a maximum matching of G containing one of e and f.

12. (a) Give an example of an infinite class of graphs G such that a matching M in G is maximal if and only if M is maximum.
    (b) Prove that if M is a matching of a graph G such that if $|M| < \frac{1}{2}\alpha'(G)$, then M is not a maximal matching of G.

Section 12.2. 1-Factors

13. (a) Prove or disprove: If G is a cubic graph containing three bridges not lying on a single path, then G does not have a 1-factor.
    (b) What does the solution to (a) tell us with regard to Errera's theorem (Theorem 12.9)?

14. Prove that every cubic graph with at most two bridges contains a 1-factor.

15. (a) Use Tutte's theorem (Theorem 12.7) to show that the graph G shown in Figure 12.12 does not have a 1-factor.
    (b) Petersen's theorem (Theorem 12.8) states that if G is a bridgeless cubic graph, then G has a 1-factor. Show that Petersen's theorem can be extended somewhat by proving that if G is a bridgeless graph having exactly one vertex of degree 7 and all others of degree 3, then G has a 1-factor.
    (c) Show that the result in (b) cannot be extended any further by giving an example of a bridgeless graph G having exactly two vertices of degree 7 and all others of degree 3 but G has no 1-factor.

16. (a) Prove that if a connected cubic graph has a 1-factor and a bridge, then the 1-factor must contain the bridge.

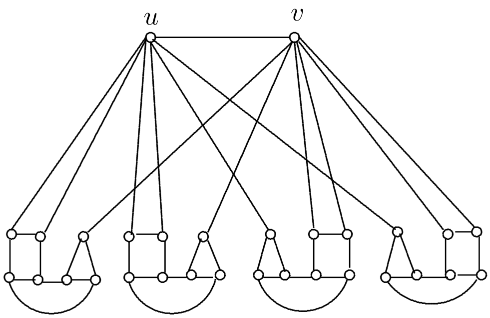


Figure 12.12: The graph G in Exercise 15

(b) Suppose that a connected cubic graph G has a minimum edge-cut consisting of two edges e and f. What can we conclude about any 1-factor of G? May it contain neither e nor f? May it contain both e and f? May it contain exactly one of e and f?

17. (a) Let G be a graph every vertex of which has odd degree and let $\{V_1, V_2\}$ be a partition of $V(G)$, where $[V_1, V_2]$ is the set of edges joining V_1 and V_2. Prove that $|V_1|$ and $|[V_1, V_2]|$ are of the same parity.

    (b) Prove that every $(2k+1)$-regular, $2k$-edge-connected graph, $k \geq 1$, contains a 1-factor.

18. Prove that if G is an r-regular, $(r-2)$-edge-connected graph ($r \geq 3$) of even order containing at most $r-1$ distinct edge-cuts of cardinality $r-2$, then G has a 1-factor.

19. Let T be the tree of Figure 12.13(a), all of whose vertices have degree 1 or 3, and let G be the cubic graph in Figure 12.13(b) constructed from T.

    (a) What is the minimum number of paths in T containing all bridges of T?

    (b) Does G contain a 1-factor?

20. A graph G is **factor-critical** if $G - v$ contains a 1-factor for every vertex v of G. Prove that a graph G of order n is factor-critical if and only if n is odd and $k_o(G-S) \leq |S|$ for every nonempty proper subset S of $V(G)$.

Section 12.3. Independence and Covers

21. Show that a graph G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|V(H)|$ for every subgraph H of G.

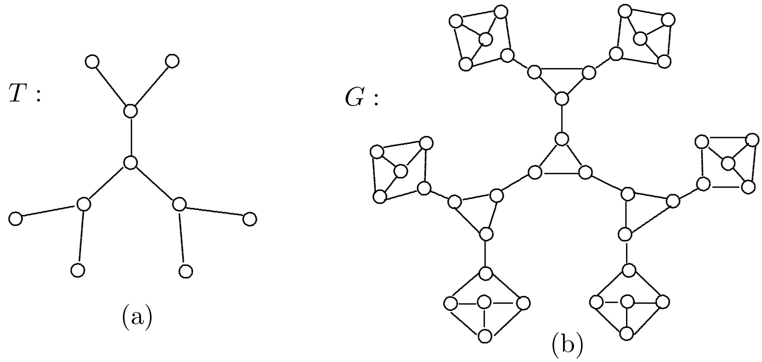


Figure 12.13: Graphs in Exercise 19

22. Let G be a connected graph of order n. Prove that

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$$

and show that these bounds are sharp.

23. Let G be a graph of order n without isolated vertices. Prove that

$$\left\lceil \frac{n}{2} \right\rceil \leq \beta'(G) \leq \left\lfloor \frac{n \cdot \Delta(G)}{1+\Delta(G)} \right\rfloor$$

and show that these bounds are sharp.

24. Characterize those nonempty graphs with the property that every two distinct maximal independent sets of vertices are disjoint.

25. Gallai's theorem states that $\alpha(G)+\beta(G)=n$ and $\alpha'(G)+\beta'(G)=n$ if G is a graph of order n having no isolated vertices. What do $\alpha(G)+\beta(G)$ and $\alpha'(G)+\beta'(G)$ equal if G is a graph of order n having k isolated vertices?

26. Prove or disprove: A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G)=\beta'(G)$.

Section 12.4. Domination

27. Determine the domination numbers of the 3-cube Q_3 and the 4-cube Q_4.

28. (a) Determine and verify a formula for $\gamma(C_n)$.

    (b) Determine and verify a formula for $\gamma(P_n)$.

29. State and prove a characterization of those graphs G with $\gamma(G)=1$.

30. Prove that if G is a graph of order n, then $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$.

31. Prove that if the diameter of a connected graph G of order n is at least 3, then $\gamma(\overline{G}) = 2$.

32. (a) Does there exist a graph G without isolated vertices such that $\gamma(G) = \beta(G)$ but $\gamma(G)$ is strictly less than each of the numbers $\alpha(G)$, $\alpha'(G)$ and $\beta'(G)$?
    (b) The question in (a) suggests three other questions. State and answer these questions.

33. Use Theorem 12.20 to give an alternative proof of Corollary 12.17: *If G is a graph of order n without isolated vertices, then $\gamma(G) \leq n/2$.*

34. Show that if G is a graph of order $n \geq 2$, then $3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1$.

35. (a) Prove or disprove: If a graph G contains a minimum dominating set that is also independent, then $\gamma(G) = i(G)$.
    (b) Show that a graph need not have any minimum dominating set that is independent.

36. (a) Prove that if G is a connected bipartite graph of order n, then $i(G) \leq n/2$.
    (b) Show that the bound given in (a) is sharp.

37. For each integer $k \geq 3$, show that there exists a graph G such that $i(G) = k$ and $\gamma(G) = 3$.

38. (a) Prove that if G is a graph of order n without isolated vertices, then $\gamma(G) + i(G) \leq n$.
    (b) Investigate the sharpness of the bound given in (a).

39. Prove or disprove: Every nontrivial connected graph has two disjoint minimal dominating sets.

40. Prove or disprove: If G is a nontrivial connected graph, then $i(L(G)) = \gamma(L(G))$.

41. A set S of vertices of a graph G is a **total dominating set** of G if for every vertex v of G, there is a vertex $u \in S$ such that $v \in N(u)$. The **total domination number** $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G.

    (a) Prove for every nontrivial connected graph G, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.
    (b) Show that the bounds in (a) are sharp by showing that there are two infinite classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of graphs such that (1) for each $H \in \mathcal{C}_1$, $\gamma(H) = \gamma_t(H)$ and (2) for each $F \in \mathcal{C}_2$, $\gamma_t(F) = 2\gamma(F)$.

42. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a dominating set for a nontrivial graph G. Then every vertex of G is dominated by one or more vertices of S. Associated with each vertex u of G is an ordered k-tuple $\text{dom}(u) = (a_1, a_2, \ldots, a_k)$, where $a_i = 1$ if u is dominated by v_i and $a_i = 0$ otherwise. A dominating set is called an **irregular dominating set** of G if $\text{dom}(u) \neq \text{dom}(v)$ for every two distinct vertices u and v of G.

    (a) Give an example of a nontrivial graph G_1 that does not have an irregular dominating set.

    (b) Give an example of a nontrivial graph G_2 that has an irregular dominating set.

    (c) Characterize those nontrivial graphs having an irregular dominating set.

43. Is the converse of Theorem 12.23 true?

Chapter 13

Factorization and Decomposition

Over the years there have been numerous conjectures and open problems dealing with collections of subgraphs of a given graph G where each edge of G belongs to exactly one subgraph in the collection. While many of these conjectures have been verified, others remain unresolved. Collections of subgraphs with this property are often divided into two categories, depending on whether the subgraphs are required to be spanning subgraphs of G.

13.1 Factorization

Recall that a factor of a graph G is a spanning subgraph of G. A graph G is said to be **factorable** into the factors $F_1, F_2, \ldots, F_t$ if these factors are (pairwise) edge-disjoint and $\cup_{i=1}^{t} E(F_i) = E(G)$. If G is factored into $F_1, F_2, \ldots, F_t$, then $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ is called a **factorization** of G.

If there exists a factorization $\mathcal{F}$ of a graph G such that each factor in $\mathcal{F}$ is a k-factor (for a fixed positive integer k), then G is **k-factorable**. If G is a k-factorable graph, then necessarily G is r-regular for some integer r that is a multiple of k.

If a graph G is factorable into $F_1, F_2, \ldots, F_t$, where each $F_i \cong H$ for some graph H, then we say that G is **H-factorable** and that G has an **isomorphic factorization** into (copies of) the factor H. Certainly, if a graph G is H-factorable, then the size of H divides the size of G. A graph G of order $n = 2k$ is therefore 1-factorable if and only if G is kK_2-factorable.

One problem in this area that has received a great deal of attention is whether certain graphs are 1-factorable. Of course, only regular graphs of even order can be 1-factorable. Trivially, every 1-regular graph is 1-factorable. Since a 2-regular graph contains a 1-factor if and only if every component is an even cycle, it is precisely these 2-regular graphs that are 1-factorable. The situation for r-regular graphs, $r \geq 3$, in general, or even 3-regular graphs in particular, is considerably more complicated. By Petersen's theorem (Theorem 12.8), every bridgeless cubic graph contains a 1-factor. Consequently, every bridgeless cubic graph can be factored into a 1-factor and a 2-factor. In 1884 Peter Guthrie Tait wrote that every cubic graph is 1-factorable but this result was *not true without limitation.* Evidently, Petersen interpreted this rather vague statement to mean that every bridgeless cubic graph is 1-factorable. Not every bridgeless cubic graph is 1-factorable, however. As Petersen himself observed [187], the graph shown in Figure 13.1 is not 1-factorable (see Exercise 1). This graph would later be called the Petersen graph, of course.

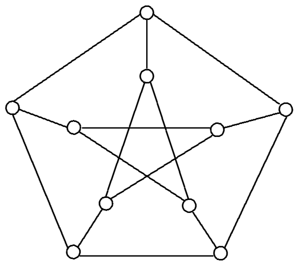


Figure 13.1: The Petersen graph: a bridgeless cubic graph that is not 1-factorable

The next two results describe two well-known classes of 1-factorable graphs, the first of which is due to König [146].

Theorem 13.1 *Every r-regular bipartite graph, $r \geq 1$, is 1-factorable.*

Proof. We proceed by induction on r. The result is obvious if $r = 1$. Assume that every $(r-1)$-regular bipartite graph is 1-factorable where $r - 1 \geq 1$ and let G be an r-regular bipartite graph. By Theorem 12.4, G contains a 1-factor F_1. Then $G - E(F_1)$ is an $(r-1)$-regular bipartite graph. By the induction hypothesis, $G - E(F_1)$ can be factored into $r-1$ 1-factors, say $F_2, F_3, \ldots, F_r$. Then $\{F_1, F_2, \ldots, F_r\}$ is a 1-factorization of G. ■

As a consequence of Theorem 13.1 then, the 4-regular bipartite graph $K_{4,4}$ can be factored into four 1-factors. If $U = \{A, K, Q, J\}$ and $W = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ are the two partite sets of $K_{4,4}$, then one possible 1-factorization of $K_{4,4}$ is shown in Figure 13.2.

The 1-factorization of $K_{4,4}$ shown in Figure 13.2 gives rise to a partition of the edge set of $K_{4,4}$ into four perfect matchings F_1, F_2, F_3, F_4 whose edges can be listed as follows:

F_1	$A\ \spadesuit$	$K\ \heartsuit$	$Q\ \diamondsuit$	$J\ \clubsuit$
F_2	$J\ \diamondsuit$	$Q\ \clubsuit$	$K\ \spadesuit$	$A\ \heartsuit$
F_3	$K\ \clubsuit$	$A\ \diamondsuit$	$J\ \heartsuit$	$Q\ \spadesuit$
F_4	$Q\ \heartsuit$	$J\ \spadesuit$	$A\ \clubsuit$	$K\ \diamondsuit$

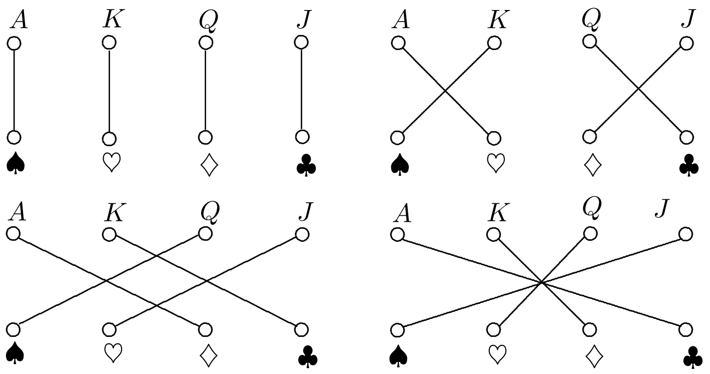


Figure 13.2: A 1-factorization of $K_{4,4}$

This, of course, is very suggestive of the 16 playing cards of the four denominations ace, king, queen, jack and the four suits spades, hearts, diamonds, clubs. In the table above, each denomination and each suit appears exactly once in each row and in each column. This card problem dates back to the year 1723. Thus, the elements of the set U as well as the elements of W can be arranged in a 4×4 table so that every element appears exactly once in each row and column. More generally, when the elements of an n-element set are arranged in an $n \times n$ table so that every element appears exactly once in each row and column, a **Latin square** results. So the table above gives rise to a pair of Latin squares. For example, if Γ is a (finite) group of order n with $\Gamma = \{g_1, g_2, \ldots, g_n\}$, say, and an $n \times n$ table is constructed where the entry in row i, column j of the table is the element $g_i g_j$ of Γ, then a Latin square results. The topic of Latin squares has a long history and is an area of much interest in the subject of combinatorics.

1-Factorable Complete Graphs

We now describe a second class of 1-factorable graphs.

Theorem 13.2 *For each positive integer k, the complete graph K_{2k} is 1-factorable.*

Proof. The result is obvious for $k = 1$, so we assume that $k \geq 2$. Denote the vertex set of K_{2k} by $\{v_0, v_1, \ldots, v_{2k-1}\}$ and arrange the vertices $v_1, v_2, \ldots, v_{2k-1}$ cyclically in a regular $(2k-1)$-gon, placing v_0 in the center. Now join every two vertices by a straight line segment, producing K_{2k}. For $i = 1, 2, \ldots, 2k-1$, define the 1-factor F_i to consist of the edge v_0v_i together with all those edges perpendicular to v_0v_i. Then $\{F_1, F_2, \cdots, F_{2k-1}\}$ is a 1-factorization of K_{2k}. ■

The construction described in the proof of Theorem 13.2 is illustrated in Figure 13.3 for the graph K_6.

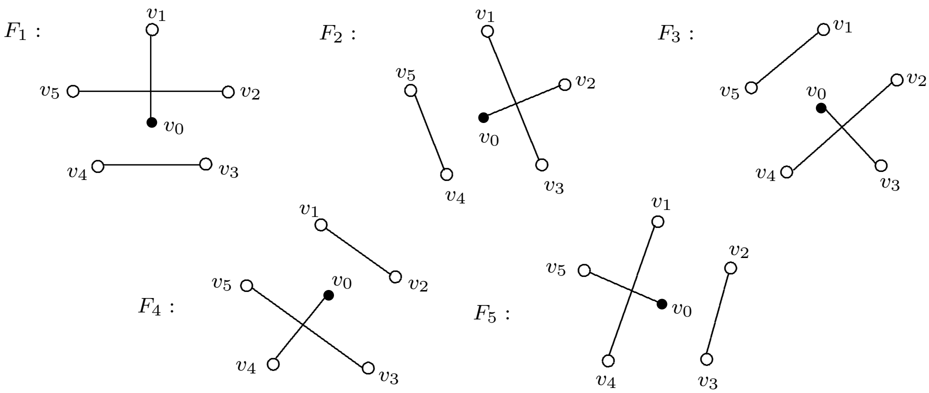


Figure 13.3: A 1-factorization of K_6

In the 1-factorization of K_{2k} described in the proof of Theorem 13.2, the 1-factor F_1 consists of the edge v_0v_1 and all edges perpendicular to v_0v_1, namely, $v_2v_{2k-1}, v_3v_{2k-2}, \ldots, v_kv_{k+1}$. If the k edges of F_1 are rotated clockwise through an angle of $2\pi/(2k-1)$ radians, then the 1-factor F_2 is obtained. In general, if the edges of F_1 are rotated clockwise through an angle of $2\pi j/(2k-1)$ radians, where $0 \leq j \leq 2k-2$, then the 1-factor F_{j+1} is produced. A factorization of a graph obtained in this manner is referred to as a **cyclic factorization**.

Such a factorization can be viewed in another way. Let K_{2k} be drawn as described in the proof of Theorem 13.2. We now label each edge of K_{2k} with one of the integers $0, 1, \ldots, k-1$. Indeed, we will assign $2k-1$ edges of K_{2k} the label i for $i = 0, 1, \ldots, k-1$. Every edge of the type v_0v_i $(1 \leq i \leq 2k-1)$ is labeled 0. Now let C denote the cycle $(v_1, v_2, \ldots, v_{2k-1}, v_1)$. For $1 \leq s < t \leq 2k-1$, the edge v_sv_t is assigned the distance label $d_C(v_s, v_t)$. Observe that $1 \leq d_C(v_s, v_t) \leq k-1$. Thus, the $2k-1$ edges of C are labeled 1; in general, then, $2k-1$ edges of K_{2k} are labeled the integer i for $0 \leq i \leq k-1$. Observe, further, that F_1 contains k edges, one of which is labeled i for $0 \leq i \leq k-1$. Moreover, when an edge of F_1 labeled i $(0 \leq i \leq k-1)$ is rotated clockwise through an angle of $2\pi j/(2k-1)$ radians, $0 \leq j \leq 2k-2$, an edge of F_{j+1} also labeled i is obtained. Hence, a 1-factorization of K_{2k} is produced.

The 1-Factorization Conjecture

We saw in Dirac's theorem (Corollary 6.2) that if G is a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for every vertex v of G, then G is Hamiltonian. Consequently, if G is an r-regular graph of even order $n \geq 4$ such that $r \geq n/2$, then G contains a Hamiltonian cycle C. Since C is an even cycle, C can be factored into two 1-factors. If there exists a 1-factorization of $G - E(C)$, then G is 1-factorable. This is certainly the case if $r = 3$. For which values of r and n this can be done is not known. There is a conjecture, however, dealing with this topic that is believed to have originated in a 1986 paper [52] by Amanda G. Chetwynd and Anthony J. W. Hilton.

The 1-Factorization Conjecture *If G is an r-regular graph of even order n such that (1) $r \geq n/2$ if $n \equiv 2 \pmod 4$ or (2) $r \geq (n-2)/2$ if $n \equiv 0 \pmod 4$, then G is 1-factorable.*

Employing a lengthy proof, Béla Csaba, Daniela Kühn, Allan Lo, Deryk Osthus and Andrew Treglown [63] were successful in 2014 in showing that there exists an integer n_0 such that the 1-Factorization Conjecture is true for every even integer $n \geq n_0$.

Theorem 13.3 *For sufficiently large even integers n, the 1-Factorization Conjecture holds.*

The bound on r in the 1-Factorization Theorem cannot be improved. For example, suppose that $n \equiv 2 \pmod 4$. Then $n = 2k$ for some odd integer $k \geq 3$. The graph $G = 2K_k$ is a $(k-1)$-regular graph of order n, where $k - 1 = \frac{n}{2} - 1$. Since G does not have a 1-factor, G is certainly not 1-factorable. Thus, it is not true that if G is an r-regular graph of even order n for which $r \geq \frac{n}{2} - 1$, then G is 1-factorable.

2-Factorable Graphs

Determining whether a graph is 2-factorable or, more specifically, can be factored into Hamiltonian cycles has also been a problem of great interest. An obvious necessary condition for a graph G to be 2-factorable is that G is $2k$-regular for some positive integer k. Julius Petersen [186] showed that this condition is sufficient as well as necessary.

Theorem 13.4 *A graph G is 2-factorable if and only if G is $2k$-regular for some positive integer k.*

Proof. We have already noted that if G is a 2-factorable graph, then G is regular of positive even degree. Conversely, suppose that G is $2k$-regular for some integer $k \geq 1$. Assume, without loss of generality, that G is connected. Hence, G is Eulerian and so contains an Eulerian circuit C.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We define a bipartite graph H with partite sets $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$, where

$$E(H) = \{u_i w_j : \ v_j \text{ immediately follows } v_i \text{ on } C\}.$$

The graph H is k-regular and so, by Theorem 13.1, H is 1-factorable. Hence, $\{F_1, F_2, \cdots, F_k\}$ is a 1-factorization of H.

Corresponding to each 1-factor F_ℓ of H is a permutation α_ℓ on the set $\{1, 2, \ldots, n\}$, defined by $\alpha_\ell(i) = j$ if $u_i w_j \in E(F_\ell)$. Let α_ℓ be expressed as a product of disjoint permutation cycles. There is no permutation cycle of length 1 in this product; for if (i) were a permutation cycle, then this would imply that $\alpha_\ell(i) = i$. However, this further implies that $u_i w_i \in E(F_\ell)$ and that $v_i v_i \in E(C)$, which is impossible. Also, there is no permutation cycle of length 2 in this product; for if $(i\ j)$ were a permutation cycle, then $\alpha_\ell(i) = j$ and $\alpha_\ell(j) = i$. This would indicate that $u_i w_j, u_j w_i \in E(F_\ell)$ and that v_j both immediately follows and precedes v_i on C, contradicting the fact that no edge is repeated on a circuit. Thus, the length of every permutation cycle in α_ℓ is at least 3.

Each permutation cycle in α_ℓ therefore gives rise to a cycle in G, and the product of the disjoint permutation cycles in α_ℓ produces a collection of mutually disjoint cycles in G containing all vertices of G, that is, a 2-factor in G. Since the 1-factors in H are mutually edge-disjoint, the resulting 2-factors in G are mutually edge-disjoint. Hence, G is 2-factorable. ■

To illustrate the proof of Theorem 13.4, let $G = K_{2,2,2}$ be the 4-regular graph of order 6 shown in Figure 13.4(a). An Eulerian circuit of G is

$$C = (v_1, v_2, v_3, v_1, v_5, v_2, v_6, v_5, v_4, v_6, v_3, v_4, v_1).$$

Following the proof of Theorem 13.4, we construct the bipartite graph H shown in Figure 13.4(b) having partite sets $U = \{u_1, u_2, \ldots, u_6\}$ and $W = \{w_1, w_2, \ldots, w_6\}$. Since $v_1 v_2 \in E(C)$, it follows that $u_1 w_2 \in E(H)$. Similarly, $u_2 w_3, u_3 w_1 \in E(H)$ and so on. Figure 13.4(c) shows one possible 1-factorization of H into the two 1-factors F_1 and F_2. Corresponding to F_1 and F_2 are the permutations below:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix} = (1\ 2\ 3)(4\ 6\ 5)$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 1 & 2 & 3 \end{pmatrix} = (1\ 5\ 2\ 6\ 3\ 4).$$

These two permutations then give rise to the 2-factors G_1 and G_2 shown in Figure 13.4(d). Thus, $\{G_1, G_2\}$ is a 2-factorization of G.

Hamiltonian Factorable Graphs

It is an immediate consequence of Theorem 13.4 that there exists a factorization of every regular graph G of positive even degree in which every factor is a union of cycles. This brings up the problem of whether there exists a factorization of G in which every factor is a single cycle, that is, a Hamiltonian cycle of G. A **Hamiltonian factorization** of a graph G is a factorization of G such that every factor is a Hamiltonian cycle of G and a graph possessing such a factorization is **Hamiltonian factorable**. Certainly, every Hamiltonian

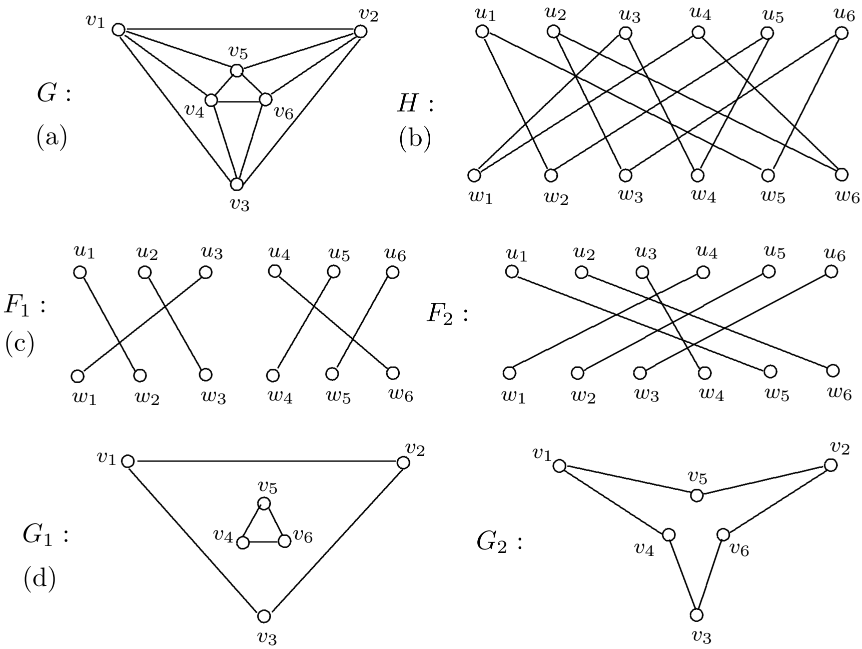


Figure 13.4: Constructing a 2-factorization of $G = K_{2,2,2}$

factorable graph is a 2-connected regular graph of positive even degree. The converse of this statement is not true, however, as the graph H of Figure 13.5 shows. Any Hamiltonian cycle of H necessarily contains both edges uv and xy and so H does not contain two edge-disjoint Hamiltonian cycles.

H :

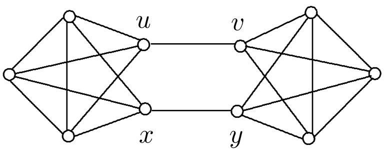


Figure 13.5: A 2-factorable graph that is not Hamiltonian factorable

Complete graphs of odd order are not only 2-factorable, they are Hamiltonian factorable. The following result and construction are credited to Walecki (see [5]).

Theorem 13.5 *For every positive integer k, the graph K_{2k+1} is Hamiltonian factorable.*

Proof. Since the result is clear for $k = 1$, we may assume that $k \geq 2$. Let $V(K_{2k+1}) = \{v_0, v_1, \ldots, v_{2k}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{2k}$ cyclically in a regular $2k$-gon and place v_0 in some convenient position. Join every two vertices by a straight line segment, thereby producing K_{2k+1}. We define the edge set of F_1 to consist of v_0v_1, v_0v_{k+1}, all edges parallel to v_1v_2 and all edges parallel to $v_{2k}v_2$ (see F_1 in Figure 13.6 for the case $k = 3$). In general, for $i = 1, 2, \ldots, k$, we define the edge set of the factor F_i to consist of v_0v_i, v_0v_{k+i}, all edges parallel to v_iv_{i+1} and all edges parallel to $v_{i-1}v_{i+1}$, where the subscripts are expressed modulo $2k$. Since

$$(v_0, v_i, v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \cdots, v_{k+i-1}, v_{k+i+1}, v_{k+i}, v_0)$$

is a Hamiltonian cycle for each i $(1 \le i \le k)$, it follows that $\{F_1, F_2, \ldots, F_k\}$ is a Hamiltonian factorization of K_{2k+1}. ■

The Hamiltonian factorization described in the proof of Theorem 13.5 is illustrated in Figure 13.6 for the complete graph K_7.

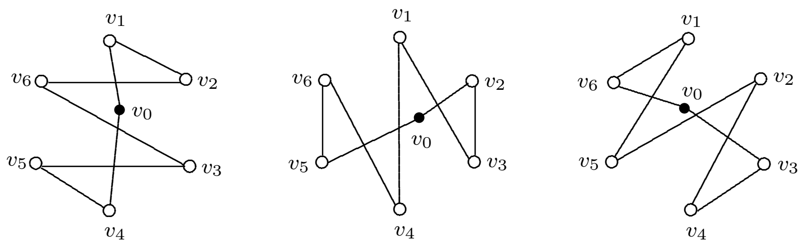


Figure 13.6: A Hamiltonian factorization of K_7

The factorization described in the proof of Theorem 13.5 is therefore a cyclic factorization. That is, if we place the vertex v_0 in the center of the regular $2k$-gon and rotate the edges of the Hamiltonian cycle F_1 clockwise through an angle of $2\pi/2k = \pi/k$ radians, then the Hamiltonian cycle F_2 is produced. Indeed, if we rotate the edges of F_1 clockwise through an angle of $\pi j/k$ radians for any integer j with $1 \le j \le k-1$, then the Hamiltonian cycle F_{j+1} is produced and the desired Hamiltonian factorization of K_{2k+1} is obtained (see Exercise 7).

Another factorization result now follows readily from Theorem 13.5 (see Exercise 10).

Corollary 13.6 *For each positive integer k, the complete graph K_{2k} can be factored into k Hamiltonian paths.*

Using the construction employed in the proof of Theorem 13.5, we can obtain yet another factorization result (see Exercise 12).

Theorem 13.7 *For each positive integer k, the graph K_{2k} can be factored into $k-1$ Hamiltonian cycles and a 1-factor.*

From Theorems 13.5 and 13.7, we then have the following corollary.

Corollary 13.8 *Every complete graph of order at least* 2 *can be factored into Hamiltonian cycles and at most one* 1*-factor.*

Nash-Williams [176] conjectured that a factorization of the type stated in Corollary 13.8 not only exists in every complete graph but in every r-regular graph of order n if r is sufficiently large.

The Hamiltonian Factorization Conjecture *If G is an r-regular graph of order n such that $r \geq n/2$, then G can be factored into Hamiltonian cycles and at most one* 1*-factor.*

Csaba, Kühn, Lo, Osthus and Treglown [63] showed that the Hamiltonian Factorization Conjecture holds if n is sufficiently large.

Theorem 13.9 *The Hamiltonian Factorization Conjecture holds for every sufficiently large integer n.*

Theorem 13.9 tells us, of course, that for sufficiently large integers n, regular graphs G of order n with $\delta(G) \geq n/2$ contain many edge-disjoint Hamiltonian cycles. Csaba, Kühn, Lo, Osthus and Treglown [63] showed that this is true as well for graphs that are not regular by answering a question posed by Nash-Williams [175].

Theorem 13.10 *For every sufficiently large integer n, every graph G of order n with $\delta(G) \geq n/2$ contains at least $(n-2)/8$ edge-disjoint Hamiltonian cycles.*

The number of edge-disjoint Hamiltonian cycles guaranteed by Theorem 13.10 cannot, in general, be improved (see Exercise 18).

13.2 Decomposition

The concept of decomposition in graph theory is very similar to that of factorization. A **decomposition** $\mathcal{D}$ of a graph G is a collection $\{H_1, H_2, \ldots, H_t\}$ of nonempty subgraphs such that $H_i = G[E_i]$ for some (nonempty) subset E_i of $E(G)$, where $\{E_1, E_2, \ldots, E_t\}$ is a partition of $E(G)$. Thus, no subgraph H_i in a decomposition of G contains isolated vertices. If $\mathcal{D}$ is a decomposition of G, then we say G is decomposed into the subgraphs $H_1, H_2, \ldots, H_t$. Indeed, if $\mathcal{D}$ is a decomposition of a graph G where each subgraph H_i is a spanning subgraph of G, then $\{H_1, H_2, \ldots, H_t\}$ is a factorization of G. On the other hand, every factorization of a nonempty graph G also gives rise to a decomposition of G.

If $\mathcal{D} = \{H_1, H_2, \ldots, H_t\}$ is a decomposition of a graph G such that $H_i \cong H$ for some graph H for each i $(1 \leq i \leq t)$, then $\mathcal{D}$ is an H**-decomposition** of G. If there exists an H-decomposition of a graph G, then G is said to be H**-decomposable**. The graph $G = K_{2,2,2}$ (the graph of the octahedron) is H-decomposable for the graph H shown in Figure 13.7. An H-decomposition of G is also shown in Figure 13.7.

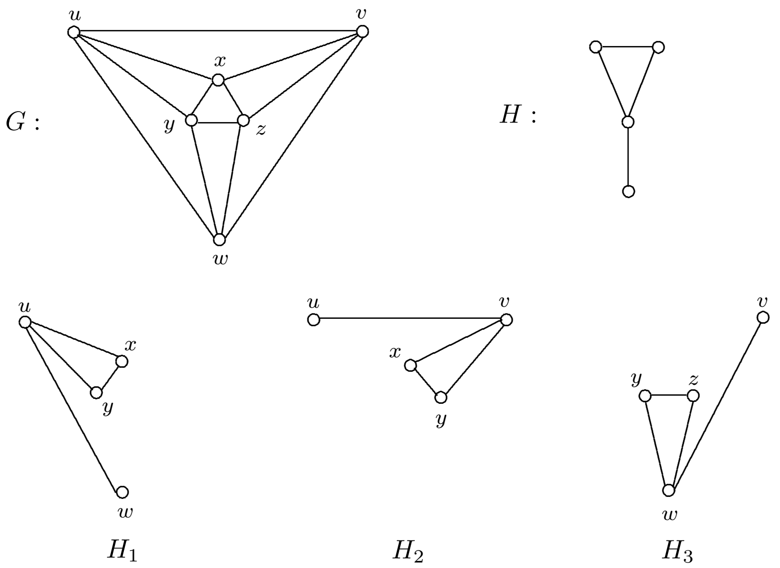


Figure 13.7: An H-decomposable graph

The decomposition shown in Figure 13.7 is a *cyclic* decomposition. In general, a **cyclic decomposition** of a graph G into k copies of a subgraph H is obtained by

(a) drawing G in an appropriate manner,

(b) selecting a suitable subgraph H_1 of G that is isomorphic to H and

(c) rotating the vertices and edges of H_1 through an appropriate angle $k-1$ times to produce the k copies of H in the decomposition.

If G is an H-decomposable graph for some graph H, then certainly H is a subgraph of G and the size of H divides the size of G. Although this last condition is necessary, it is not sufficient. For example, the graph $K_{1,4}$ is a subgraph of the graph G of Figure 13.7 and the size 4 of $K_{1,4}$ divides the size 12 of G, but yet G is not $K_{1,4}$-decomposable (see Exercise 20).

The basic problem here is whether a given graph G is H-decomposable for some subgraph H of G such that the size of H divides the size of G. Of course, every nonempty graph is K_2-decomposable. For a connected graph G to be P_3-decomposable, the size of G must be even. It turns out that this condition is sufficient as well as necessary for a graph to be P_3-decomposable.

Theorem 13.11 *A nontrivial connected graph G is P_3-decomposable if and only if G has even size.*

Proof. We have already noted that if G is P_3-decomposable, then G has even size. For the converse, assume that G has even size. Suppose first that G is Eulerian, where the edges of G are encountered in the order $e_1, e_2, \ldots, e_m$. Then each of the sets $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\ldots$,$\{e_{m-1}, e_m\}$ induces a copy of P_3; so G is P_3-decomposable. Otherwise, G has $2k$ odd vertices for some $k \geq 1$. By Theorem 5.3, $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_k$, where for each i, $G[E_i]$ is an open trail T_i of even length connecting odd vertices of G. (That is, G can be decomposed into k open trails of even length connecting odd vertices.) Then, as with the Eulerian circuit above, the edges of each trail T_i can be paired off so that each pair of consecutive edges on each trail induce a copy of P_3. Thus, G is P_3-decomposable. ■

Many theorems and conjectures dealing with decompositions involve H-decompositions for some graph H. Richard Wilson [258] proved that for every graph H without isolated vertices, there exist infinitely many positive integers n such that K_n is H-decomposable.

Theorem 13.12 *Let H be a graph of size m without isolated vertices and let*

$$d = \gcd\{\deg v : \ v \in V(H)\}.$$

Then there exists a positive integer N such that whenever (i) $n \geq N$, (ii) $m \mid \binom{n}{2}$ and (iii) $d \mid (n-1)$, then K_n is H-decomposable.

There is a conjecture involving decompositions of a different nature [2].

The Ascending Subgraph Decomposition Conjecture *Every graph of size $\binom{k+1}{2}$ for some positive integer k has a decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ where G_i has size i for $1 \leq i \leq k$ and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, k-1$.*

To illustrate the Ascending Subgraph Decomposition Conjecture, we consider the graph $G = C_5 \,\square\, K_2$ of size $15 = \binom{6}{2} = \sum_{i=1}^{5} i$, shown in Figure 13.8. The decomposition $\mathcal{D} = \{G_1, G_2, G_3, G_4, G_5\}$, also drawn in this figure, shows that this conjecture holds for this graph G. While this conjecture has been verified for many classes of graphs, very little of a general nature is known.

13.3 Cycle Decomposition

As we have noted, the vast majority of factorization and decomposition conjectures and results deal with factoring or decomposing complete graphs into a specific graph or graphs. A problem of particular interest concerns determining those complete graphs that are K_3-decomposable or, equivalently, C_3-decomposable. This problem has a curious history.

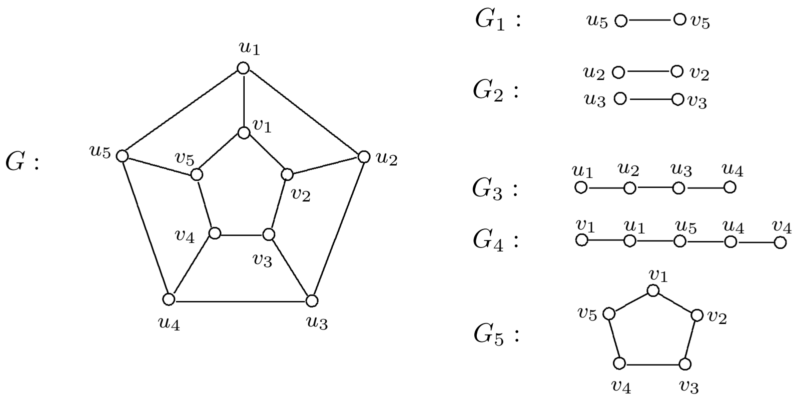


Figure 13.8: An ascending subgraph decomposition of $G = C_5 \square K_2$

Steiner Triple Systems

A **Steiner triple system of order** n consists of a set S of n elements and a collection T of 3-element subsets of S, called **triples**, such that every pair of elements of S belongs to exactly one triple in T. For example,

$$S = \{0, 1, \ldots, 6\} \text{ and } T = \{013, 124, 235, 346, 450, 561, 602\}$$

is a Steiner triple system of order 7. A Steiner triple system of order n corresponds to a K_3-decomposition of K_n. In particular, the Steiner triple system of order 7 mentioned above results in the cyclic K_3-decomposition of K_7 shown in Figure 13.9.

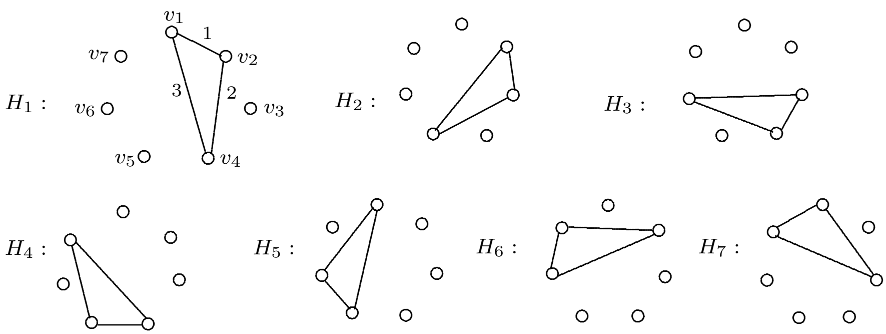


Figure 13.9: A K_3-decomposition of K_7

Steiner triple systems are named for the Swiss mathematician Jakob Steiner (1796–1863). Although Steiner asked for which integers n such a triple system of order n exists, he was not the first to ask this question. The problem of determining those integers n for which there exists a Steiner triple system of order n was initially posed by the Reverend Wesley S. B. Woolhouse as Prize question 1733 in the *Lady's and Gentlemen's Diary* of 1844. This problem was solved three years later by the Reverend Thomas Penygton Kirkman [143].

Theorem 13.13 *There exists a Steiner triple system of order $n \geq 3$ if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.*

Kirkman's Schoolgirl Problem

While the necessity for the existence of a Steiner triple system of order n stated in Theorem 13.13 is quite straightforward (see Exercise 21), the sufficiency requires the construction of appropriate Steiner triple systems. Despite the fact that Kirkman's work preceded Steiner's, these systems are named for Steiner, not Kirkman. Since $7 \equiv 1 \pmod 6$, there is a Steiner triple system of order 7, as we saw above. Furthermore, $9 \equiv 3 \pmod 6$ and so there is a Steiner triple system of order 9. Likewise, there is a Steiner triple system of order 13 and a Steiner triple system of order 15. A Steiner triple system of order 15 must then consist of 35 triples. Also, in the *Lady's and Gentlemen's Diary* of 1850, Kirkman posed an even more challenging question which led to a famous problem.

Kirkman's Schoolgirl Problem: *A school mistress has fifteen schoolgirls whom she wishes to take on a daily walk. The girls are to walk in five rows of three girls each. It is required that every two girls should walk in the same row exactly once a week. Can this be done?*

If we think about this a bit, we see that the question can be rephrased as follows: Is there a $5K_3$-factorization of K_{15}? If we label the vertices of K_{15} by the schoolgirls, numbered $1, 2, \ldots, 15$ say, then we see that a solution is given below.

Sunday	{1, 2, 3}, {4, 8, 12}, {5, 10, 14}, {6, 9, 15}, {7, 11, 13}
Monday	{1, 4, 5}, {2, 9, 11}, {3, 13, 15}, {6, 8, 14}, {7, 10, 12}
Tuesday	{1, 6, 7}, {2, 8, 10}, {3, 12, 14}, {4, 9, 13}, {5, 11, 15}
Wednesday	{1, 8, 9}, {2, 5, 7}, {3, 13, 14}, {4, 10, 15}, {6, 11, 12}
Thursday	{1, 10, 11}, {2, 12, 15}, {3, 4, 6}, {5, 8, 13}, {7, 9, 14}
Friday	{1, 12, 13}, {2, 5, 6}, {3, 9, 10}, {4, 11, 14}, {7, 8, 15}
Saturday	{1, 14, 15}, {2, 4, 7}, {3, 8, 11}, {5, 9, 12}, {6, 10, 13}

Although there is a $5K_3$-factorization of K_{15}, it turns out that there is no cyclic $5K_3$-factorization of K_{15} (which makes it more difficult to construct such a factorization). Suppose that a graph G of order n and size m has a

2-factorization in which every component of each 2-factor is a triangle. Then G is tK_3-factorable for some positive integer t. Therefore, $n = 3t$. Since the degree of every vertex of G is 2 in each 2-factor of G, every vertex has even degree in G. Therefore, $n-1$ is even and so $n = 3t$ is odd, which implies that t is odd. Therefore, $t = 2k+1$ for some nonnegative integer k and so $n = 6k+3$. Hence, G is $(2k+1)K_3$-factorable.

A **Kirkman triple system** of order n is an n-element set S, a collection T of triples of S and a partition $\mathcal{P}$ of T such that

(1) every two distinct elements of S belong to a unique triple in T and

(2) every element of S belongs to a unique triple in each element of $\mathcal{P}$.

Consequently, if there is a Kirkman triple system of order n, then $n = 6k+3$ for some nonnegative integer k. In fact, there is a Kirkman triple system of order $6k+3$ if and only if there is a $(2k+1)K_3$-factorization of K_{6k+3}. In 1971 Dijen Ray-Chaudhuri and Richard Wilson [195] established the existence of a Kirkman triple system of order $n = 6k+3$ for every nonnegative integer k.

Theorem 13.14 *There exists a Kirkman triple system of order $n \geq 3$ if and only if $n \equiv 3 \pmod 6$.*

According to Veblen's Theorem (Theorem 5.4), a nontrivial connected graph G can be decomposed into cycles if and only if G is Eulerian. Therefore, the complete graph K_n has a cycle decomposition if and only if $n \geq 3$ is odd. We saw in Theorem 13.5 that K_n is C_n-decomposable if $n \geq 3$ is odd and in Theorem 13.14 that K_n is C_3-decomposable if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. These two results can be stated in a different manner, namely: For an odd integer $n \geq 3$ and $k = 3$ or $k = n$, the complete graph K_n is C_k-decomposable if and only if $k \mid \binom{n}{2}$. This brings up the question of for which other integers k and n, the complete graph K_n is C_k-decomposable. In two separate papers, one by Brian Alspach and Heather Gavlas [6] in 2001 and the other by Matea Šajna [213] in 2002, this question was answered. These results, combined with those stated above, are summarized in the following theorem.

Theorem 13.15 *For an odd integer $n \geq 3$ and an integer k with $3 \leq k \leq n$, the complete graph K_n is C_k-decomposable if and only if $k \mid \binom{n}{2}$.*

For example, the order of the complete graph K_9 is the odd integer $n = 9$ and its size is $\binom{n}{2} = \binom{9}{2} = 36$. Since $6 \mid \binom{9}{2}$, it follows by Theorem 13.15 that K_9 is C_6-decomposable. A C_6-decomposition of K_9, where $V(K_9) = \{v_1, v_2, \ldots, v_9\}$, is shown in Figure 13.10.

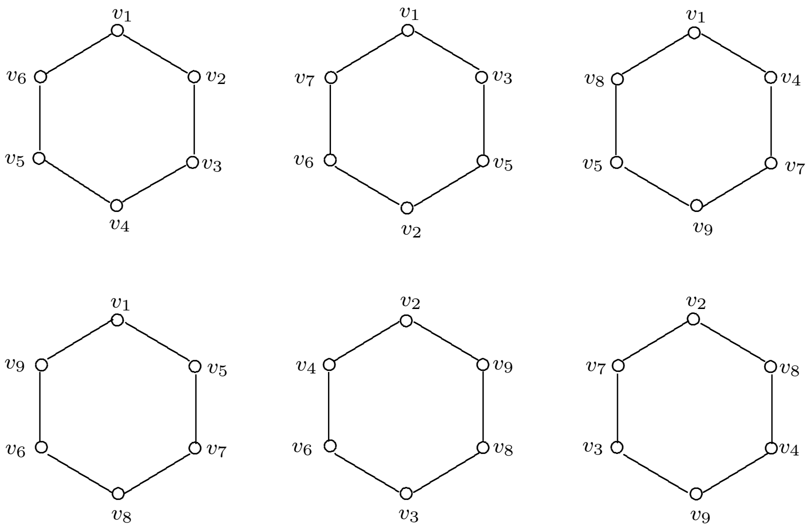


Figure 13.10: A C_6-decomposition of K_9

Alspach's Conjecture

In 1981 Alspach [4] conjectured that an even stronger result may exist concerning cycle decompositions of complete graphs.

Alspach's Conjecture *Let $n \geq 3$ be an odd integer and let $k_1, k_2, \ldots, k_t$ be t integers such that $3 \leq k_i \leq n$ for each i $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} k_i = \binom{n}{2}$. Then the complete graph K_n can be decomposed into the cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_t}$.*

In 2012, this conjecture was verified by Darryn Bryant, Daniel Horsley and William Pettersson [39].

Theorem 13.16 *For every t integers $k_1, k_2, \ldots, k_t$ and odd integer $n \geq 3$ such that $3 \leq k_i \leq n$ for each i $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} k_i = \binom{n}{2}$, the complete graph K_n can be decomposed into the cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_t}$.*

For example, for the odd integer $n = 7$ and the five integers $k_1 = 3$, $k_2 = k_3 = k_4 = 4$ and $k_5 = 6$, it follows that $3 \leq k_i \leq 7$ for $1 \leq i \leq 5$ and $\sum_{i=1}^{5} k_i = \binom{7}{2} = 21$. Therefore, by Theorem 13.16, the complete graph K_7 can be decomposed into the cycles C_3, C_4, C_4, C_4, C_6. One such decomposition is shown in Figure 13.11.

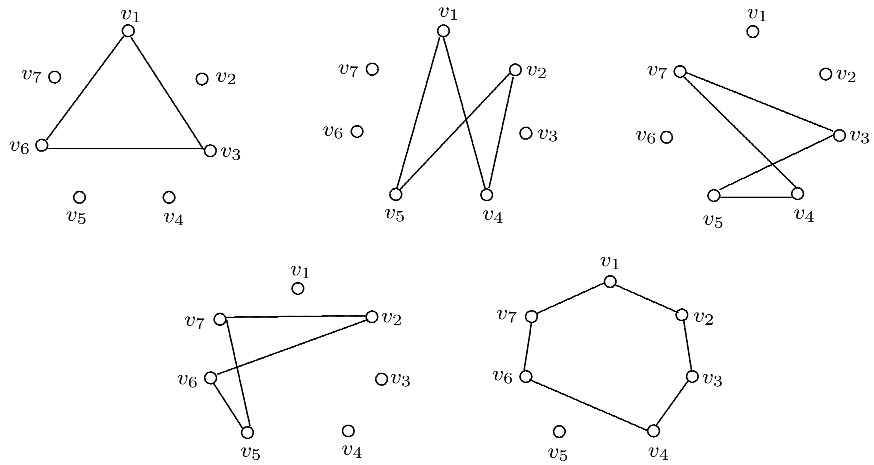


Figure 13.11: A decomposition $\{C_3, C_4, C_4, C_4, C_6\}$ of K_7

The fact that the hypothesis of Theorem 13.16 requires the graph in question to be an Eulerian complete graph is critical to its conclusion. For example, the noncomplete Eulerian graph G in Figure 13.12 has size $15 = \binom{6}{2}$ but the sizes of cycles in cycle decompositions of G cannot be specified. In fact, there are only two cycle decompositions of G, namely $\{C_5, C_{10}\}$ and $\{C_3, C_3, C_3, C_3, C_3\}$.

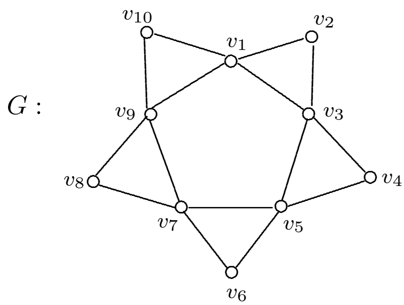


Figure 13.12: An Eulerian graph with
exactly two cycle decompositions

The mathematicians who are responsible for Theorems 13.15 and 13.16 also obtained corresponding theorems for complete graphs of even order (see [6, 213, 39]), resulting in the following.

Theorem 13.17 *For every t integers $k_1, k_2, \ldots, k_t$ and even integer $n \geq 4$ such that $3 \leq k_i \leq n$ for each i $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} k_i = \binom{n}{2} - \frac{n}{2} = \frac{n^2-2n}{2}$, the complete graph K_n can be decomposed into a 1-factor and the cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_t}$.*

By Theorem 5.3, if G is a connected graph with $2k \geq 2$ odd vertices for some positive integer k, then G has a decomposition into k open trails. In fact, if G has such a decomposition where r of the trails have odd length and another such decomposition where t of the trails have odd length (where, necessarily, r and t are of the same parity) and s is an integer such that $r < s < t$ and $s - r$ is even, then there is a decomposition of G into k open trails, exactly s of which are of odd length (see Exercise 31). Moreover, if G is an Eulerian graph having a decomposition into circuits, r of which are of odd length and another such decomposition, t of which are of odd length, then for every integer s with $r < s < t$ such that $s - r$ is even, there is a circuit decomposition of G, where exactly s of the circuits are of odd length (see Exercise 32). This has led to the following open problem [48].

The Eulerian Cycle Decomposition Problem *For an integer $n \geq 3$, determine an expression $f(n)$ in terms of n such that if G is an Eulerian graph of order n with $\delta(G) \geq f(n)$ for which G has a cycle decomposition with r odd cycles and a cycle decomposition with t odd cycles, then G has a cycle decomposition with s odd cycles for every integer s with $r < s < t$ where $s - r$ is even.*

The Eulerian graph in Figure 13.12 shows that $f(n) > n/5$ if $f(n)$ is a linear function of n and $f(n) \geq 4$ if $f(n)$ is a constant.

13.4 Graceful Graphs

A **graph labeling** is an assignment of labels (typically positive integers or nonnegative integers) to the vertices or edges (or both) of a graph, often satisfying some prescribed requirements. Many labelings can be traced to a 1967 paper titled "On certain valuations of the vertices of a graph", authored by Alexander Rosa [211]. Rosa received his Ph.D. in 1968 from the Slovak Academy of Science under Anton Kotzig, whom we will encounter shortly. The labeling that has drawn the most attention is Rosa's β-valuation, which was later called a graceful labeling by Solomon W. Golomb [106]. This terminology has become standard.

A nonempty graph G of size m is **graceful** if it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, m\}$ in such a way that the induced edge labeling, which assigns the integer $|i - j|$ to the edge joining vertices labeled i and j, results in the m edges of G being labeled $1, 2, \ldots, m$. Such a labeling is called a **graceful labeling**. Thus, a graceful graph is a graph that admits a graceful labeling.

All of the graphs K_3, K_4, $K_4 - e$ and C_4 are graceful, as is illustrated in Figure 13.13. Here, the vertex labels are placed within the vertices and the induced edge labels are placed near the relevant edges.

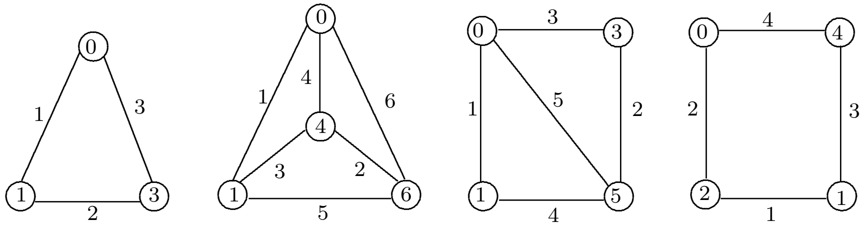


Figure 13.13: Graceful graphs

All connected graphs of order at most 4 are graceful. There are exactly three connected graphs of order 5 that are not graceful, namely C_5, K_5 and $K_1 \vee 2K_2$. These graphs are shown in Figure 13.14. That the first two of these graphs are not graceful will be shown in Theorems 13.21 and 13.22. Exercise 33 concerns the third graph.

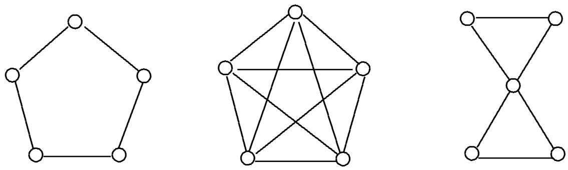


Figure 13.14: The three connected graphs of order 5 that are not graceful

For each graceful graph H of size m, the complete graph K_{2m+1} is H-decomposable. In fact, there is a cyclic decomposition of K_{2m+1} into copies of H. This observation is also due to Rosa [211].

Theorem 13.18 *If H is a graceful graph of size m, then the complete graph K_{2m+1} can be cyclically decomposed into copies of H.*

Proof. Since H is graceful, there is a graceful labeling of H, that is, the vertices of H can be labeled from a subset of $\{0, 1, \ldots, m\}$ so that the induced edge labels are $1, 2, \ldots, m$. Let $V(K_{2m+1}) = \{v_0, v_1, \ldots, v_{2m}\}$ where the vertices of K_{2m+1} are arranged cyclically in a regular $(2m+1)$-gon, denoting the resulting $(2m+1)$-cycle by C. A vertex labeled i for each integer i $(0 \le i \le m)$ in H is placed at v_i in K_{2m+1} and this is done for each vertex of H. Every edge of H is drawn as a straight line segment in K_{2m+1}, denoting the resulting copy of H in K_{2m+1} by H_1. Hence, $V(H_1) \subseteq \{v_0, v_1, \ldots, v_m\}$.

Each edge v_sv_t of K_{2m+1} $(0 \le s, t \le 2m)$ is labeled $d_C(v_s, v_t)$, where then $1 \le d_C(v_s, v_t) \le m$. Consequently, K_{2m+1} contains exactly $2m+1$ edges labeled i for each i $(1 \le i \le m)$ and H_1 contains exactly one edge labeled i $(1 \le i \le m)$. Whenever an edge of H_1 is rotated through an angle (clockwise, say) of $2\pi k/(2m+1)$ radians, where $1 \le k \le 2m$, an edge of the same label is obtained. Denote the subgraph obtained by rotating H_1 through a clockwise angle of

$2\pi k/(2m+1)$ radians by H_{k+1}. Then $H_{k+1} \cong H$ and a cyclic decomposition of K_{2m+1} into $2m+1$ copies of H results. ■

As an illustration of Theorem 13.18, we consider the graceful graph $H = P_3$. A graceful labeling of H is shown in Figure 13.15 as well as the resulting cyclic H-decomposition of K_5.

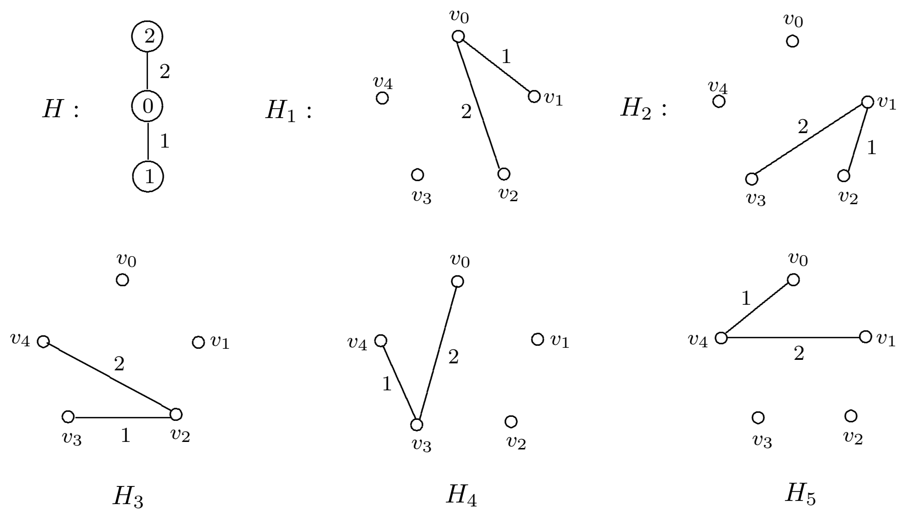


Figure 13.15: A cyclic decomposition of K_5 into copies of the graceful graph P_3

Although K_{2m+1} has a cyclic decomposition into copies of every graceful graph H of size m, it is not necessary for H to be graceful in order for K_{2m+1} to have a cyclic H-decomposition. For example, we have seen that C_5 is not graceful; yet K_{11} has a cyclic C_5-decomposition. The first subgraph C_5 in such a decomposition is depicted in Figure 13.16.

The Graceful Tree Conjecture

While many graphs are known to be nongraceful, none of these is a tree. Indeed, it has been conjectured by Anton Kotzig that every nontrivial tree is graceful.

The Graceful Tree Conjecture *Every nontrivial tree is graceful.*

If the Graceful Tree Conjecture is true and T is a tree of size m, then by Theorem 13.18, K_{2m+1} is T-decomposable. Regardless of whether every tree is graceful, Gerhard Ringel [202] made the following conjecture concerning decompositions of complete graphs into trees.

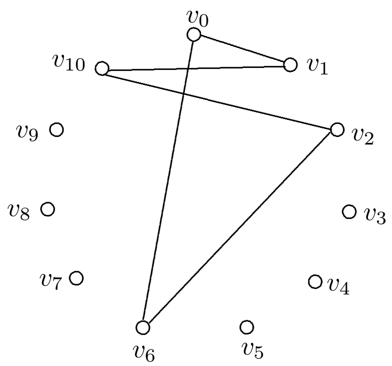


Figure 13.16: A cyclic decomposition of K_{11} into copies of the nongraceful graph C_5

Ringel's Conjecture *For every tree T of size m, the complete graph K_{2m+1} is T-decomposable.*

Not only does the truth of Kotzig's conjecture imply the truth of Ringel's conjecture, the truth of Kotzig's conjecture also implies the truth of the following conjecture, due jointly to Ringel and Kotzig.

The Ringel-Kotzig Conjecture *For every tree T of size m, K_{2m+1} can be cyclically decomposed into copies of T.*

Although there are no general sufficient conditions for a graph to be graceful, there are necessary conditions.

Theorem 13.19 *If G is a graceful graph of size m, then there exists a partition of $V(G)$ into two subsets V_e and V_o such that the number of edges joining V_e and V_o is $\lceil m/2 \rceil$.*

Proof. Let a graceful labeling of G be given. Denote the set of vertices labeled with an even integer by V_e and the set of vertices labeled with an odd integer by V_o. All edges labeled with an odd integer must then join a vertex of V_e and a vertex of V_o. Since there are $\lceil m/2 \rceil$ such edges, the result follows. ■

Theorem 13.19 will be revisited again in Chapter 19. The following necessary condition for an Eulerian graph to be graceful is due to Rosa [211].

Theorem 13.20 *If G is a graceful Eulerian graph of size m, then*

$$m \equiv 0 \pmod 4 \text{ or } m \equiv 3 \pmod 4.$$

Proof. Let $C = (v_0, v_1, \ldots, v_{m-1}, v_m = v_0)$ be an Eulerian circuit of G, and let a graceful labeling of G be given that assigns the integer a_i $(0 \le a_i \le m)$ to v_i for $0 \le i \le m$, where, of course, $a_i = a_j$ if $v_i = v_j$. Thus, the label of the edge $v_{i-1}v_i$ is $|a_i - a_{i-1}|$. Observe that

$$|a_i - a_{i-1}| \equiv (a_i - a_{i-1}) \pmod 2$$

for $1 \le i \le m$. Thus, the sum of the labels of the edges of G is

$$\sum_{i=1}^{m} |a_i - a_{i-1}| \equiv \sum_{i=1}^{m} (a_i - a_{i-1}) \equiv 0 \pmod 2,$$

that is, the sum of the edge labels of G is even. However, the sum of the edge labels is

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2};$$

so, $m(m+1)/2$ is even. Consequently, $4 \mid m(m+1)$, which implies that $4 \mid m$ or $4 \mid (m+1)$ so that $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$. ■

Classes of Graceful Graphs

We now determine which graphs belonging to certain well-known classes are graceful. Rosa [211] determined those cycles that are graceful.

Theorem 13.21 *The cycle C_n is graceful if and only if*

$$n \equiv 0 \pmod 4 \text{ or } n \equiv 3 \pmod 4.$$

Proof. Since C_n is an Eulerian graph, it follows by Theorem 13.20 that if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$, then C_n is not graceful; so, it remains only to show that if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$, then C_n is graceful. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Assume first that $n \equiv 0 \pmod 4$. We assign v_i the label a_i, where

$$a_i = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd} \\ n + 1 - \frac{i}{2} & \text{if } i \text{ is even and } i \le n/2 \\ n - \frac{i}{2} & \text{if } i \text{ is even and } i > n/2. \end{cases}$$

It remains to observe that this labeling is graceful. This labeling is illustrated in Figure 13.17(a) for $n = 12$.

Next, assume that $n \equiv 3 \pmod 4$. In this case, we assign v_i the label b_i, where

$$b_i = \begin{cases} n + 1 - \frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i-1}{2} & \text{if } i \text{ is odd and } i < (n-1)/2 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } i > (n-1)/2. \end{cases}$$

This is a graceful labeling of C_n. An illustration is given in Figure 13.17(b) for $n = 11$. ■

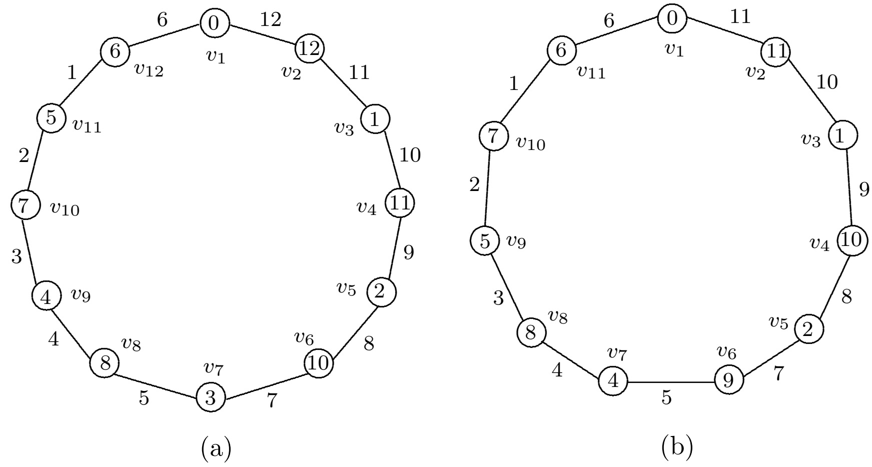


Figure 13.17: Graceful labelings of C_{12} and C_{11}

If G is a graceful graph of order n and size m, then, of course, the vertices of G can be labeled with the elements of a set $\{a_1, a_2, \ldots, a_n\} \subseteq \{0, 1, \ldots, m\}$ so that the induced edge labels are precisely $1, 2, \ldots, m$. This means one vertex in some pairs of adjacent vertices is labeled 0 and the other vertex in the pair is labeled m. Also, if we were to replace each vertex label a_i by $m - a_i$, then we have a new graceful labeling, called the **complementary labeling** of G.

We saw in Figure 13.13 that the complete graphs K_3 and K_4 are graceful. It is very easy to show that K_2 is graceful. The following result of Golomb [106] shows that there are no other graceful complete graphs.

Theorem 13.22 *The complete graph K_n $(n \geq 2)$ is graceful if and only if $n \leq 4$.*

Proof. We have already observed that K_n is graceful if $2 \leq n \leq 4$. Assume then that $n \geq 5$ and suppose, to the contrary, that K_n is graceful. Hence, there exists a graceful labeling of the vertices of K_n from an n-element subset of $\{0, 1, \ldots, m\}$, where $m = \binom{n}{2}$.

We have already seen that every graceful labeling of a graph of size m requires 0 and m to be vertex labels. Since some edge of K_n must be labeled $m-1$, some vertex of K_n must be labeled 1 or $m-1$. We may assume, without loss of generality, that a vertex of K_n is labeled 1; otherwise, we may use the complementary labeling.

To produce an edge labeled $m - 2$, we must have adjacent vertices labeled 0, $m-2$ or 1, $m-1$ or 2, m. If a vertex is labeled 2 or $m-1$, then we have two edges labeled 1, which is impossible. Thus, some vertex of K_n must be labeled $m-2$.

Since we now have vertices labeled $0, 1, m-2$ and m, we have edges labeled $1, 2, m-3, m-2, m-1$ and m. To have an edge labeled $m-4$, we must have a vertex labeled 4 because all other choices result in two edges with the same label.

Now we have vertices labeled $0, 1, 4, m-2$ and m, which results in edges labeled $1, 2, 3, 4, m-6, m-4, m-2, m-1$ and m. However, it is quickly seen that there is no vertex label that will produce the edge label $m-5$ without also producing a duplicate edge label. Hence, no graceful labeling of K_n exists. ∎

Unlike the classes of graphs we have considered, where some graphs are graceful and others are not, every complete bipartite graph is graceful.

Theorem 13.23 *Every complete bipartite graph is graceful.*

Proof. Let $K_{s,t}$ have partite sets U and W, where $|U| = s$ and $|W| = t$. Label the vertices of U by $0, 1, \ldots, s-1$ and label the vertices of W by $s, 2s, \ldots, (t-1)s, ts$ (see Figure 13.18). This is a graceful labeling. ∎

$K_{s,t}$:

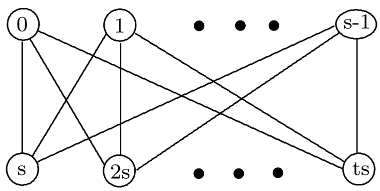


Figure 13.18: A graceful labeling of $K_{s,t}$

Exercises for Chapter 13

Section 13.1. Factorization

1. Show that the Petersen graph is not 1-factorable.

2. Give an example of a connected graph G of composite size having the property that whenever F is a factor of G and the size of F divides the size of G, then G is F-factorable.

3. (a) Prove that Q_n is 1-factorable for all $n \geq 1$.
   (b) Prove that Q_n is k-factorable if and only if $k \mid n$.

4. It was shown (following Theorem 13.3) that the bound on r in the 1-Factorization Theorem cannot be improved when $n \equiv 2 \pmod 4$ by observing that the graph $G = 2K_{n/2}$ contains no 1-factor and is therefore not 1-factorable. Show that if $n \geq 10$ and $n \equiv 2 \pmod 4$, then a 2-connected r-regular graph G of order n, where $r = \frac{n}{2} - 1$, need not be 1-factorable.

5. Show for even integers $n \geq 8$ with $n \equiv 0 \pmod 4$ that the bound on r in the 1-Factorization Theorem cannot be improved.

6. Use the proof of Theorem 13.4 to give a 2-factorization of the 4-regular graph C_8^2 (the square of the 8-cycle C_8).

7. Use the proof of Theorem 13.5 to produce a Hamiltonian factorization of K_9.

8. Use Theorems 13.2 and 13.5 to prove the following: Let r and n be integers with $0 \leq r \leq n-1$. Then there exists an r-regular graph of order n if and only if r and n are not both odd.

9. Let k be a nonnegative even integer and $n \geq 5$ an odd integer with $k \leq n-3$. Prove that there exists a graph G of order n, all of whose vertices have degree k or $k+2$.

10. Prove Corollary 13.6: *For each positive integer k, the complete graph K_{2k} can be factored into k Hamiltonian paths.*

11. Show that K_{2k+1} cannot be factored into Hamiltonian paths.

12. Give a constructive proof of Theorem 13.7: *For each positive integer k, the graph K_{2k} can be factored into $k-1$ Hamiltonian cycles and a 1-factor.*

13. Show for every positive integer a and odd positive integer b, that there exists a factorization of K_{2a+b+1} into a Hamiltonian cycles and b 1-factors.

14. Show for every positive integer k that the complete graph K_{6k+4} is 3-factorable, where each 3-factor is Hamiltonian.

15. By Theorem 13.4 (and Exercise 6) the 4-regular graph C_8^2 is 2-factorable. Show that C_8^2 is, in fact, Hamiltonian factorable.

16. By Dirac's theorem (Corollary 6.2), every 4-regular graph of order 8 is Hamiltonian. Suppose that a 4-regular graph G of order 8 has a Hamiltonian cycle C with the property that $H = G - E(C) = C_3 + C_5$. Show that G is Hamiltonian factorable.

17. Theorem 13.10 is an improvement of Dirac's theorem (Corollary 6.2) for sufficiently large integers n. Show that there is no equivalent improvement of Ore's theorem (Theorem 6.1) by describing an infinite family of graphs G of order n with $\sigma_2(G) \geq n$ such that G does not contain $(n-2)/8$ edge-disjoint Hamiltonian cycles.

18. Let $\{A, B\}$ be a partition of the vertex set of a graph G of order $n = 8k+2 \geq 10$ where $|A| = 4k$ and $|B| = 4k+2$. The edge set of G is defined by $G[A] = \overline{K}_{4k}$, $G[B] = (2k+1)K_2$ and all possible edges joining a vertex of A and a vertex of B.

    (a) Show that G is Hamiltonian.

    (b) Show that G contains at most $\lfloor |B|/4 \rfloor = k = (n-2)/8$ edge-disjoint Hamiltonian cycles.

Section 13.2. Decomposition

19. Show that there is a cyclic P_5-decomposition of the graph of the octahedron (see Figure 13.7).

20. Show that the graph of the octahedron (see Figure 13.7) is not $K_{1,4}$-decomposable.

21. Prove that if there exists a Steiner triple system of order $n \geq 3$, then either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.

22. Prove that nine schoolgirls can take four daily walks in three rows of three girls each so that no two girls walk in the same row twice.

23. For each odd integer $n \geq 5$, show that there is an ascending subgraph decomposition of K_n for which

    (a) each subgraph in the decomposition is a star;

    (b) each subgraph in the decomposition is a path.

24. Show that there is an ascending subgraph decomposition of $4P_8$ such that each subgraph in the decomposition is 1-regular.

25. Give an example of an ascending subgraph decomposition of $K_{3,7}$ for which each subgraph in the decomposition is connected.

26. Give an example of an ascending subgraph decomposition of $K_6 + K_4$.

Section 13.3. Cycle Decomposition

27. Figure 13.10 shows a C_6-decomposition of K_9. Display a C_k-decomposition of K_9 for all other possible values of k.

28. Give a cycle decomposition of K_7 such that the decomposition contains a maximum number of cycles of distinct lengths.

29. The graph G of Figure 13.19 is an Eulerian graph of order 12 and size 30. Give an example of a cycle decomposition $\mathcal{D}_k$ of G containing exactly k odd cycles for every possible value of k.

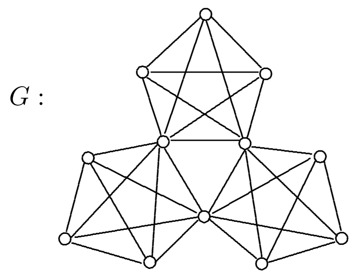


Figure 13.19: The graph in Exercise 29

30. Give an example of a cycle decomposition of K_7 such that each cycle in the decomposition is one of two lengths and that the number of cycles of each length is the same.

31. Let G be a connected graph with $2k \geq 2$ odd vertices for some positive integer k. According to Theorem 5.3, there is a decomposition of G into k open trails, at most one of which has odd length.

    (a) Give an example of a connected graph G of even size with $2k \geq 2$ odd vertices such that for every decomposition of G into k open trails, the trails have even length.

    (b) Give an example of a connected graph G of even size with $2k \geq 2$ odd vertices such that for every even integer t with $0 \leq t \leq k$, there is a decomposition of G into k open trails, exactly t of which are of odd length.

32. Let G be an Eulerian graph. Among all circuit decompositions of G, suppose that the maximum number of circuits of odd length is t. Prove that G has a circuit decomposition in which r of the circuits have odd length for every integer r with $0 \leq r \leq t$ such that r and t are of the same parity.

Section 13.4. Graceful Graphs

33. Show that the graph of order 5 in Figure 13.14 obtained by identifying a vertex in two triangles is not graceful.

34. Determine whether the graph of order 6 obtained by identifying a vertex in a triangle and a vertex in a 4-cycle is graceful.

35. (a) Use the fact that K_3 is graceful to find a K_3-decomposition of K_7.

    (b) Find a noncomplete regular connected K_3-decomposable graph.

36. Find an F-decomposition of K_{12} where $F = 2P_2 + 2P_3$.

37. Find a P_6-decomposition of K_{10}.

38. For each integer $k \geq 1$, show that

    (a) K_{2k+1} is $K_{1,k}$-decomposable.

    (b) K_{2k} is $K_{1,k}$-decomposable.

39. Let m be an even integer and let G be a graceful graph of size m. Show that K_{3m+1} can be decomposed into $3m+1$ copies of G and an m-regular graph of order $3m+1$.

40. (a) Use the drawing of the Petersen graph shown in Figure 13.20 to find cyclic F_i-decompositions for $i = 1, 2, 3$.

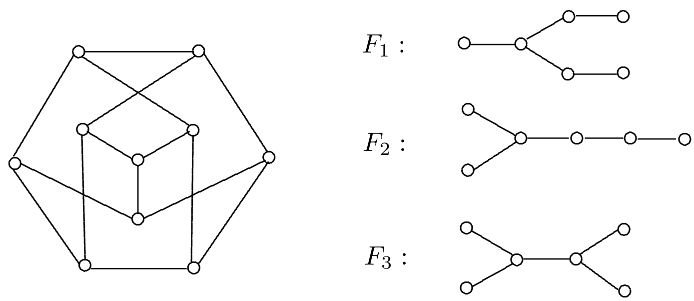


Figure 13.20: The Petersen graph and graphs F_1, F_2 and F_3 in Exercise 40

    (b) Does there exist a decomposition $\{F_1, F_2, F_3\}$ of the Petersen graph?

41. Find all graphs F of size 3 that are subgraphs of the Petersen graph P for which P is F-decomposable.

42. Determine graceful labelings of C_{15} and C_{16}.

43. By Theorem 13.21, the cycle C_6 is not graceful. Show, however, that there is a cyclic C_6-decomposition of K_{13}.

44. (a) By Theorem 13.21, the graph $G = C_9$ is not graceful. Thus, there is no labeling of the vertices of G with distinct elements from the set $\{0, 1, \ldots, 9\}$ in such a way that the induced edge labeling (which prescribes the integer $|i - j|$ to the edge joining vertices labeled i and j) assigns distinct labels from the set $\{1, 2, \ldots, 9\}$ to the edges of G. Show, however, that there is a labeling of the vertices of G with distinct elements of the set $\{0, 1, \ldots, 18\}$ such that the set of edge labels of G produced by the induced edge labeling has a nonempty intersection with each of the sets $S_1 = \{1, 18\}$, $S_2 = \{2, 17\}$, ..., $S_9 = \{9, 10\}$.

    (b) What information about G can be obtained from (a)?

    (c) From (a) and (b), what more general information can be obtained?

45. Determine graceful labelings of P_6, P_7, P_9 and P_{10}.

46. Show that the following classes of trees are graceful:

    (a) double stars, (b) caterpillars.

47. (a) Show that no disconnected forest is graceful.

    (b) Give an example of a disconnected graph without isolated vertices having order n and size m with $m \geq n - 1$ that is not graceful.

48. Prove for every integer $k \geq 2$ that there exists a graceful graph having chromatic number k.

49. Let G be a graceful graph of order n and size m where there is a graceful labeling of G that assigns the labels $a_1, a_2, \ldots, a_n$ to the vertices of G. Show that the complementary labeling that replaces the vertex label a_i by $m - a_i$ is also a graceful labeling of G.

Chapter 14

Vertex Colorings

This and the next four chapters will be devoted to perhaps the best known and most studied area of graph theory: graph colorings. In 1993 Noga Alon of Tel Aviv University in Israel, well known for his work in combinatorics and theoretical computer science, wrote:

Graph coloring is arguably the most popular subject in graph theory.

Graph colorings has become a subject of great interest, largely because of its intriguing history (which will be discussed in Chapter 16), its diverse theoretical results, its unsolved problems and its numerous applications. The problems dealing with graph colorings that have received the most attention involve coloring the vertices of a graph. Furthermore, the problems concerning vertex colorings that have been studied most often are those concerning proper vertex colorings. It is these colorings that will be discussed in this chapter.

14.1 The Chromatic Number of a Graph

By a **vertex coloring** of a graph G is meant an assignment of colors to the vertices of G, one color to each vertex. Vertex colorings in which adjacent vertices are colored differently are **proper vertex colorings**. Since the only vertex colorings we will consider here are proper vertex colorings, we ordinarily refer to these simply as **colorings** of G. While the colors used can be elements of any set, we typically use positive integers, say $1, 2, \ldots, k$, for some positive integer k. Thus, a (proper) coloring can be considered as a function $c : V(G) \to \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G. If each color used is one of k given colors, then we refer to the coloring as a k**-coloring**. In a k-coloring, we may then assume that it is the colors $1, 2, \ldots, k$ that are being used.

Suppose that c is a k-coloring of a graph G, where each color is one of the integers $1, 2, \ldots, k$ as mentioned above. If V_i $(1 \leq i \leq k)$ is the set of vertices in G colored i (where one or more of these sets may be empty), then each nonempty set V_i is called a **color class** and the nonempty elements of $\{V_1, V_2, \ldots, V_k\}$ produce a partition of $V(G)$. Because no two adjacent vertices of G are assigned the same color by c, each nonempty color class V_i $(1 \leq i \leq k)$ is an independent set of vertices of G.

A graph G is k**-colorable** if there exists a coloring of G from a set of k colors. In other words, G is k-colorable if there exists a k-coloring of G. The minimum positive integer k for which G is k-colorable is the **chromatic number** of G and is denoted by $\chi(G)$. (The symbol χ is the Greek letter *chi.*) The chromatic number of a graph G is therefore the minimum number of independent sets into which $V(G)$ can be partitioned. A graph G with chromatic number k is a k**-chromatic graph**. Therefore, if $\chi(G) = k$, then there exists a k-coloring of G but not a $(k-1)$-coloring. In fact, a graph G is k-colorable if and only if $\chi(G) \leq k$. Certainly, every graph of order n is n-colorable. Necessarily, if a k-coloring of a k-chromatic graph G is given, then all k colors must be used.

Two different colorings of a graph H are shown in Figure 14.1. The coloring in Figure 14.1(a) is a 5-coloring and the coloring in Figure 14.1(b) is a 4-coloring. Because the order of H is 10, the graph H is k-colorable for every integer k with $4 \leq k \leq 10$. Since H is 4-colorable, $\chi(H) \leq 4$. There is, however, no 3-coloring of H as we now explain. In any coloring c of H, the vertices u_1, u_2 and v_2 must be colored differently as they induce a triangle. Hence, we may assume that u_1 is colored 1, u_2 is colored 2 and v_2 is colored 3 as indicated in Figure 14.1(b). If we attempt to color H with only the three colors $1, 2$ and 3, then we are forced to color the vertices of H as follows:

$$c(u_1) = 1,\ c(u_2) = 2,\ c(v_2) = 3,\ c(v_3) = 1,\ c(u_3) = 3,$$
$$c(v_4) = 2,\ c(u_4) = 1,\ c(v_5) = 3,\ c(u_5) = 2,\ c(v_1) = 2.$$

However, the adjacent vertices v_1 and u_5 are now both colored 2, which is impossible. So, there is no 3-coloring of H. Therefore, $\chi(H) \geq 4$ and so $\chi(H) = 4$.

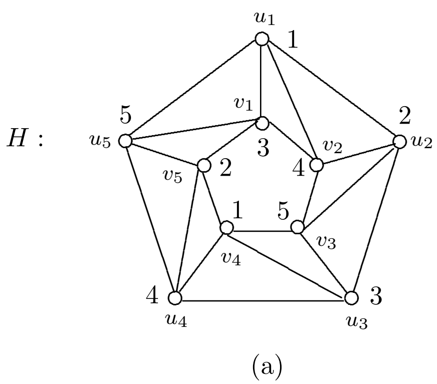


Figure 14.1: Colorings of a graph H

The argument used to verify that the graph H of Figure 14.1 has chromatic number 4 is a common one. In general, to show that some graph G has chromatic number k, say, we need to show that there exists a k-coloring of G (and so $\chi(G) \leq k$) and to show that every coloring of G requires at least k colors (and so $\chi(G) \geq k$).

There is no general formula for the chromatic number of a graph. Indeed, the problem of showing that a graph is k-colorable for some $k \geq 3$ is **NP**-complete. Consequently, we will often be concerned with and must be content with (1) determining the chromatic number of some specific graphs of interest or of graphs belonging to some classes of interest and (2) determining upper and/or lower bounds for the chromatic number of a graph. Certainly, for every graph G of order n,

$$1 \leq \chi(G) \leq n.$$

Clearly, $\chi(K_n) = n$. Furthermore, if G is a graph of order n that is not complete, then assigning the color 1 to two nonadjacent vertices of G and distinct colors to the remaining $n-2$ vertices of G produces an $(n-1)$-coloring of G. Therefore:

A graph G of order n has chromatic number n if and only if $G = K_n$.

Since distinct colors are needed to color any two adjacent vertices of a graph, we also have:

A graph G of order n has chromatic number 1 if and only if $G = \overline{K}_n$.

Thus, for a graph G to have chromatic number at least 2, G must have at least one edge. Also, for a nonempty graph G to have chromatic number 2, there must be some way to partition $V(G)$ into two independent subsets V_1 (the vertices of G colored 1) and V_2 (the vertices of G colored 2). Since every edge of G must join a vertex of V_1 and a vertex of V_2, the graph G is bipartite. That is:

A nonempty graph G has chromatic number 2 if and only if G is bipartite.

From these observations, we have the following.

Theorem 14.1 *A nontrivial graph G is 2-colorable if and only if G is bipartite.*

As a consequence of Theorem 2.6, we also have the following:

The chromatic number of a graph G is at least 3 if and only if G has an odd cycle.

A rather obvious, but often useful, lower bound for the chromatic number of a graph involves the chromatic numbers of its subgraphs.

Theorem 14.2 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

Proof. Suppose that $\chi(G) = k$. Then there exists a k-coloring c of G. Since c assigns distinct colors to every two adjacent vertices of G, the coloring c also assigns distinct colors to every two adjacent vertices of H. Therefore, H is k-colorable and so $\chi(H) \leq k = \chi(G)$. ■

As a result of Theorem 14.2, if a graph G contains K_k as a subgraph, then $\chi(G) \geq k$. The **clique number** $\omega(G)$ of a graph G is the order of the largest **clique** (complete subgraph) of G. (The symbol ω is the Greek letter *omega*.) A clique of order k is called a k**-clique**. The following result is then an immediate consequence of Theorem 14.2.

Corollary 14.3 *For every graph G, $\chi(G) \geq \omega(G)$.*

The lower bound for the chromatic number of a graph in Corollary 14.3 is related to a much studied class of graphs called "perfect graphs", which will be visited in the next chapter.

If a graph G is the union or the join of graphs $G_1, G_2, \ldots, G_k$, then the chromatic number of G can be easily expressed in terms of the chromatic numbers of these k graphs (see Exercise 13).

Theorem 14.4 *For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 + G_2 + \cdots + G_k$,*

$$\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}.$$

The following is then an immediate consequence of Theorem 14.4.

Corollary 14.5 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}.$$

There is a result analogous to Corollary 14.5 that expresses the chromatic number of a graph in terms of the chromatic numbers of its blocks (see Exercise 14).

Theorem 14.6 *If G is a graph with blocks $B_1, B_2, \ldots, B_k$, then*

$$\chi(G) = \max\{\chi(B_i) : 1 \leq i \leq k\}.$$

Corollary 14.5 and Theorem 14.6 tell us that we can restrict our attention to 2-connected graphs when studying the chromatic number of graphs. In the case of joins, we have the following (see Exercise 15).

Theorem 14.7 *For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 \vee G_2 \vee \cdots \vee G_k$,*

$$\chi(G) = \sum_{i=1}^{k} \chi(G_i).$$

A special case of Theorem 14.7 is the following:

Every complete k-partite graph has chromatic number k.

Certainly every even cycle is 2-chromatic since these are bipartite graphs and the chromatic number of every odd cycle is at least 3. The coloring c defined on the vertices of an odd cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ by

$$c(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i < n \\ 2 & \text{if } i \text{ is even} \\ 3 & \text{if } i = n \end{cases}$$

is a 3-coloring. Thus, we have the following.

Theorem 14.8 *For every integer $n \geq 3$,*

$$\chi(C_n) = \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 3 & \textit{if } n \textit{ is odd.} \end{cases}$$

Many bounds (both upper and lower bounds) have been developed for the chromatic number of a graph. Two of the most elementary bounds for the chromatic number of a graph G involve the independence number $\alpha(G)$, which, recall, is the maximum cardinality of an independent set of vertices of G. The lower bound is especially useful.

Theorem 14.9 *If G is a graph of order n, then*

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1.$$

Proof. Suppose that $\chi(G) = k$ and let there be given a k-coloring of G with resulting color classes V_1, V_2, ..., V_k. Since

$$n = |V(G)| = \sum_{i=1}^{k} |V_i| \leq k\alpha(G),$$

it follows that

$$\frac{n}{\alpha(G)} \leq \chi(G).$$

Next, let U be a maximum independent set of vertices of G and assign the color 1 to each vertex of U. Assigning distinct colors different from 1 to each vertex of $V(G) - U$ produces a proper coloring of G. Hence,

$$\chi(G) \leq |V(G) - U| + 1 = n - \alpha(G) + 1,$$

as desired. ■

According to Theorem 14.9, for the complete 3-partite graph $G = K_{1,2,3}$, which has order $n = 6$ and independence number $\alpha(G) = 3$, we have

$$n/\alpha(G) = 2 \leq \chi(G) \leq 4 = n - \alpha(G) + 1.$$

Since G is a complete 3-partite graph, it follows that $\chi(G) = 3$ and so neither bound in Theorem 14.9 is attained in this case. On the other hand, $\omega(G) = 3$ and so $\chi(G) = \omega(G)$.

For the graph G of order 10 shown in Figure 14.2(a), we have $\alpha(G) = 2$ and $\omega(G) = 4$. By Corollary 14.3, $\chi(G) \geq 4$; while, according to Theorem 14.9, $5 \leq \chi(G) \leq 9$. However, the 5-coloring of G in Figure 14.2(b) shows that $\chi(G) \leq 5$ and so $\chi(G) = 5$.

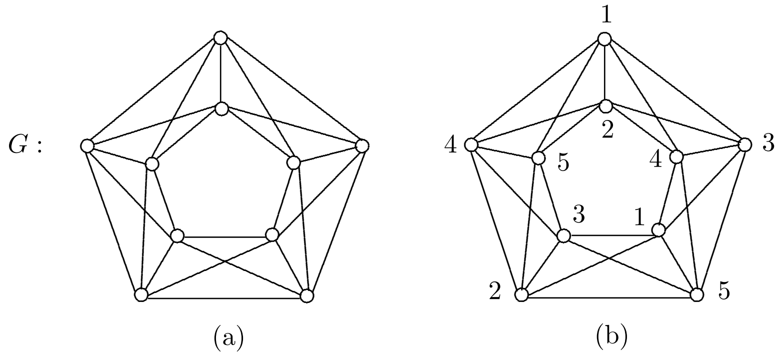


Figure 14.2: A 5-chromatic graph G with $\alpha(G) = 2$ and $\omega(G) = 4$

Applications of Vertex Colorings

Many problems, diverse in nature, have solutions that involve modeling the situation by a graph and then determining the chromatic number of the graph. We look at three examples of this now.

Example 14.10

The service department of an automobile dealer has advertised maintenance specials for a particular afternoon. One of these is a 30-minute maintenance special and the other is a 60-minute maintenance special. Eight customers, denoted by $c_1, c_2, \ldots, c_8$, have made appointments, listed below:

$$\begin{array}{llll} c_1: & 1:30-2:00 \text{ pm} & c_2: & 1:40-2:40 \text{ pm} \\ c_3: & 1:40-2:10 \text{ pm} & c_4: & 1:50-2:50 \text{ pm} \\ c_5: & 2:30-3:00 \text{ pm} & c_6: & 2:15-3:15 \text{ pm} \\ c_7: & 2:55-3:55 \text{ pm} & c_8: & 3:10-3:40 \text{ pm} \end{array}$$

This situation can be modeled by the graph G shown in Figure 14.3(a), where $V(G)$ is the set of customers and two vertices (customers) are adjacent if they have overlapping appointments. The obvious question to ask here is: What is the minimum number of mechanics needed to provide the necessary maintenance?

Solution. Certainly, a mechanic cannot work on two cars whose owners have overlapping appointments. In other words, we seek the chromatic number of G. The vertices c_1, c_2, c_3 and c_4 induce a complete graph of order 4. In fact, the clique number of G is 4. Hence, $\chi(G) \geq \omega(G) = 4$. Color the vertices c_i $(1 \leq i \leq 4)$ with color i and color c_5, c_6, c_7 and c_8 with the colors $1, 3, 2$ and 4, respectively as shown in Figure 14.3(b). Thus, $\chi(G) = 4$. Since c_1 and c_5 are colored 1, mechanic #1 can service the cars of the customers c_1 and c_5, and so on. In general, we have the following

mechanic #1: c_1, c_5 mechanic #2: c_2, c_7
mechanic #3: c_3, c_6 mechanic #4: c_4, c_8.

♦

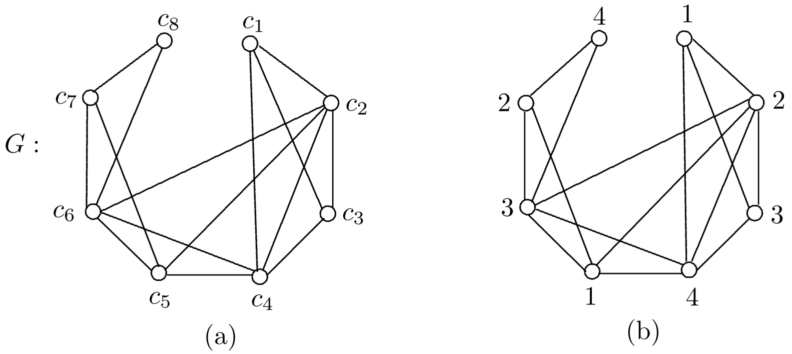


Figure 14.3: The graph G in Example 14.10 and a 4-coloring of G

Example 14.11

Eight faculty members, denoted by $f_1, f_2, \ldots, f_8$, have been invited to a workshop to discuss seven topics of interest. It is decided that it would be more efficient to divide these faculty members into seven committees of three each, namely,

$$S_1 = \{f_1, f_2, f_3\},\ S_2 = \{f_2, f_3, f_4\},\ S_3 = \{f_4, f_5, f_6\},\ S_4 = \{f_5, f_6, f_7\},$$
$$S_5 = \{f_1, f_7, f_8\},\ S_6 = \{f_1, f_5, f_7\},\ S_7 = \{f_2, f_5, f_8\}.$$

If each committee must meet during one of the time periods

9 - 10 am, 10 - 11 am, 11am - 12 noon, 2 - 3 pm, 3 - 4 pm, 4 - 5 pm,

then what is the earliest time that all committee meetings can be completed?

Solution. No two committees can meet during the same time period if some faculty member belongs to both committees. Define a graph G by $V(G) = \{S_1, S_2, \ldots, S_7\}$, where two vertices S_i and S_j are adjacent if $S_i \cap S_j \neq \emptyset$

(and so S_i and S_j must meet at different times). The graph G is shown in Figure 14.4(a). The answer to this question is $\chi(G)$. Here, $\omega(G) = 4$ and so $\chi(G) \geq 4$. The 5-coloring in Figure 14.4(b) shows that $\chi(G) \leq 5$. The subgraph induced by $\{S_3, S_4, S_6, S_7\}$ is K_4 and so any vertex coloring of G must assign distinct colors to these four vertices, say S_3-1, S_4-2, S_6-3, S_7-4. If only four colors are used to color G, then we must have S_5 colored 1 and S_2 colored 2. However then, S_1 cannot be colored by any of 1, 2, 3 or 4. Since no 4-coloring of G is possible, $\chi(G) = 5$. Possible meeting times for these committees are therefore

9 - 10 am: S_1 10 - 11 am: S_6 11am - 12 noon: S_7

2 - 3 pm: S_2, S_4 3 - 4 pm: S_3, S_5

and so all committee meetings are over by 4 pm. ♦

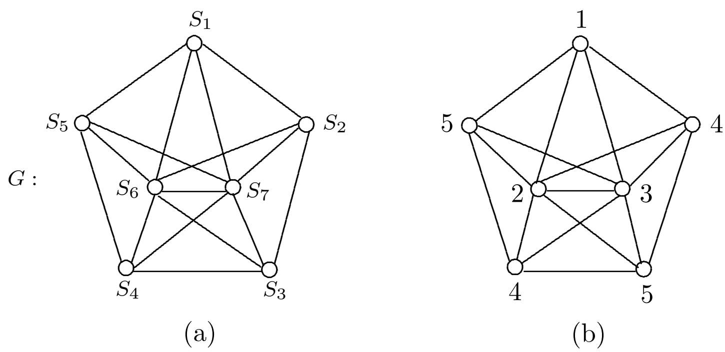


Figure 14.4: The graph G in Example 14.11 and a 5-coloring of G

Example 14.12

Figure 14.5 shows six traffic lanes $L_1, L_1, \ldots, L_6$ at the intersection of two streets. A traffic light is located at the intersection. During each phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection into certain permitted lanes. What is the minimum number of phases needed for the traffic light so that all cars may proceed safely through the intersection?

Solution. A graph G is constructed with vertex set $V(G) = \{L_1, L_2, \ldots, L_6\}$, where L_i is adjacent to L_j $(i \neq j)$ if cars in lanes L_i and L_j cannot proceed safely through the intersection at the same time. (See Figure 14.6(a).) The minimum number of phases needed for the traffic light so that all cars may proceed, in time, through the intersection is $\chi(G)$. Since $\{L_1, L_2, L_4\}$ induces a triangle, $\chi(G) \geq 3$. Since there is a 3-coloring of G (see Figure 14.6(b)), it follows that $\chi(G) = 3$. For example, since L_1, L_5 and L_6 belong to the same color class, cars in those three lanes may proceed safely through the intersection

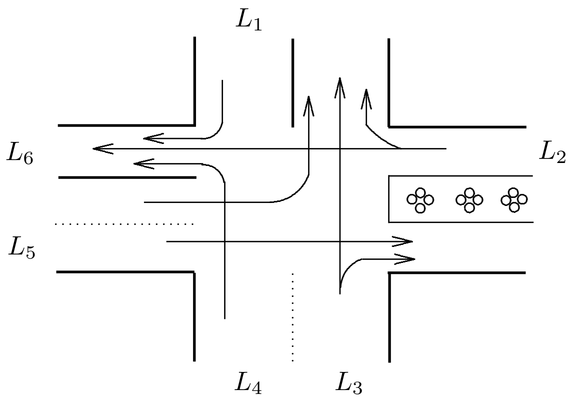


Figure 14.5: Traffic lanes at street intersections in Example 14.12

at the same time. Three possible phases for all cars to proceed safely through the intersection are therefore

Phase #1: L_1, L_6,   Phase #2: L_2, L_5,   Phase #3: L_3, L_4.

♦

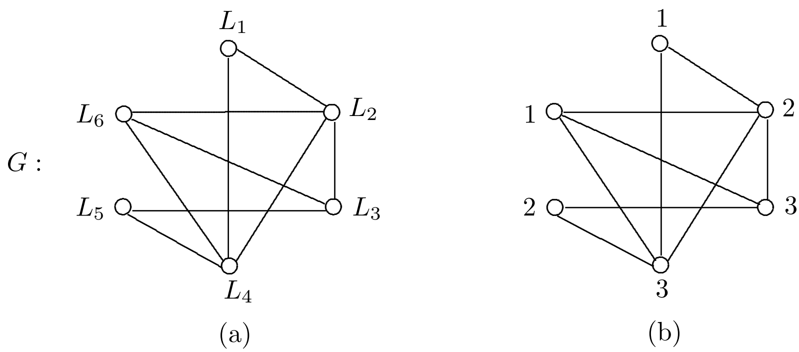


Figure 14.6: The graph G of Example 14.12 and a 3-coloring of G

14.2 Color-Critical Graphs

For every k-chromatic graph G with $k \geq 2$ and every vertex v of G, either $\chi(G - v) = k$ or $\chi(G - v) = k - 1$. Furthermore, for every edge e of G, either $\chi(G - e) = k$ or $\chi(G - e) = k - 1$. In fact, if $\chi(G - e) = k - 1$ and u is a vertex incident with e, then $\chi(G - u) = k - 1$. Graphs that are k-chromatic (but just barely) are often of great interest. A graph G is called **color-critical** if $\chi(H) < \chi(G)$ for every proper subgraph H of G. If G is a color-critical k-chromatic graph, then G is called **critically k-chromatic** or simply **k-critical**. The graph K_2 is the only 2-critical graph. In fact, K_n is n-critical for every integer $n \geq 2$. The odd cycles are the only 3-critical graphs. There is no characterization of k-critical graphs for any integer $k \geq 4$.

Let G be a k-chromatic graph, where $k \geq 2$, and suppose that H is a k-chromatic subgraph of minimum size in G having no isolated vertices. Then for every proper subgraph F of H, $\chi(F) < \chi(H)$, that is, H is a k-critical subgraph of G. From this observation, it follows that every k-chromatic graph, $k \geq 2$, contains a k-critical subgraph. By Corollary 14.5, every k-critical graph, $k \geq 2$, must be connected; while by Theorem 14.6, every k-critical graph, $k \geq 3$, must be 2-connected and therefore 2-edge-connected. The following theorem tells us even more.

Theorem 14.13 *Every k-critical graph, $k \geq 2$, is $(k-1)$-edge-connected.*

Proof. The only 2-critical graph is K_2, which is 1-edge-connected; while the only 3-critical graphs are odd cycles, each of which is 2-edge-connected. Since the theorem holds for $k = 2, 3$, we may assume that $k \geq 4$.

Suppose that there is a k-critical graph G, $k \geq 4$, that is not $(k-1)$-edge-connected. This implies by Theorem 4.2 that there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that the number of edges joining the vertices of V_1 and the vertices of V_2 is at most $k-2$. Since G is k-critical, the two induced subgraphs

$$G_1 = G[V_1] \text{ and } G_2 = G[V_2]$$

are $(k-1)$-colorable. Let there be given colorings of G_1 and G_2 from the same set of $k-1$ colors and suppose that E' is the set of edges of G that join the vertices in V_1 and the vertices in V_2. It cannot occur that every edge in E' joins vertices of different colors, for otherwise, G itself is $(k-1)$-colorable. Hence, there are some edges in E' joining vertices that are assigned the same color. We now show that there exists a permutation of the colors assigned to the vertices of V_1 that results in a proper coloring of G in which every edge of E' joins vertices of different colors, which again shows that G is $(k-1)$-colorable, producing a contradiction.

Let $U_1, U_2, \ldots, U_t$ denote the color classes of G_1 for which there is some vertex in U_i $(1 \leq i \leq k-2)$ adjacent to a vertex of V_2. Suppose that there are k_i edges joining the vertices of U_i and the vertices of V_2. Then each $k_i \geq 1$ and

$$\sum_{i=1}^{t} k_i \leq k-2.$$

If, for every vertex $u_1 \in U_1$, the neighbors of u_1 are assigned a color different from that assigned to u_1, then the color of the vertices in U_1 is not altered. If, on the other hand, some vertex $u_1 \in U_1$ is adjacent to a vertex in V_2 that is colored the same as u_1, then the $k-1$ colors used to color the vertices of G_1

may be permuted so that no vertex in U_1 is adjacent to a vertex of V_2 having the same color. This is possible since there are at most k_1 colors to avoid when coloring the vertices of U_1 but there are at least $k-1-k_1 \geq 1$ colors available for this purpose. If, upon giving this new coloring to the vertices of G_1, each vertex $u_2 \in U_2$ is adjacent only to the vertices in V_2 assigned a color different from that of u_2, then no (additional) permutation of the colors of V_1 is performed. Suppose, however, that there is some vertex $u_2 \in U_2$ that is assigned the same color as one of its neighbors in V_2. In this case, we may once again permute the $k-1$ colors used to color the vertices of V_1, where the color assigned to the vertices in U_1 is not changed. This too is possible since there are at most k_2+1 colors to avoid when coloring the vertices of U_2 but the number of colors available for U_2 is at least

$$(k-1)-(k_2+1) \geq (k-1)-(k_2+k_1) \geq 1.$$

This procedure is continued until a $(k-1)$-coloring of G is produced, which, as we noted, is impossible. ∎

As a consequence of Theorem 14.13, $\chi(G) \leq 1+\lambda(G)$ for every color-critical graph G, where, recall, $\lambda(G)$ denotes the edge-connectivity of G. A related theorem of David W. Matula [164] provides an upper bound for the chromatic number of an arbitrary graph in terms of the edge connectivity of its subgraphs.

Theorem 14.14 *For every graph G,*

$$\chi(G) \leq 1+\max\{\lambda(H)\},$$

where the maximum is taken over all subgraphs H of G.

Proof. Suppose that F is a color-critical subgraph of G with $\chi(G)=\chi(F)$. By Theorem 14.13,

$$\chi(G)=\chi(F) \leq 1+\lambda(F) \leq 1+\max\{\lambda(H)\},$$

where the maximum is taken over all subgraphs H of G. ∎

Since the edge connectivity of a graph never exceeds its minimum degree (by Theorem 4.4), we have the following corollary of Theorem 14.13.

Corollary 14.15 *If G is a color-critical graph, then*

$$\chi(G) \leq 1+\delta(G).$$

We have noted that every k-critical graph, $k \geq 3$, is 2-connected. While every 4-critical graph must be 3-edge-connected (by Theorem 14.13), a 4-critical graph need not be 3-connected. The 4-critical graph G of Figure 14.7 has connectivity 2. In fact, $S=\{u,v\}$ is a minimum vertex-cut of G.

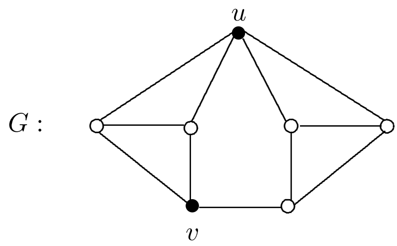


Figure 14.7: A critically 4-chromatic graph

14.3 Bounds for the Chromatic Number

Since determining whether a graph is k-chromatic for $k \geq 3$ is known to be an **NP**-complete problem, it is not surprising that much of the research emphasis on coloring has centered on finding bounds (both lower bounds and upper bounds) for the chromatic number of a graph. For a graph G of order n with clique number $\omega(G)$ and independence number $\alpha(G)$, we have already seen that $\omega(G)$ and $n/\alpha(G)$ are lower bounds for $\chi(G)$ while $n - \alpha(G) + 1$ is an upper bound for $\chi(G)$. Of course, n is also an upper bound for $\chi(G)$. In particular,

$$\omega(G) \leq \chi(G) \leq n.$$

Bruce Reed [198] showed that $\chi(G)$ can never be closer to n than to $\omega(G)$.

Theorem 14.16 *For every graph G of order n,*

$$\chi(G) \leq \left\lfloor \frac{n + \omega(G)}{2} \right\rfloor.$$

Proof. We proceed by induction on the nonnegative integer $|V(G)| - \omega(G)$. If G is a graph such that $|V(G)| - \omega(G) = 0$, then $G = K_n$ for some positive integer n and so $\chi(G) = \omega(G) = n$. Thus, $\chi(G) = \left\lfloor \frac{n+\omega(G)}{2} \right\rfloor$. This verifies the basis step of the induction.

Assume for a positive integer k and every graph H (having order n') such that $n' - \omega(H) < k$ that $\chi(H) \leq \left\lfloor \frac{n'+\omega(H)}{2} \right\rfloor$. Let G be a graph of order n such that $n - \omega(G) = k \geq 1$. Since G is not complete, G contains two nonadjacent vertices u and v. Let $H = G - u - v$. Then H has order $n - 2$ and either $\omega(H) = \omega(G)$ or $\omega(H) = \omega(G) - 1$. In either case, $0 \leq (n-2) - \omega(H) < k$ and so

$$\chi(H) \leq \left\lfloor \frac{n - 2 + \omega(H)}{2} \right\rfloor$$

by the induction hypothesis. Hence, there exists a $\left\lfloor \frac{n-2+\omega(H)}{2} \right\rfloor$-coloring of H. Assigning u and v the same new color implies that

$$\chi(G) \leq \left\lfloor \frac{n - 2 + \omega(H)}{2} \right\rfloor + 1 = \left\lfloor \frac{n + \omega(H)}{2} \right\rfloor \leq \left\lfloor \frac{n + \omega(G)}{2} \right\rfloor. \qquad \blacksquare$$

The Greedy Coloring Algorithm

Suppose that our goal is to assign colors to the vertices of some graph G to obtain a proper coloring. Ideally, of course, we would like to use as few colors as possible, preferably $\chi(G)$ colors. Since there are no known efficient algorithms that will solve this problem with $\chi(G)$ colors in all instances, this suggests seeking an efficient algorithm that does not use an excessive number of colors.

In Chapter 3, we encountered Kruskal's algorithm for obtaining a minimum spanning tree in a connected weighted graph. This procedure selects the best current option at each step without regard to future consequences. Kruskal's algorithm is a greedy algorithm that succeeds in producing a minimum spanning tree in every instance. We now describe a greedy algorithm for our coloring problem. While this method may not result in coloring the vertices of G using the minimum number of colors, it does provide several upper bounds for the chromatic number of G.

Let G be a graph of order n whose vertices are listed in some specified order. In a greedy coloring of G, the vertices are successively colored with positive integers according to an algorithm that assigns to the vertex under consideration the smallest available color. Hence, if the vertices of G are listed in the order $v_1, v_2, \ldots, v_n$, then the resulting greedy coloring c assigns the color 1 to v_1, that is, $c(v_1) = 1$. If v_2 is not adjacent to v_1, then also define $c(v_2) = 1$; while if v_2 is adjacent to v_1, then define $c(v_2) = 2$. In general, suppose that the first j vertices $v_1, v_2, \ldots, v_j$ in the sequence have been colored, where $1 \leq j < n$, and t is the smallest positive integer not used in coloring any neighbor of v_{j+1} from among $v_1, v_2, \ldots, v_j$. We then define $c(v_{j+1}) = t$. When the algorithm ends, the vertices of G have been assigned colors from the set $\{1, 2, \ldots, k\}$ for some positive integer k. Thus, $\chi(G) \leq k$ and so k is an upper bound for the chromatic number of G.

While there is always an ordering of the vertices of a graph G so that this algorithm gives a vertex coloring of G with $\chi(G)$ colors (see Exercise 41(a)), it is also possible that this algorithm gives a vertex coloring of G using k colors such that k is considerably larger than $\chi(G)$ (see Exercise 41(b)). This algorithm is now stated more formally.

Algorithm 14.17 (**The Greedy Coloring Algorithm**) *Suppose that the vertices of a graph G are listed in the order $v_1, v_2, \ldots, v_n$.*

1. *The vertex v_1 is assigned the color* 1.

2. *Once the vertices $v_1, v_2, \ldots, v_j$ have been assigned colors, where $1 \leq j < n$, the vertex v_{j+1} is assigned the smallest color that is not assigned to any neighbor of v_{j+1} belonging to the set $\{v_1, v_2, \ldots, v_j\}$.*

As an illustration of the greedy coloring algorithm, suppose that we consider the graph C_6 of Figure 14.8. If we list the vertices of C_6 in the order u, w, v, y, z, x, then the greedy coloring algorithm yields the coloring c of G defined by

$$c(u) = 1,\ c(w) = 1,\ c(v) = 2,\ c(y) = 1,\ c(z) = 2,\ c(x) = 2.$$

This gives $\chi(C_6) \leq 2$. Of course, $\chi(C_6) = 2$ and so with this ordering of the vertices of C_6, a $\chi(C_6)$-coloring is produced. On the other hand, if the vertices of G are listed in the order u, x, v, w, z, y, then the greedy coloring algorithm yields the coloring c' of G defined by

$$c'(u) = 1,\ c'(x) = 1,\ c'(v) = 2,\ c'(w) = 3,\ c'(z) = 2,\ c'(y) = 3.$$

This gives a 3-coloring of C_6, which, of course, is not the chromatic number of C_6.

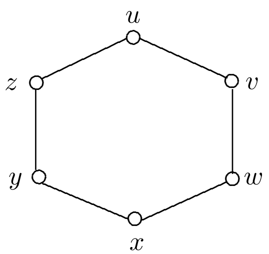


Figure 14.8: The graph C_6

From the second listing of the vertices of C_6, we see that $\chi(C_6) \leq 3$. As the proof of the following theorem shows, no greater upper bound for $\chi(C_6)$ is possible using the greedy coloring algorithm.

Theorem 14.18 *For every graph G,*

$$\chi(G) \leq 1 + \Delta(G).$$

Proof. Suppose that the vertices of G are listed in the order $v_1, v_2, \ldots, v_n$ and the greedy coloring algorithm is applied. Then v_1 is assigned the color 1 and for $2 \leq i \leq n$, the vertex v_i is either assigned the color 1 or is assigned the color $k+1$, where k is the largest integer such that all of the colors $1, 2, \ldots, k$ are used to color the neighbors of v_i in the set $S = \{v_1, v_2, \ldots, v_{i-1}\}$. Since at most $\deg v_i$ neighbors of v_i belong to S, the largest value of k is $\deg v_i$. Hence, the color assigned to v_i is at most $1 + \deg v_i$. Thus,

$$\chi(G) \leq \max_{1 \leq i \leq n} \{1 + \deg v_i\} = 1 + \Delta(G),$$

as desired. ■

The following theorem of George Szekeres and Herbert S. Wilf [230] gives an upper bound for the chromatic number of a graph that is an improvement over that stated in Theorem 14.18 but which is more difficult to compute.

Theorem 14.19 *For every graph G,*

$$\chi(G) \leq 1 + \max\{\delta(H)\},$$

where the maximum is taken over all subgraphs H of G.

Proof. Let $\chi(G) = k$ and let F be a k-critical subgraph of G. By Corollary 14.15, $\delta(F) \geq k - 1$. Thus,

$$k - 1 \leq \delta(F) \leq \max\{\delta(H)\},$$

where the maximum is taken over all subgraphs H of G. Therefore, $\chi(G) = k \leq 1 + \max\{\delta(H)\}$. ∎

Since $\delta(H') \leq \delta(H)$ for each spanning subgraph H' of an induced subgraph H of G, it follows that in Theorem 14.19 we may restrict the subgraphs H of G only to those that are induced.

For an r-regular graph G, both Theorems 14.18 and 14.19 give the same upper bound for $\chi(G)$, namely $1 + r$. On the other hand, if T is a tree of order at least 3, then Theorem 14.18 gives $1 + \Delta(T) \geq 3$ for an upper bound for $\chi(T)$, while Theorem 14.19 gives the improved and best possible upper bound $\chi(T) \leq 2$ since every subgraph of a tree contains a vertex of degree 1 or less.

When applying the greedy coloring algorithm to a graph G, there are, in general, fewer colors to avoid when coloring a vertex if the vertices of higher degree are listed early in the ordering of the vertices of G. The following result is due to Dominic J. A. Welsh and Martin B. Powell [252].

Theorem 14.20 *Let G be a graph of order n whose vertices are listed in the order $v_1, v_2, \ldots, v_n$ so that $\deg v_1 \geq \deg v_2 \geq \cdots \geq \deg v_n$. Then*

$$\chi(G) \leq 1 + \min_{1 \leq i \leq n} \{\max\{i-1, \deg v_i\}\} = \min_{1 \leq i \leq n} \{\max\{i, 1 + \deg v_i\}\}.$$

Proof. Suppose that $\chi(G) = k$ and let H be a k-critical subgraph of G. Hence, the order of H is at least k and $\delta(H) \geq k - 1$ by Corollary 14.15. Therefore, for $1 \leq i \leq k$,

$$\max\{i, 1 + \deg v_i\} \geq k;$$

while for $k + 1 \leq i \leq n$,

$$\max\{i, 1 + \deg v_i\} \geq k + 1.$$

Consequently, $\chi(G) = k \leq \min_{1 \leq i \leq n} \{\max\{i, 1 + \deg v_i\}\}$. ∎

Brooks' Theorem

While, as we saw in Theorem 14.18, $1 + \Delta(G)$ is an upper bound for the chromatic number of a connected graph G, Rowland Leonard Brooks (1916–1993) showed [38] that $\chi(G) = 1 + \Delta(G)$ occurs only in very special cases.

Theorem 14.21 (**Brooks' Theorem**) *For every connected graph G that is not an odd cycle or a complete graph,*

$$\chi(G) \leq \Delta(G).$$

Proof. Let $\chi(G) = k \geq 2$ and let H be a k-critical subgraph of G. Thus, H is 2-connected and $\Delta(H) \leq \Delta(G)$. Suppose first that either $H = K_k$ or H is an odd cycle. Then $G \neq H$ since G is neither an odd cycle nor a complete graph. Since G is connected, $\Delta(G) \geq k$ if $H = K_k$; while $\Delta(G) \geq 3$ if H is an odd cycle. If $H = K_k$, then $k = \chi(H) = \chi(G) \leq \Delta(G)$; while if H is an odd cycle, then $3 = \chi(H) = \chi(G) \leq \Delta(G)$. Therefore, in both cases, $\chi(G) \leq \Delta(G)$, as desired. Hence, we may assume that H is a k-critical subgraph that is neither an odd cycle nor a complete graph. This implies that $k \geq 4$.

Suppose that H has order n. Since $\chi(G) = k \geq 4$ and H is not complete, $n > k$ and so $n \geq 5$. Since H is 2-connected, either H is 3-connected or H has connectivity 2. We consider these two cases.

Case 1. *H is 3-connected.* Since H is not complete, there are two vertices u and w of H such that $d_H(u, w) = 2$. Let (u, v, w) be a $u - w$ geodesic in H. Since H is 3-connected, $H - u - w$ is connected. Let $v = u_1, u_2, \ldots, u_{n-2}$ be the vertices of $H - u - w$, so listed that each vertex u_i $(2 \leq i \leq n-2)$ is adjacent to some vertex preceding it. Let $u_{n-1} = u$ and $u_n = w$. Consequently, for each set

$$U_j = \{u_1, u_2, \ldots, u_j\},\ 1 \leq j \leq n,$$

the induced subgraph $H[U_j]$ is connected.

We now apply a greedy coloring to H with respect to the reverse ordering

$$w = u_n, u = u_{n-1}, u_{n-2}, \ldots, u_2, u_1 = v \tag{14.1}$$

of the vertices of H. Since w and u are not adjacent, each is assigned the color 1. Furthermore, each vertex u_i $(2 \leq i \leq n-2)$ is assigned the smallest color in the set $\{1, 2, \ldots, \Delta(H)\}$ that was not used to color a neighbor of u_i that preceded it in the sequence (14.1). Since each vertex u_i has at least one neighbor following it in the sequence (14.1), u_i has at most $\Delta(H) - 1$ neighbors preceding it in the sequence and so a color is available for u_i. Moreover, the vertex $u_1 = v$ is adjacent to two vertices colored 1 (namely $w = u_n$ and $u = u_{n-1}$) and so at most $\Delta(H) - 1$ colors are assigned to the neighbors of v, leaving a color for v. Hence,

$$\chi(G) = \chi(H) \leq \Delta(H) \leq \Delta(G). \tag{14.2}$$

Case 2. $\kappa(H) = 2$. Since H is k-critical and $k \geq 4$, it follows by Theorem 14.13 that H is $(k-1)$-edge-connected and so $\delta(H) \geq \lambda(H) \geq k-1 \geq 3$. Furthermore, H is not complete. Thus, there is a vertex x in H such that $3 \leq \deg_H x \leq n-2$. Since $\kappa(H) = 2$, either $\kappa(H-x) = 2$ or $\kappa(H-x) = 1$. If $\kappa(H-x) = 2$, then x belongs to no minimum vertex-cut of H, which implies that H contains a vertex y such that $d_H(x, y) = 2$. Proceeding as in Case 1 with $u = x$ and $w = y$, we see that there is a coloring of H with at most $\Delta(H)$ colors and so once again we have (14.2), that is, $\chi(G) \leq \Delta(G)$.

Finally, we may assume that $\kappa(H-x) = 1$. Thus, $H-x$ has end-blocks B_1 and B_2, containing cut-vertices x_1 and x_2, respectively, of $H-x$. Since H is 2-connected, there exist vertices $y_1 \in V(B_1) - \{x_1\}$ and $y_2 \in V(B_2) - \{x_2\}$ such that x is adjacent to both y_1 and y_2. Proceeding as in Case 1 with $u = y_1$ and $w = y_2$, we obtain a coloring of H with at most $\Delta(H)$ colors, once again giving us (14.2) and so $\chi(G) \leq \Delta(G)$. ■

We have now seen a number of upper bounds for the chromatic number of a graph G of order n and two lower bounds, namely, the clique number $\omega(G)$ and another $n/\alpha(G)$ in terms of its order and independence number. Dennis Paul Geller [102] gave a lower bound for the chromatic number of a graph in terms of its order and size.

Theorem 14.22 *If G is a graph of order n and size m, then*

$$\chi(G) \geq \frac{n^2}{n^2 - 2m}.$$

Proof. Suppose that $\chi(G) = k$. Let c be a k-coloring of G resulting in color classes $V_1, V_2, \ldots, V_k$ with $|V_i| = n_i$ for $1 \leq i \leq k$. Then the largest possible size of G occurs when G is a complete k-partite graph with partite sets $V_1, V_2, \ldots, V_k$ and the cardinalities of these partite sets are as equal as possible (or each $|V_i|$ is as close to $\frac{n}{k}$ as possible for $1 \leq i \leq k$). There are $\binom{k}{2}$ pairs of partite sets and at most $\left(\frac{n}{k}\right)^2$ edges joining the vertices in each pair. This implies that

$$m \leq \binom{k}{2} \frac{n^2}{k^2} \text{ and } 2m \leq \frac{(k-1)n^2}{k}.$$

Since $n^2 - 2m \geq n^2 - \frac{(k-1)n^2}{k}$, it follows that

$$\frac{n^2}{n^2 - 2m} \leq \frac{n^2}{n^2 - \frac{(k-1)n^2}{k}} = k = \chi(G),$$

giving the desired result. ■

If G is a graph of order n and size $n^2/4$, then certainly $\chi(G) \geq 2$. Actually, this cannot be improved since $K_{\frac{n}{2},\frac{n}{2}}$ is a bipartite graph of order n and size $n^2/4$. On the other hand, if G is a graph of order n whose size is greater than $n^2/4$, then $\chi(G) \geq 3$ since by Theorem 1.9, G has a triangle.

The Nordhaus–Gaddum Theorem

We have seen that $\chi(K_n) = n$ and $\chi(\overline{K_n}) = 1$. So, $\chi(K_n) + \chi(\overline{K_n}) = n + 1$. By Theorem 14.1, if G is a graph such that $\overline{G}$ is a nonempty bipartite graph, then $\chi(\overline{G}) = 2$, while $\chi(G) \leq n - 1$. Thus, $\chi(G) + \chi(\overline{G}) \leq n + 1$. Edward A. Nordhaus and Jerry W. Gaddum [178] showed that this inequality holds for all graphs G of order n when they established two pairs of inequalities involving the sum and product of the chromatic numbers of a graph and its complement. Such inequalities established for any parameter have become known as **Nordhaus–Gaddum inequalities**. The following proof is due to Hudson V. Kronk (see [49]).

Theorem 14.23 **(The Nordhaus–Gaddum Theorem)** *If G is a graph of order n, then*

(i) $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$

(ii) $n \leq \chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$.

Proof. Suppose that $\chi(G) = k$ and $\chi(\overline{G}) = \ell$. Let a k-coloring c of G and an ℓ-coloring $\overline{c}$ of $\overline{G}$ be given. Using these colorings, we obtain a coloring of K_n. With each vertex v of G (and of $\overline{G}$), we associate the ordered pair $(c(v), \overline{c}(v))$. Since every two vertices of K_n are either adjacent in G or in $\overline{G}$, they are assigned different colors in that subgraph of K_n. Thus, this is a coloring of K_n using at most $k\ell$ colors. Therefore,

$$n = \chi(K_n) \leq k\ell = \chi(G) \cdot \chi(\overline{G}).$$

This establishes the lower bound in (ii). Since the geometric mean of two positive real numbers never exceeds their arithmetic mean, it follows that

$$\sqrt{n} \leq \sqrt{\chi(G) \cdot \chi(\overline{G})} \leq \frac{\chi(G) + \chi(\overline{G})}{2}. \tag{14.3}$$

Consequently,

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}),$$

which verifies the lower bound in (i).

To verify the upper bound in (i), let $p = \max\{\delta(H)\}$, where the maximum is taken over all subgraphs H of G. Hence, the minimum degree of every subgraph of G is at most p. By Theorem 14.19, $\chi(G) \leq 1 + p$.

We claim that the minimum degree of every subgraph of $\overline{G}$ is at most $n-p-1$. Assume, to the contrary, that there is a subgraph H of G such that $\delta(\overline{H}) \geq n-p$ for the subgraph $\overline{H}$ in $\overline{G}$. Thus, every vertex of H has degree $p - 1$ or less in G. Let F be a subgraph of G such that $\delta(F) = p$. So, every vertex of F has degree p or more in G. This implies that no vertex of F belongs to H. Since the order of F is at least $p+1$, the order of H is at most $n-(p+1) = n-p-1$.

This, however, contradicts the fact that $\delta(\overline{H}) \geq n - p$. Thus, as claimed, the minimum degree of every subgraph of $\overline{G}$ is at most $n-p-1$. By Theorem 14.19, $\chi(\overline{G}) \leq 1 + (n - p - 1) = n - p$ and so

$$\chi(G) + \chi(\overline{G}) \leq (1 + p) + (n - p) = n + 1.$$

This verifies the upper bound in (i). By (14.3),

$$\chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2,$$

verifying the final inequality. ■

As the proof of Theorem 14.23 indicates, the key inequality of the four inequalities listed is $\chi(G) + \chi(\overline{G}) \leq n + 1$. Bonnie M. Stewart [226] and Hans-Joachim Finck [90] showed that no improvement in Theorem 14.23 is possible.

Theorem 14.24 *Let n be a positive integer. For every two positive integers a and b such that*

$$2\sqrt{n} \leq a + b \leq n + 1 \text{ and } n \leq ab \leq \left(\tfrac{n+1}{2}\right)^2$$

there is a graph G of order n such that $\chi(G) = a$ and $\chi(\overline{G}) = b$.

Proof. Let $n_1, n_2, \ldots, n_a$ be a positive integers such that $\sum_{i=1}^{a} n_i = n$ and $n_1 \leq n_2 \leq \ldots \leq n_a = b$. Since $a + b - 1 \leq n \leq ab$, such integers n_i $(1 \leq i \leq a)$ exist. The graph $G = K_{n_1,n_2,\ldots,n_a}$ has order n and $\overline{G} = K_{n_1} + K_{n_2} + \cdots + K_{n_a}$. Hence, $\chi(G) = a$ and $\chi(\overline{G}) = b$. ■

Of the several bounds for the chromatic number of a graph G, the clique number $\omega(G)$ is the best known and simplest lower bound for $\chi(G)$, while $1 + \Delta(G)$ is the best known and simplest upper bound for $\chi(G)$. If $\Delta(G) \geq 3$ and G is not complete, then $\Delta(G)$ is an improved upper bound for $\chi(G)$ by Brooks' theorem (Theorem 14.21). Bruce Reed [198] conjectured that $\chi(G)$ is always at least as close to $\omega(G)$ as to $1 + \Delta(G)$.

Reed's Conjecture *For every graph G,*

$$\chi(G) \leq \frac{\omega(G) + 1 + \Delta(G)}{2}.$$

As we saw in Theorem 14.9, for a graph G of order n, the number $n-\alpha(G)+1$ is also an upper bound for $\chi(G)$. By Theorem 14.16, $(n + \omega(G))/2$ is also an upper bound for $\chi(G)$. The following improved upper bound for $\chi(G)$ is due to Robert C. Brigham and Ronald D. Dutton [36].

Theorem 14.25 *For every graph G of order n,*

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Proof. We proceed by induction on n. When $n = 1$, $G = K_1$ and $\chi(G) = \omega(G) = \alpha(G) = 1$ and so

$$\chi(G) = \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Thus, the basis step holds for the induction.

Assume that the inequality holds for all graphs of order less than n where $n \geq 2$ and let G be a graph of order n. If $G = \overline{K}_n$, then $\chi(G) = \omega(G) = 1$ and $\alpha(G) = n$; so

$$\chi(G) = \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Hence, we may assume that $G \neq \overline{K}_n$. Thus $1 \leq \alpha(G) \leq n - 1$. Let V_0 be a maximum independent set of vertices in G. Therefore, $|V_0| = \alpha(G)$. Let $G_1 = G - V_0$, where $\alpha(G_1) = \alpha_1$ and $\omega(G_1) = \omega_1$. Furthermore, let $V(G_1) = V_1$, where, then, $|V_1| = n - \alpha(G)$. We consider two cases.

Case 1. *G_1 is a complete graph.* Thus, $V(G)$ can be partitioned into V_0 and V_1, where $G[V_0] = \overline{K}_{\alpha(G)}$ and $G_1 = G[V_1] = K_{n-\alpha(G)}$. Therefore, either $\chi(G) = \omega(G) = n - \alpha(G)$ or $\chi(G) = \omega(G) = n - \alpha(G) + 1$. We now consider these two subcases.

Subcase 1.1. $\chi(G) = \omega(G) = n - \alpha(G)$. So,

$$\begin{aligned}\chi(G) &= n - \alpha(G) = \frac{(n - \alpha(G)) + (n - \alpha(G))}{2} \\ &= \frac{\omega(G) + n - \alpha(G)}{2} < \frac{\omega(G) + n + 1 - \alpha(G)}{2}.\end{aligned}$$

Subcase 1.2. $\chi(G) = \omega(G) = n - \alpha(G) + 1$. Here,

$$\begin{aligned}\chi(G) &= n - \alpha(G) + 1 = \frac{(n - \alpha(G) + 1) + (n - \alpha(G) + 1)}{2} \\ &= \frac{\omega(G) + (n - \alpha(G) + 1)}{2}.\end{aligned}$$

Case 2. *G_1 is not a complete graph.* In this case, $\alpha_1 \geq 2$. Since $\chi(G) \leq \chi(G_1) + 1$, it follows by the induction hypothesis that

$$\begin{aligned}\chi(G) &\leq \chi(G_1) + 1 \leq \frac{\omega_1 + (n - \alpha(G)) + 1 - \alpha_1}{2} + 1 \\ &\leq \frac{\omega(G) + (n - \alpha(G)) + 1 - \alpha_1}{2} + 1 \leq \frac{\omega(G) + (n - \alpha(G) + 1)}{2},\end{aligned}$$

completing the proof. ■

Applying Theorem 14.25 to both a graph G of order n and its complement $\overline{G}$ (where then $\omega(\overline{G}) = \alpha(G)$ and $\alpha(\overline{G}) = \omega(G)$), we have

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}$$

and

$$\chi(\overline{G}) \leq \frac{\omega(\overline{G}) + n + 1 - \alpha(\overline{G})}{2}.$$

Adding these inequalities gives us an alternative proof of the major inequality stated in the Nordhaus-Gaddum theorem (Theorem 14.23): For every graph G of order n,

$$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

The Gallai-Roy-Vitaver Theorem

Bounds for the chromatic number of a graph G can also be given in terms of the length $\ell(D)$ of a longest (directed) path in an orientation D of G. Suppose that G is a k-chromatic graph, where a k-coloring of G is given using the colors $1, 2, \ldots, k$. An orientation D of G can be constructed by directing each edge uv of G from u to v if the color assigned to u is smaller than the color assigned to v. Thus, the length of every directed path in D is at most $k - 1$. In particular, $\ell(D) \leq \chi(G) - 1$. So, there exists an orientation D of G such that $\chi(G) \geq 1 + \ell(D)$. On the other hand, Tibor Gallai [100], Bernard Roy [212] and L. M. Vitaver [245] independently discovered the following result.

Theorem 14.26 (**The Gallai-Roy-Vitaver Theorem**) *For every orientation D of a graph G,*

$$\chi(G) \leq 1 + \ell(D).$$

Proof. Let D be an orientation of G and let D' be a spanning acyclic subdigraph of D of maximum size. A coloring c is defined on G by assigning to each vertex v of G the color 1 plus the length of a longest path in D' whose terminal vertex is v. Then, as we proceed along the vertices in any directed path in D', the colors are strictly increasing.

Let (u, v) be an arc of D. If (u, v) belongs to D', then $c(u) < c(v)$. On the other hand, if (u, v) is not in D', then adding (u, v) to D' creates a directed cycle, which implies that $c(v) < c(u)$. Consequently, $c(u) \neq c(v)$ for every two adjacent vertices u and v of G and so

$$c : V(G) \to \{1, 2, \ldots, 1 + \ell(D)\}$$

is a proper coloring of G. Therefore, $\chi(G) \leq 1 + \ell(D)$. ■

The following result is therefore a consequence of Theorem 14.26 and the observation that precedes it.

Corollary 14.27 *Let G be a graph and let $\ell = \min\{\ell(D)\}$, where the minimum is taken over all orientations D of G. Then*

$$\chi(G) = 1 + \ell.$$

There is also a corollary to Corollary 14.27.

Corollary 14.28 *Every orientation of a graph G contains a directed path with at least $\chi(G)$ vertices.*

Exercises for Chapter 14

Section 14.1. The Chromatic Number of a Graph

1. The vertices of a graph G are colored with three colors in such a way that each vertex is adjacent to vertices all of which are colored with exactly one of the two remaining colors. Show that $\chi(G) = 2$.

2. Prove that the size of every k-chromatic graph is at least $\binom{k}{2}$.

3. Let G be a graph with $\chi(G) = k$ and let a k-coloring of G be given. Let H be the graph with $V(H) = V(G)$ such that $uv \in E(H)$ if u and v are assigned different colors in G. Determine $\chi(H)$.

4. Prove that if S is a color class resulting from a k-coloring of a k-chromatic graph G, where $k \geq 2$, then there is a component H of $G - S$ such that $\chi(H) = k - 1$.

5. Show, for every connected graph G of order n and diameter d, that
$$\chi(G) \leq n - d + 1.$$

6. For each integer $k \geq 3$, give an example of a regular k-chromatic graph G such that $G \neq K_k$.

7. Let G be a k-chromatic graph, where $k \geq 2$, and let r be a positive integer such that $r \geq \Delta(G)$. Prove that there exists an r-regular k-chromatic graph H such that G is an induced subgraph of H.

8. Determine the largest positive integer k such that $\chi(H) = \chi(G) = k$, where H is obtained from a nonempty graph G by subdividing each edge of G exactly once.

9. Let G be a nonempty graph and let H be a graph obtained from G by subdividing a single edge of G exactly once.
   (a) Show that $|\chi(G) - \chi(H)| \leq 1$.
   (b) Show that each of the following is possible:
      (i) $\chi(H) = \chi(G)$
      (ii) $\chi(H) = \chi(G) - 1$
      (iii) $\chi(H) = \chi(G) + 1$.

10. A **balanced coloring** of a graph G is an assignment of colors to the vertices of G such that (i) every two adjacent vertices are assigned different colors and (ii) the numbers of vertices assigned any two different colors differ by at most 1. The smallest number of colors used in a balanced coloring of G is the **balanced chromatic number** $\chi_b(G)$ of G.

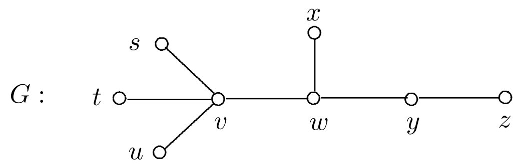


Figure 14.9: The graph in Exercise 10(b)

(a) Prove that the balanced chromatic number is defined for every graph G.

(b) Determine $\chi_b(G)$ for the graph G in Figure 14.9.

11. Let G be an r-regular graph, $r \geq 3$, such that $3 \leq \chi(G) \leq r+1$. Prove or disprove the following:

    (a) For every edge e of G, there exists an odd cycle containing e.

    (b) For every edge e of G, there exists an odd cycle not containing e.

12. (a) Show for every graph G that
$$\chi(G) = \max\{\chi(H) : H \text{ is a subgraph of } G\}.$$

    (b) The problem in (a) should suggest a question to you. Ask and answer this question.

13. Prove Theorem 14.4: *For graphs* $G_1, G_2, \ldots, G_k$ *and* $G = G_1 + G_2 + \cdots + G_k$, $\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}$.

14. Prove Theorem 14.6: *If* G *is a graph with blocks* $B_1, B_2, \ldots, B_k$, *then* $\chi(G) = \max\{\chi(B_i) : 1 \leq i \leq k\}$.

15. Prove Theorem 14.7: *For graphs* $G_1, G_2, \ldots, G_k$ *and* $G = G_1 \vee G_2 \vee \cdots \vee G_k$, $\chi(G) = \sum_{i=1}^{k} \chi(G_i)$.

16. Let G be a graph of order n with independence number $\alpha(G) = 2$.

    (a) Prove that if G is disconnected, then G contains $K_{\lceil \frac{n}{2} \rceil}$ as a subgraph.

    (b) Prove that if G is connected, then G contains a path (u, v, w) such that $uw \notin E(G)$ and every vertex in $G - \{u, v, w\}$ is adjacent to either u or w (or both).

17. By Theorem 14.9, it follows that for every graph G of order n,
$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1.$$
Prove or disprove: The chromatic number of a graph G can never be closer to $n - \alpha(G) + 1$ than to $\frac{n}{\alpha(G)}$.

18. For a given positive integer n, determine all graphs of order n for which the two bounds in Theorem 14.9 are equal.

19. Let $k \geq 2$ be an integer. For each integer i with $1 \leq i \leq 2k+1$, let G_i be a copy of K_k. The graph G of order $2k^2+k$ is obtained from the graphs $G_1, G_2, \ldots, G_{2k+1}, G_{2k+2} = G_1$ by joining each vertex in G_i to every vertex in G_{i+1} $(1 \leq i \leq 2k+1)$. Compute the bounds $n/\alpha(G)$, $n+1-\alpha(G)$ and $\omega(G)$ for $\chi(G)$ for an arbitrary $k \geq 2$. Determine $\chi(G)$.

20. Give an example of a graph G of order n for which $n/\alpha(G)$ is an integer and for which $\chi(G)$ is none of the numbers $n/\alpha(G)$, $n-\alpha(G)+1$ and $\omega(G)$.

21. Give an example of a graph G with $\chi(G) = \alpha(G)$ in which no $\chi(G)$-coloring of G results in a color class containing $\alpha(G)$ vertices.

22. Does there exist a k-chromatic graph G in which no color class of a k-coloring of G contains at least $\alpha(G) - 2$ vertices?

23. Let $k \geq 2$ be an integer. Prove that if G is a k-colorable graph of order n such that $\delta(G) > \left(\frac{k-2}{k-1}\right) n$, then G is k-chromatic. Is the bound sharp?

24. Prove that a connected graph G has chromatic number at least 3 if and only if for every vertex v of G, there exist two adjacent vertices u and w in G such that $d(u,v) = d(v,w)$.

25. Recall that an independent set S of vertices in a graph G is maximal if S is not a proper subset of any other independent set of vertices in G. The **upper chromatic number** $\overline{\chi}(G)$ of a graph G is the maximum number of maximal independent sets of vertices into which $V(G)$ can be partitioned.

    (a) Show that the upper chromatic number of a graph need not be defined.

    (b) Show that if the upper chromatic number $\overline{\chi}(G)$ of a graph G is defined, then $\overline{\chi}(G) \geq \chi(G)$.

    (c) Give an example of a graph G for which $\overline{\chi}(G)$ is defined and determine $\overline{\chi}(G)$ for this graph G.

26. (See Exercise 25 for the definitions of maximal independent sets of vertices and the upper chromatic number $\overline{\chi}(G)$ of a graph G.)

    (a) Let G be a graph with $\overline{\chi}(G) = k$ and let $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ into k maximal independent sets. Let each vertex of V_i $(1 \leq i \leq k)$ be assigned the color i. Show that if any vertex of G is assigned a different color in $\{1, 2, \ldots, k\}$, then the resulting coloring is not proper.

(b) Let G be a graph of order n having upper chromatic number $\overline{\chi}(G)$. Show that the domination number $\gamma(G)$ satisfies the inequality $\gamma(G) \cdot \overline{\chi}(G) \le n$.

(c) Show that for every integer k, there exists a graph G for which $\overline{\chi}(G)$ is defined and $\overline{\chi}(G) - \chi(G) = k$.

27. (a) Let v be a vertex of a k-chromatic graph G and let $U_1, U_2, \ldots, U_\ell$ be the maximal independent sets of G containing v. Prove that for some k-coloring of G, one of the resulting k color classes is U_i for some i with $1 \le i \le \ell$.

(b) Let G be a k-chromatic graph, $k \ge 2$, and S an independent set of vertices in G. Prove that $\chi(G[V(G) - S]) = k - 1$ if and only if S is a color class in some k-coloring of G.

28. For a nonempty graph G, let v be a vertex of G and let $U_1, U_2, \ldots, U_\ell$ be the maximal independent sets of G containing v.

(a) Prove that $\chi(G) = 1 + \min_{1 \le i \le \ell} \chi(G[V(G) - U_i])$.

(b) What is the relationship between $\chi(G)$ and

$$1 + \max_{1 \le i \le \ell} \chi(G[V(G) - U_i])?$$

29. Prove that every Hamiltonian-connected graph of order at least 3 has chromatic number at least 3.

30. Determine (and prove) a necessary and sufficient condition for a connected graph to have a 2-colorable line graph.

31. Two dentists are having new offices designed for themselves. In the common waiting room for their patients, they have decided to have an aquatic area containing fish tanks. Because some fish require a coldwater environment while others are more tropical and because some fish are aggressive with other types of fish, not all fish can be placed in a single tank. It is decided to have ten exotic fish, denoted by $F_1, F_2, \ldots, F_{10}$, where the fish that cannot be placed in the same tank as F_i $(1 \le i \le 10)$ are listed to the right of F_i.

F_1: F_2, F_3, F_4, F_5, F_{10} F_2: F_1, F_3, F_6 F_3: F_1, F_2, F_4
F_4: F_1, F_3, F_5 F_5: F_1, F_4, F_9 F_6: F_2, F_7, F_{10}
F_7: F_6, F_8, F_{10} F_8: F_7, F_9, F_{10} F_9: F_5, F_8, F_{10}
F_{10}: F_1, F_6, F_7, F_8, F_9.

What is the minimum number of tanks required?

32. Figure 14.10 shows traffic lanes $L_1, L_1, \ldots, L_7$ at the intersection of two streets. A traffic light is located at the intersection. During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection in permissible directions. What is the minimum number of phases needed for the traffic light so that (eventually) all cars may proceed through the intersection?

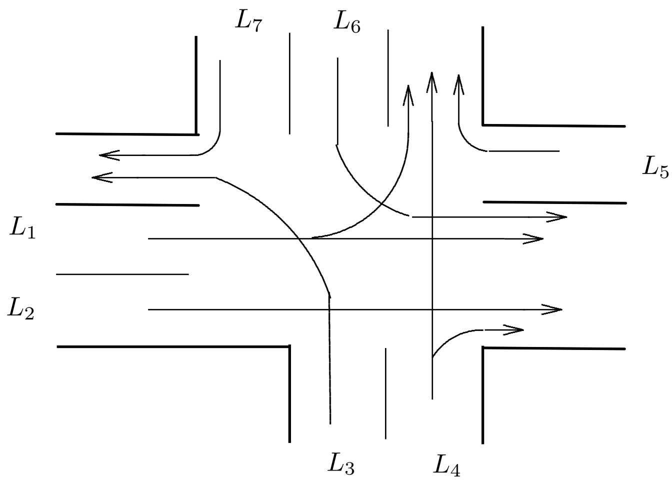


Figure 14.10: Traffic lanes at street intersections in Exercise 32

Section 14.2. Color-Critical Graphs

33. Let G be a k-chromatic graph where $k \geq 2$.

    (a) Prove that for every vertex v of G, either $\chi(G-v) = k$ or $\chi(G-v) = k-1$.

    (b) Prove that for every edge e of G, either $\chi(G-e) = k$ or $\chi(G-e) = k-1$.

34. Let G be a k-chromatic graph such that $\chi(G-e) = k-1$ for some edge $e = uv$ of G. Prove that $\chi(G-u) = \chi(G-v) = k-1$.

35. Show that the odd cycles are the only 3-critical graphs.

36. Determine all k-critical graphs with $k \geq 3$ such that $G-v$ is $(k-1)$-critical for every vertex v of G.

37. Prove or disprove: If G and H are color-critical graphs, then the join $G \vee H$ of G and H is color-critical.

38. It has been mentioned that every k-critical graph, $k \geq 3$, is 2-connected. Show that there exists a k-critical graph having connectivity 2 for every integer $k \geq 3$.

39. Prove or disprove the following:

    (a) There exists no graph G with $\chi(G) = 3$ without isolated vertices such that $\chi(G - v) = 2$ for exactly 75% of the vertices v of G.

    (b) There exists no graph G with $\chi(G) = 3$ without isolated vertices and containing a 3-critical component such that $\chi(G - v) = 2$ for exactly 75% of the vertices v of G.

40. A graph G is defined to be **k-degenerate**, $k \geq 0$, if $\delta(H) \leq k$ for every induced subgraph H of G. The 0-degenerate graphs are the empty graphs, the 1-degenerate graphs are the forests and every planar graph is 5-degenerate (by Corollary 10.5). A k-degenerate graph is **maximal k-degenerate** if, for every two nonadjacent vertices u and v of G, the graph $G + uv$ is not k-degenerate.

    For $k \geq 0$, let $\mathcal{P}_k$ denote the family of k-degenerate graphs. Then $\chi_{\mathcal{P}_k}(G)$ is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a k-degenerate subgraph of G. A graph is said to be **ℓ-critical with respect to** $\chi_{\mathcal{P}_k}$, $\ell \geq 2$, if $\chi_{\mathcal{P}_k}(G) = \ell$ and $\chi_{\mathcal{P}_k}(G - v) = \ell - 1$ for every $v \in V(G)$.

    (a) Prove that if G is a maximal k-degenerate graph of order n, where $n \geq k + 1$, then $\delta(G) = k$.

    (b) Determine $\chi_{\mathcal{P}_k}(K_n)$.

    (c) Prove that if G is a graph that is ℓ-critical with respect to $\chi_{\mathcal{P}_k}$, then $\delta(G) \geq (k + 1)(\ell - 1)$.

Section 14.3. Bounds for the Chromatic Number

41. (a) Show, for every k-chromatic graph G, that there exists an ordering of the vertices of G such that the greedy coloring algorithm gives a k-coloring of G.

    (b) Show, for every positive integer p, that there exists a graph G and an ordering of the vertices of G such that the greedy coloring algorithm gives a k-coloring of G, where $k = p + \chi(G)$.

42. Since every nontrivial tree T is bipartite, $\chi(T) = 2$. Prove, for every integer $k \geq 2$, that there exists a tree T_k with $\Delta(T_k) = k$ and an ordering s of the vertices of T_k that produces a greedy coloring of T_k using $k + 1$ colors.

43. Let T be the tree of Figure 14.11.

(a) What is the greedy coloring c produced by the ordering s: v_1, v_2, ..., v_8 of the vertices of T?

(b) Does there exist a different ordering of the vertices of T giving a greedy coloring that uses fewer colors?

(c) Does there exist a different ordering of the vertices of T giving a greedy coloring that uses more colors than that in (a)?

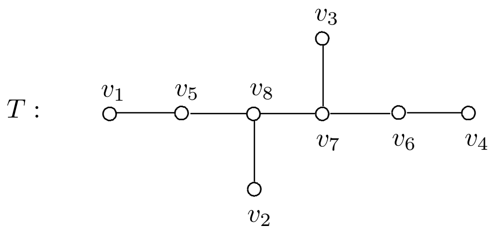


Figure 14.11: The tree T in Exercise 43

44. (a) What upper bound for $\chi(G)$ is given by Theorem 14.19 for the graph G in Figure 14.12?

    (b) What is $\chi(G)$ for this graph G?

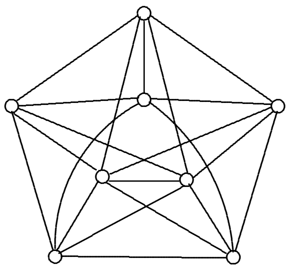


Figure 14.12: The graph G in Exercise 44

45. For the double star T containing two vertices of degree 4, what upper bound for $\chi(T)$ is given by Theorems 14.18, 14.19 and 14.20?

46. What bound is given for $\chi(G)$ by Theorem 14.19 in the case that G is (a) a tree? (b) an outerplanar graph?

47. Among all 5-regular graphs, let s be the smallest chromatic number of such a graph and let t be the largest chromatic number. For each integer k with $s \leq k \leq t$, show that there exists a 5-regular graph G_k such that $\chi(G_k) = k$.

48. Prove a result analogous to Theorem 14.21 for disconnected graphs.

49. Let G be a connected cubic graph of order $n > 4$ having girth 3. Determine $\chi(G)$.

50. Let G be a nontrivial connected graph of order n that is not regular. Let $v_n \in V(G)$ such that $\deg v_n = \delta(G)$. Let v_{n-1} be a vertex adjacent to v_n and v_{n-2} a vertex adjacent to v_n or v_{n-1}. Continue this arriving at $v_n, v_{n-1}, \ldots, v_1$. Apply the Greedy Coloring Algorithm to the sequence $v_1, v_2, \ldots, v_n$. What upper bound is obtained for $\chi(G)$?

51. A graph G of order n is kn-regular, where k is a rational number with $\frac{1}{2} \leq k < 1$. What lower bound does Theorem 14.22 give for $\chi(G)$?

52. Show that there is no graph of order 6 and size 13 that has chromatic number 3.

53. (a) If G is a graph of order n and size $n^2/4$, what bound does Theorem 14.22 give about $\chi(G)$? How sharp is this bound?
    (b) If G is a graph of order n and size m where $m > n^2/4$, what bound does Theorem 14.22 give about $\chi(G)$? How sharp is this bound?

54. What does the bound in Theorem 14.22 say for a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order n where $n_1 = n_2 = \cdots = n_k$?

55. Use Corollary 14.27 to determine the chromatic number of every nonempty bipartite graph.

56. By a proper k-coloring of a graph G is meant a function $c : V(G) \to \mathbb{N}_k = \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$. This can also be defined as a function $c : V(G) \to \mathcal{P}(\mathbb{N}_k)$, the power set of $\mathbb{N}_k$, such that $|c(x)| = 1$ for each $x \in V(G)$ and $c(u) \cap c(v) = \emptyset$ if $d(u, v) = 1$, where no condition is placed on pairs u, v of vertices if $d(u, v) \geq 2$. A **2-tone k-coloring** of G is a function $c : V(G) \to \mathcal{P}(\mathbb{N}_k)$ such that $|c(x)| = 2$ for each $x \in V(G)$ and $c(u) \cap c(v) = \emptyset$ if $d(u, v) = 1$ and $|c(u) \cap c(v)| \leq 1$ if $d(u, v) = 2$, where there is no condition placed on $c(u) \cap c(v)$ if $d(u, v) \geq 3$. The **2-tone chromatic number** $\chi_2(G)$ of G is the minimum k for which G has a 2-tone k-coloring.

    (a) Prove that if the maximum degree of a graph G is Δ, then
    $$\chi_2(G) \geq \left\lceil \frac{5 + \sqrt{8\Delta + 1}}{2} \right\rceil.$$
    (b) Compute $\chi_2(C_7)$ and $\chi_2(C_8)$, thereby showing that there are graphs for which the bound in (a) is attained and graphs for which the bound in (a) is not attained.

Chapter 15

Perfect Graphs and List Colorings

In this chapter we continue our discussion of proper vertex colorings. We consider the so-called perfect graphs, for which the clique number plays an important role. We also consider proper vertex colorings of graphs in which the colors allowed for each vertex are specified in advance.

15.1 Perfect Graphs

In Corollary 14.3, we saw that the clique number $\omega(G)$ of a graph G is a lower bound for $\chi(G)$. While there are many examples of graphs G for which $\chi(G) = \omega(G)$, such as complete graphs and bipartite graphs, there are also many graphs whose chromatic number exceeds its clique number, such as odd cycles of length 5 or more and the Petersen graph, shown in Figure 15.1 with a 3-coloring. In these instances, the graphs have chromatic number 3 and clique number 2. As we are about to see, a graph with clique number 2 can have an arbitrarily large chromatic number.

Triangle-Free Graphs with Large Chromatic Number

For a given graph F, recall that a graph G is called F-free if no induced subgraph of G is isomorphic to F. Recall also that a K_3-free graph is commonly called a triangle-free graph. Consequently, every nonempty triangle-free graph has clique number 2. The graph G of Figure 15.2 is triangle-free (and so $\omega(G) = 2$) but $\chi(G) = 4$. A 4-coloring of this graph is shown in Figure 15.2. Hence, $\chi(G)$ exceeds $\omega(G)$ by 2 in this case. This graph is called the **Grötzsch graph**, named for the German mathematician Herbert Grötzsch. It is known to be the unique smallest graph (in terms of order) that is both 4-chromatic

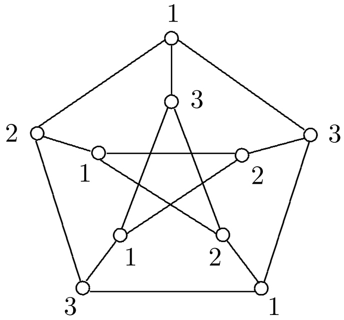


Figure 15.1: The Petersen graph: A 3-chromatic graph with clique number 2

and triangle-free. The fact that a graph can be triangle-free and yet have a large chromatic number has been established by a number of mathematicians, including Blanche Descartes [66], John B. Kelly and Leroy M. Kelly [141] and Alexander Zykov [263]. The proof of this fact that we present here, however, is due to Jan Mycielski [174].

G :

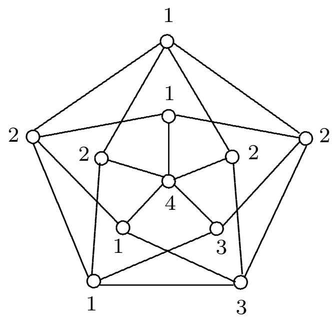


Figure 15.2: The Grötzsch graph: A 4-chromatic triangle-free graph

Theorem 15.1 *For every positive integer k, there exists a triangle-free k-chromatic graph.*

Proof. Since no graph with chromatic number 1 or 2 contains a triangle, the theorem is obviously true for $k = 1$ and $k = 2$. To verify the theorem for $k \geq 3$, we proceed by induction on k. As we have already observed, $\chi(C_5) = 3$ and C_5 is triangle-free; so the statement is true for $k = 3$.

Assume that there exists a triangle-free graph with chromatic number k, where $k \geq 3$. We show that there exists a triangle-free $(k+1)$-chromatic graph. Let H be a triangle-free graph with $\chi(H) = k$, where $V(H) = \{v_1, v_2, \ldots, v_n\}$. We construct a graph G from H by adding $n+1$ new vertices $u, u_1, u_2, \ldots, u_n$, joining u to each vertex u_i $(1 \leq i \leq n)$ and joining u_i to each neighbor of v_i

in H. (See Figure 15.3 for the case where $k = 3$ and $H = C_5$. Here the resulting graph G is the Grötzsch graph of Figure 15.3.)

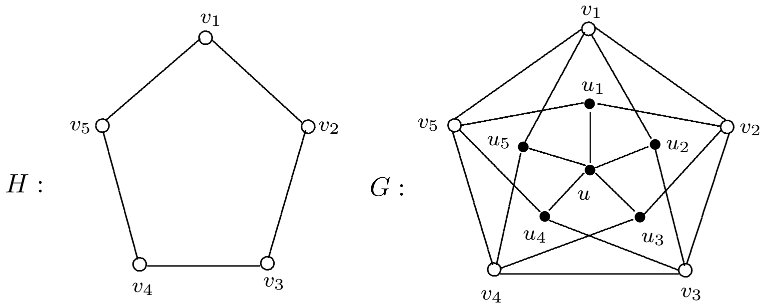


Figure 15.3: The Mycielski construction

We claim that G is a triangle-free $(k+1)$-chromatic graph. First, we show that G is triangle-free. Since $S = \{u_1, u_2, \ldots, u_n\}$ is an independent set of vertices of G and u is adjacent to no vertex of H, it follows that u belongs to no triangle in G. Hence, if there is a triangle T in G, then two of the three vertices of T must belong to H and the third vertex must belong to S, say $V(T) = \{u_i, v_j, v_k\}$. Since u_i is adjacent to v_j and v_k, it follows that v_i is adjacent to v_j and v_k. Since v_j and v_k are adjacent, H contains a triangle, which is a contradiction. Thus, as claimed, G is triangle-free.

Next, we show that $\chi(G) = k+1$. Since H is a subgraph of G and $\chi(H) = k$, it follows that $\chi(G) \geq k$. Let a k-coloring of H be given and assign to u_i the same color that is assigned to v_i for $1 \leq i \leq n$. Assigning the color $k+1$ to u produces a $(k+1)$-coloring of G and so $\chi(G) \leq k+1$. Hence, either $\chi(G) = k$ or $\chi(G) = k+1$. Suppose that $\chi(G) = k$. Then there is a k-coloring of G with colors $1, 2, \ldots, k$, where u is assigned the color k, say. Necessarily, none of the vertices $u_1, u_2, \ldots, u_n$ is assigned the color k; that is, each vertex of S is assigned one of the colors $1, 2, \ldots, k-1$. Since $\chi(H) = k$, one or more vertices of H are assigned the color k. For each vertex v_i of H colored k, recolor it with the color assigned to u_i. This produces a $(k-1)$-coloring of H, which is impossible. Thus, $\chi(G) = k+1$. ∎

If the Mycielski construction (described in the proof of Theorem 15.1) is applied to the Grötzsch graph (Figures 15.2 and 15.3), then a 5-chromatic triangle-free graph of order 23 is produced. Using a computer search, Tommy Jensen and Gordon F. Royle [137] showed that the smallest order of a 5-chromatic triangle-free graph is actually 22. Applying the Mycielski construction to this graph produces a 6-chromatic triangle-free graph of order 45. Whether 45 is the smallest order of such a graph is unknown.

k-Chromatic Graphs with Large Girth

Theorem 15.1 can be described in terms of the clique number and chromatic number of a graph: For every positive integer k, there exists a nonempty graph G with $\omega(G) = 2$ and $\chi(G) = k$. With the aid of Theorem 15.1, it can be shown that for every two integers ℓ and k with $2 \leq \ell \leq k$, there exists a graph G with $\omega(G) = \ell$ and $\chi(G) = k$ (see Exercise 2).

Recall that the girth $g(G)$ of a graph G is the length of a smallest cycle in G. Thus, Theorem 15.1 can also be described in terms of the girth and chromatic number of a graph: For every integer $k \geq 4$, there exists a nonempty graph G with $g(G) \geq 4$ and $\chi(G) = k$. This interpretation emphasizes, in some sense, the sparseness of G since G has no 3-cycle. Theorem 15.1 was extended by Erdős [79] and Lovász [158], who proved the existence of graphs with arbitrarily large chromatic number and girth.

Theorem 15.2 *For every two integers $k \geq 2$ and $\ell \geq 3$, there exists a graph G with $g(G) > \ell$ and $\chi(G) = k$.*

A proof of Theorem 15.2 will be discussed in Chapter 21.

Perfect Graphs

While much interest has been shown in graphs G for which $\chi(G) > \omega(G)$, even more interest has been shown in graphs G for which not only $\chi(G) = \omega(G)$ but $\chi(H) = \omega(H)$ for *every* induced subgraph H of G. A graph G is called **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Thus, every induced subgraph of a perfect graph is also perfect. This definition was introduced by Claude Berge [24]. This concept has led to two conjectures, both of which attracted much attention. Before describing these, we consider several classes of perfect graphs.

Certainly, if $G = K_n$, then $\chi(G) = \omega(G) = n$. Furthermore, every induced subgraph H of K_n is also a complete graph and so $\chi(H) = \omega(H)$. Thus, every complete graph is perfect. On the other hand, if $G = \overline{K}_n$ and H is any induced subgraph of G, then $\chi(H) = \omega(H) = 1$. So, every empty graph is also perfect. The bipartite graphs constitute a more interesting class of perfect graphs.

Theorem 15.3 *Every bipartite graph is perfect.*

Proof. Let G be a bipartite graph and let H be an induced subgraph of G. If H is nonempty, then $\chi(H) = \omega(H) = 2$; while if H is empty, then $\chi(H) = \omega(H) = 1$. In either case, $\chi(H) = \omega(H)$ and so G is perfect. ■

The next theorem, which is a consequence of a result due to Tibor Gallai [98], describes a related class of perfect graphs.

Theorem 15.4 *Every graph whose complement is bipartite is perfect.*

Proof. Let G be a graph of order n such that $\overline{G}$ is bipartite. Since the complement of every (nontrivial) induced subgraph of G is also bipartite, to verify that G is perfect, it suffices to show that $\chi(G) = \omega(G)$. Suppose that $\chi(G) = k$ and $\omega(G) = \ell$. Then $k \geq \ell$. Let there be given a k-coloring of G. Then each color class of G consists either of one or two vertices; for if G contains a color class with three or more vertices, then this would imply that $\overline{G}$ has a triangle, which is impossible.

Of the k color classes, suppose that p of these classes consist of a single vertex and that each of the remaining q classes consists of two vertices. Hence, $p + q = k$ and $p + 2q = n$. Let W be the set of vertices of G belonging to a singleton color class. Since every two vertices of W are necessarily adjacent, $G[W] = K_p$ and so $\overline{G}[W] = \overline{K}_p$.

Since no k-coloring of G results in more than q color classes having two vertices, it follows that $\overline{G}$ has a set M of q pairwise nonadjacent edges but no more than q such edges. We claim that for each edge $uv \in M$, either u is adjacent to no vertex of W or v is adjacent to no vertex of W in $\overline{G}$. Suppose that this is not the case. Then we may assume that u is adjacent to some vertex $w_1 \in W$ and v is adjacent to some vertex $w_2 \in W$ in $\overline{G}$. Since $\overline{G}$ is triangle-free, $w_1 \neq w_2$. However then, $(M - \{uv\}) \cup \{uw_1, vw_2\}$ is a set of $q + 1$ pairwise nonadjacent edges in $\overline{G}$. This, however, is impossible and so, as claimed, for each edge uv in M either u is adjacent to no vertex of W or v is adjacent to no vertex of W.

Therefore, $\overline{G}$ contains an independent set of at least $p + q = k$ vertices and so $\omega(G) = \ell \geq k$. Hence, $\chi(G) = \omega(G)$. ■

Interval Graphs

For a collection $S = \{I_1, I_2, \ldots, I_n\}$ of finite closed intervals of real numbers, the **interval graph** G representing S has vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $v_iv_j \in E(G)$ if $I_i \cap I_j \neq \emptyset$. If G is an interval graph, then every induced subgraph of G is also an interval graph (see Exercise 11). In the definition of "interval graph", we arrive at the same concept if open intervals are used rather than closed intervals. Indeed, we can use half-open (half-closed) intervals or a mixture of any of these (see Exercises 13 and 15).

For example, the graph G in Figure 15.4 is an interval graph as can be seen by considering the five intervals $I_1 = [0, 2]$, $I_2 = [1, 5]$, $I_3 = [3, 6]$, $I_4 = [4, 8]$, $I_5 = [7, 9]$, where v_i and v_j are adjacent in G, $1 \leq i, j \leq 5$, if $I_i \cap I_j \neq \emptyset$. Observe that $\chi(G) = \omega(G) = 3$ for this graph G. This graph is also a perfect graph. Indeed, every interval graph is a perfect graph.

Theorem 15.5 *Every interval graph is a perfect graph.*

Proof. Let G be an interval graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since every induced subgraph of an interval graph is also an interval graph, it suffices to show that $\chi(G) = \omega(G)$. Because G is an interval graph, there exist n closed

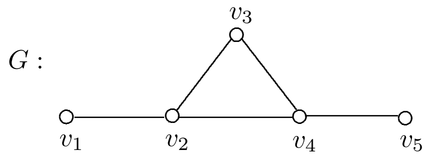


Figure 15.4: An interval graph

intervals $I_i = [a_i, b_i]$, $1 \le i \le n$, such that v_i is adjacent to v_j $(i \ne j)$ if $I_i \cap I_j \ne \emptyset$. We may assume that the intervals (and consequently, the vertices of G) have been labeled so that $a_1 \le a_2 \le \cdots \le a_n$.

We now define a vertex coloring of G by applying the greedy coloring algorithm to the vertices of G listed in the order $v_1, v_2, \ldots, v_n$. We thus assign v_1 the color 1. If v_1 and v_2 are not adjacent (that is, if I_1 and I_2 are disjoint), then assign v_2 the color 1 as well; otherwise, assign v_2 the color 2. Suppose then that we have assigned colors to $v_1, v_2, \ldots, v_r$ for some integer r with $2 \le r < n$. We now assign v_{r+1} the smallest color that has not been assigned to any neighbor of v_{r+1} in the set $\{v_1, v_2, \ldots, v_r\}$. Thus, if v_{r+1} is adjacent to no vertex in $\{v_1, v_2, \ldots, v_r\}$, then v_{r+1} is assigned the color 1. This results in a k-coloring of G for some positive integer k and so $\chi(G) \le k$. If $k = 1$, then $G = \overline{K}_n$ and $\chi(G) = \omega(G) = 1$. Hence, we may assume that $k \ge 2$.

Suppose that the vertex v_t has been assigned the color k. Since it was not possible to assign v_t any of the colors $1, 2, \ldots, k-1$, this means that the interval $I_t = [a_t, b_t]$ must have a nonempty intersection with $k-1$ intervals $I_{j_1}, I_{j_2}, \ldots, I_{j_{k-1}}$, where say $1 \le j_1 < j_2 < \cdots < j_{k-1} < t$. Thus, $a_{j_1} \le a_{j_2} \le \ldots \le a_{j_{k-1}} \le a_t$. Since $I_{j_i} \cap I_t \ne \emptyset$ for $1 \le i \le k-1$, it follows that

$$a_t \in I_{j_1} \cap I_{j_2} \cap \cdots \cap I_{j_{k-1}} \cap I_t.$$

Thus, for $U = \{v_{j_1}, v_{j_2}, \ldots, v_{j_{k-1}}, v_t\}$,

$$G[U] = K_k$$

and so $\chi(G) \le k \le \omega(G)$. Since $\chi(G) \ge \omega(G)$, we have $\chi(G) = \omega(G)$, as desired. ■

Chordal Graphs

We now consider a more general class of graphs. Recall that a chord of a cycle C in a graph is an edge that joins two nonconsecutive vertices of C. For example, wz and xz are chords in the cycle $C = (u, v, w, x, y, z, u)$ in the graph G of Figure 15.5; while in the cycle $C' = (w, x, y, z, w)$ in G, the edge xz is a chord and wz is not. The cycle $C'' = (u, v, w, z, u)$ has no chords. Obviously, no triangle contains a chord.

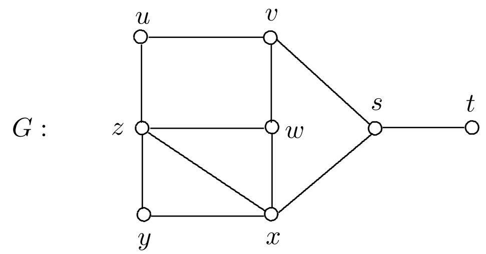


Figure 15.5: Chords in cycles

A graph G is a **chordal graph** if every cycle of length 4 or more in G has a chord. Since the cycle $C'' = (u, v, w, z, u)$ in the graph G of Figure 15.5 contains no chords, the graph G is not a chordal graph.

While every complete graph is a chordal graph, no complete bipartite graph $K_{s,t}$, where $s, t \geq 2$, is chordal, for if u_1 and v_1 belong to one partite set and u_2 and v_2 belong to the other partite set, then the cycle $(u_1, u_2, v_1, v_2, u_1)$ contains no chord. Indeed, no graph having girth 4 or more is chordal. The graphs G_1 and G_2 of Figure 15.6 are chordal graphs. For the subset $S_1 = \{u_1, v_1, x_1\}$ of $V(G_1)$ and the subset $S_2 = \{u_2, w_2, x_2\}$ of $V(G_2)$, let the graph G_3 be obtained by identifying the vertices in the complete subgraph $G_1[S_1]$ with the vertices in the complete subgraph $G_2[S_2]$, where, say, u_1 and u_2 are identified, v_1 and x_2 are identified and x_1 and w_2 are identified. The graph G_3 shown in Figure 15.6 is also a chordal graph.

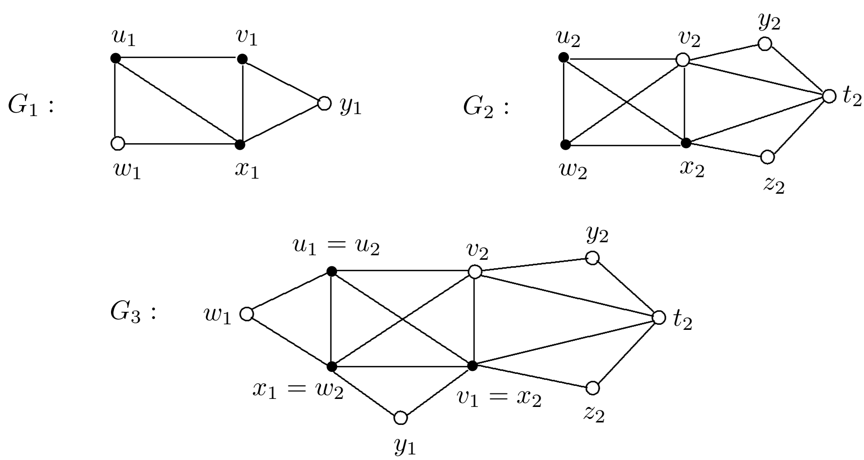


Figure 15.6: Chordal graphs

More generally, suppose that G_1 and G_2 are two graphs containing complete subgraphs H_1 and H_2, respectively, of the same order and G_3 is the graph obtained by identifying the vertices of H_1 with the vertices of H_2 (in a one-to-one manner). If G_3 contains a cycle of length 4 or more having no chord, then C must belong to G_1 or G_2. That is, if G_1 and G_2 are chordal, then G_3 is chordal. Furthermore, if G_3 is chordal, then both G_1 and G_2 are chordal.

Theorem 15.6 *Let G be a graph obtained by identifying two complete subgraphs of the same order in two graphs G_1 and G_2. Then G is chordal if and only if G_1 and G_2 are chordal.*

Proof. We have already noted that if G_1 and G_2 are two chordal graphs containing complete subgraphs H_1 and H_2, respectively, of the same order, then the graph G obtained by identifying the vertices of H_1 with the vertices of H_2 is also chordal. On the other hand, if G_1, say, were not chordal, then it would contain a cycle C of length 4 or more having no chords. However then, C would also be a cycle in G having no chords. ∎

We have now observed that every graph obtained by identifying two complete subgraphs of the same order in two chordal graphs is also chordal. These are not only sufficient conditions for a graph to be chordal, they are necessary conditions as well. The following characterization of chordal graphs is due to Andras Hajnal and János Surányi [115] and Gabriel Dirac [73].

Theorem 15.7 *A graph G is chordal if and only if G can be obtained by identifying two complete subgraphs of the same order in two chordal graphs.*

Proof. As a consequence of Theorem 15.6, we need only show that every chordal graph can be obtained from two chordal graphs by identifying two complete subgraphs of the same order in these two graphs. If G is complete, say $G = K_n$, then G is chordal and can trivially be obtained by identifying the vertices of $G_1 = K_n$ and the vertices of $G_2 = K_n$ in any one-to-one manner. Hence, we may assume that G is a connected chordal graph that is not complete.

Let S be a minimum vertex-cut of G. Now, let V_1 be the vertex set of one component of $G - S$ and let $V_2 = V(G) - (V_1 \cup S)$. Consider the two subgraphs

$$G_1 = G[V_1 \cup S] \text{ and } G_2 = G[V_2 \cup S]$$

of G. Consequently, G is obtained by identifying the vertices of S in G_1 and in G_2. We now show that $G[S]$ is complete. Since this is certainly true if $|S| = 1$, we may assume that $|S| \geq 2$.

Each vertex v in S is adjacent to at least one vertex in each component of $G - S$, for otherwise $S - \{v\}$ is a vertex-cut of G, which is impossible. Let $u, w \in S$. Hence, there are $u - w$ paths in G_1, where every vertex except u and w belongs to V_1. Among all such paths, let $P = (u, x_1, x_2, \ldots, x_s, w)$ be one of minimum length. Similarly, let $P' = (u, y_1, y_2, \ldots, y_t, w)$ be a $u - w$ path of minimum length where every vertex except u and w belongs to V_2. Hence,

$$C = (u, x_1, x_2, \ldots, x_s, w, y_t, y_{t-1}, \ldots, y_1, u)$$

is a cycle of length 4 or more in G. Since G is chordal, C contains a chord. No vertex x_i $(1 \leq i \leq s)$ can be adjacent to a vertex y_j $(1 \leq j \leq t)$ since S is a vertex-cut of G. Furthermore, no nonconsecutive vertices of P or of P' can be adjacent due to the manner in which P and P' are defined. Thus, $uw \in E(G)$, implying that $G[S]$ is complete. By Theorem 15.6, G_1 and G_2 are chordal. ■

With the aid of Theorem 15.7, we now have an even larger class of perfect graphs.

Corollary 15.8 *Every chordal graph is perfect.*

Proof. Since every induced subgraph of a chordal graph is also a chordal graph, it suffices to show that if G is a connected chordal graph, then $\chi(G) = \omega(G)$. We proceed by induction on the order n of G. If $n = 1$, then $G = K_1$ and $\chi(G) = \omega(G) = 1$. Assume, therefore, that $\chi(H) = \omega(H)$ for every chordal graph H of order less than n, where $n \geq 2$, and let G be a chordal graph of order $n \geq 2$.

If G is a complete graph, then $\chi(G) = \omega(G) = n$. Hence, we may assume that G is not complete. By Theorem 15.7, G can be obtained from two chordal graphs G_1 and G_2 by identifying two complete subgraphs of the same order in G_1 and G_2. Observe that

$$\chi(G) \leq \max\{\chi(G_1), \chi(G_2)\} = k.$$

By the induction hypothesis, $\chi(G_1) = \omega(G_1)$ and $\chi(G_2) = \omega(G_2)$. Thus,

$$\chi(G) \leq \max\{\omega(G_1), \omega(G_2)\} = k.$$

On the other hand, let S denote the set of vertices in G that belong to G_1 and G_2. Thus, $G[S]$ is complete and no vertex in $V(G_1) - S$ is adjacent to a vertex in $V(G_2) - S$. Hence,

$$\omega(G) = \max\{\omega(G_1), \omega(G_2)\} = k.$$

Thus, $\chi(G) \geq k$. Therefore, $\chi(G) = k = \omega(G)$. ■

The graph G of Figure 15.7 is not chordal but yet $\chi(H) = \omega(H)$ for every induced subgraph H of G. Hence, the converse of Corollary 15.8 is not true.

$G:$

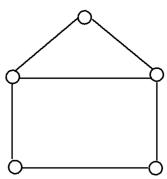


Figure 15.7: A non-chordal graph

15.2 The Perfect and Strong Perfect Graph Theorems

We've seen that if G is a graph that is either complete or bipartite, then both G and $\overline{G}$ are perfect. Indeed, in 1961 Claude Berge made the following conjecture:

The Perfect Graph Conjecture *A graph is perfect if and only if its complement is perfect.*

The Perfect Graph Theorem

In 1970, at the age of 22, László Lovász received his Ph.D. from the Hungarian Academy of Sciences. His advisor was Tibor Gallai. Only two years later, Lovász [159] verified the Perfect Graph Conjecture. His proof relied on the following lemma involving replication graphs.

Let G be a graph and let v be a vertex of G. Then the **replication graph** $R_v(G)$ of G (with respect to v) is that graph obtained from G by adding a new vertex v' to G and joining v' to the vertices in the closed neighborhood $N[v]$ of v. Figure 15.8 shows a graph G (the wheel $G = W_4$) and the replication graph $R_v(G)$ of G. We say that $R_v(G)$ is obtained from G by a **vertex replication**.

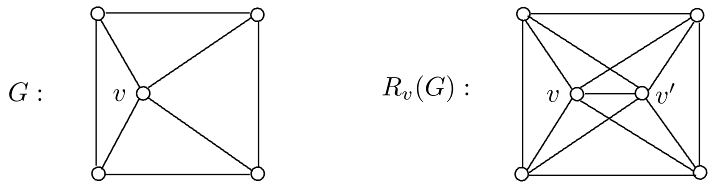


Figure 15.8: The replication graph $R_v(G)$ with respect to a vertex v of a graph G

The following result was obtained by Lovász [160] in 1972.

Theorem 15.9 (**The Replication Lemma**) *If G is a perfect graph, then $R_v(G)$ is perfect for every vertex v of G.*

Proof. Let $v \in V(G)$ and $G' = R_v(G)$. First, we show that $\chi(G') = \omega(G')$. We consider two cases, depending on whether v belongs to a maximum clique of G.

Case 1. *v belongs to a maximum clique of G.* Then $\omega(G') = \omega(G)+1$. Since

$$\chi(G') \le \chi(G) + 1 = \omega(G) + 1 = \omega(G'),$$

it follows that $\chi(G') = \omega(G')$.

Case 2. *v does not belong to any maximum clique of G.* Suppose that $\chi(G) = \omega(G) = k$. Let there be given a k-coloring of G using the colors $1, 2, \ldots, k$. We may assume that v is assigned the color 1. Let V_1 be the color class consisting of the vertices of G that are colored 1. Thus, $v \in V_1$. Since $\omega(G) = k$, every maximum clique of G must contain a vertex of each color. Since v does not belong to a maximum clique, it follows that $|V_1| \geq 2$. Let $U_1 = V_1 - \{v\}$. Because every maximum clique of G contains a vertex of U_1, it follows that $\omega(G - U_1) = \omega(G) - 1 = k - 1$. Since G is perfect, $\chi(G - U_1) = k - 1$. Let a $(k-1)$-coloring of $G - U_1$ be given, using the colors $1, 2, \ldots, k-1$. Since V_1 is an independent set of vertices, so is $U_1 \cup \{v'\}$. Assigning the color k to the vertices of $U_1 \cup \{v'\}$ produces a k-coloring of G'. Therefore,

$$k = \omega(G) \leq \omega(G') \leq \chi(G') \leq k$$

and so $\chi(G') = \omega(G')$.

It remains to show that $\chi(H) = \omega(H)$ for every induced subgraph H of G'. This is certainly the case if H is a subgraph of G. If H contains v' but not v, then $H \cong G[(V(H) - \{v'\}) \cup \{v\}]$ and so $\chi(H) = \omega(H)$. If H contains both v and v' but $H \ncong G'$, then H is the replication graph of $G[V(H) - \{v'\}]$ and the argument used to show that $\chi(G') = \omega(G')$ can be applied to show that $\chi(H) = \omega(H)$. ■

As we observed following the proof of Theorem 14.24, the independence number of a graph equals the clique number of its complement.

Theorem 15.10 (**The Perfect Graph Theorem**) *A graph is perfect if and only if its complement is perfect.*

Proof. Since $\overline{\overline{G}} = G$ for every graph G, it suffices to show that if G is a perfect graph, then $\overline{G}$ is perfect. We proceed by induction on the order n of a perfect graph G. For $n = 1$, $G = K_1$, which is perfect. Since $G = \overline{G}$, it follows that $\overline{G}$ is perfect and so the basis step of the induction is true. Assume, for an integer $n \geq 2$, that the complement of every perfect graph of order less than n is perfect. Let G be a perfect graph of order n. We show that $\overline{G}$ is perfect, that is, $\chi(F) = \omega(F)$ for every induced subgraph F of $\overline{G}$.

For each proper induced subgraph F of $\overline{G}$, there is a proper induced subgraph H of G such that $\overline{H} = F$. Since G is perfect, H is perfect. It follows by the induction hypothesis that F is perfect and so $\chi(F) = \omega(F)$. Hence, it remains to show that $\chi(\overline{G}) = \omega(\overline{G})$. Since $\chi(\overline{G}) \geq \omega(\overline{G})$, we need only show that

$$\chi(\overline{G}) \leq \omega(\overline{G}). \tag{15.1}$$

Let $\mathcal{S}$ be the set consisting of all vertex sets of cliques of G. Consequently, if $U \in \mathcal{S}$, then U is an independent set of vertices in $\overline{G}$. Let $\mathcal{T}$ be the set of all maximum independent sets in G. Hence, if $W \in \mathcal{T}$, then $|W| = \alpha(G)$. We claim that there exists some set $U \in \mathcal{S}$ having the property that

$$U \cap W \neq \emptyset \text{ for every } W \in \mathcal{T}. \tag{15.2}$$

If there is a set $U \in \mathcal{S}$ satisfying property (15.2), then

$$\omega(\overline{G} - U) = \alpha(G - U) = \alpha(G) - 1 = \omega(\overline{G}) - 1.$$

It then follows by the induction hypothesis that

$$\chi(\overline{G}) \leq \chi(\overline{G} - U) + 1 = \omega(\overline{G} - U) + 1 = \omega(\overline{G}). \tag{15.3}$$

We now show that there is a set $U \in \mathcal{S}$ satisfying property (15.2). Assume, to the contrary, that no set $U \in \mathcal{S}$ satisfies (15.2). Then, for every set $U \in \mathcal{S}$, there is some set $W_U \in \mathcal{T}$ such that $U \cap W_U = \emptyset$.

For each $x \in V(G)$, let $n_x = |\{U \in \mathcal{S} :\ x \in W_U\}|$. Next, we construct a new graph G' from G. First, every vertex x of G for which $n_x = 0$ is removed from G. If $n_x > 0$, then x is replaced in G by a complete graph G_x of order n_x. If x and y are two adjacent vertices of G, then every vertex of G_x is joined to every vertex of G_y. The graph G' is then defined by

$$V(G') = \bigcup_{x \in V(G)} V(G_x),$$

where uv is an edge of G' if and only if $u \in V(G_x)$ and $v \in V(G_y)$ such that either $x = y$ or $xy \in E(G)$.

Let $H = G[\{x \in V(G) :\ n_x > 0\}]$. Since H is an induced subgraph of the perfect graph G, it follows that H is perfect. Since the graph G' can be obtained from H by a sequence of vertex replications, it follows by Theorem 15.9 that G' is perfect and so

$$\chi(G') = \omega(G'). \tag{15.4}$$

First, we consider $\omega(G')$. By the construction of G', every maximal complete subgraph of G' has the form $G'[\cup_{x \in X} V(G_x)]$ for some $X \in \mathcal{S}$. So there exists a set $Y \in \mathcal{S}$ such that

$$\begin{aligned}\omega(G') &= \sum_{y \in Y} n_y = \sum_{y \in Y} |\{U \in \mathcal{S} : y \in W_U\}| \\ &= \sum_{U \in \mathcal{S}} |Y \cap W_U|.\end{aligned}$$

Since $G[Y]$ is complete in G and W_U is an independent set in G, it follows that $|Y \cap W_U| \leq 1$ for all $U \in \mathcal{S}$. Furthermore, it follows by our assumption that $|Y \cap W_Y| = 0$. Therefore,

$$\omega(G') \leq |\mathcal{S}| - 1. \tag{15.5}$$

Next, we consider $\chi(G')$. Observe that

$$\begin{aligned}|V(G')| &= \sum_{x \in V(G)} n_x = \sum_{x \in V(G)} |\{U \in \mathcal{S} : x \in W_U\}| \\ &= \sum_{U \in \mathcal{S}} |W_U| = |\mathcal{S}|\alpha(G).\end{aligned}$$

Since $\alpha(G') \leq \alpha(G)$ by the construction of G', it follows that

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{|V(G')|}{\alpha(G)} = |\mathcal{S}|. \tag{15.6}$$

Consequently, by (15.5) and (15.6), we have

$$\chi(G') \geq |\mathcal{S}| > |\mathcal{S}| - 1 = \omega(G'),$$

which contradicts (15.4). ■

Lovász [158] obtained another characterization of perfect graphs (see Exercise 26).

Theorem 15.11 *A graph G is perfect if and only if $|V(H)| \leq \alpha(H) \cdot \omega(H)$ for every induced subgraph H of G.*

The Strong Perfect Graph Theorem

As we have mentioned, if G is an odd cycle of length 5 or more or is the complement of such an odd cycle, then $\chi(G) \neq \omega(G)$. In 1961 Claude Berge conjectured that this fact is critical in determining whether a graph is perfect.

The Strong Perfect Graph Conjecture *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length 5 or more.*

After an intensive 28-month assault on this conjecture, its truth was established in 2002 by Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas [53].

Theorem 15.12 (**The Strong Perfect Graph Theorem**) *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length 5 or more.*

15.3 List Colorings

In recent decades there has been increased interest in colorings of graphs in which the color of each vertex is to be chosen from a specified list of allowable colors. Let G be a graph for which there is an associated set $L(v)$ of permissible colors for each vertex v of G. The set $L(v)$ is commonly called a **color list** for v. A **list coloring** of G is then a proper coloring c of G such that $c(v) \in L(v)$ for each vertex v of G. A list coloring is also referred to as a **choice function**. If

$$\mathfrak{L} = \{L(v) : \ v \in V(G)\}$$

is a collection of (not necessarily distinct) color lists for the vertices of G and there exists a list coloring for this collection $\mathfrak{L}$ of color lists, then this list coloring is called an **$\mathfrak{L}$-list-coloring** of G, and G is said to be **$\mathfrak{L}$-choosable** or **$\mathfrak{L}$-list-colorable**. A graph G is **k-choosable** or **k-list-colorable** if G is $\mathfrak{L}$-choosable for every collection $\mathfrak{L}$ of lists $L(v)$ for the vertices v of G such that $|L(v)| \geq k$ for each vertex v. The **list chromatic number** $\chi_\ell(G)$ of G is the minimum positive integer k such that G is k-choosable. Then $\chi_\ell(G) \geq \chi(G)$. The concept of list colorings was introduced by Vadim Vizing [248] in 1976 and, independently, by Paul Erdős, Arthur L. Rubin and Herbert Taylor [82] in 1979.

Suppose that G is a graph with $\Delta(G) = \Delta$. By Theorem 14.17 if we let

$$L(v) = \{1, 2, \ldots, \Delta, 1 + \Delta\}$$

for each vertex v of G, then for these color lists there is a list coloring of G. Indeed, if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\mathfrak{L} = \{L(v_i) : 1 \leq i \leq n\}$ is a collection of color lists for G where each set $L(v_i)$ consists of any $1+\Delta$ colors, then a greedy coloring of G produces a proper coloring and so G is $\mathfrak{L}$-choosable. Therefore, $\chi_\ell(G) \leq 1+\Delta(G)$. (See Exercise 27.) Summarizing these observations, we have the following:

Theorem 15.13 *For every graph G,*

$$\chi(G) \leq \chi_\ell(G) \leq 1 + \Delta(G).$$

We now consider some examples. First, $\chi(C_4) = 2$ and so $\chi_\ell(C_4) \geq 2$. Consider the cycle C_4 of Figure 15.9 and suppose that we are given any four color lists $L(v_i)$, $1 \leq i \leq 4$, with $|L(v_i)| = 2$. Let $L(v_1) = \{a, b\}$. We consider three cases.

Case 1. $a \in L(v_2) \cap L(v_4)$. In this case, assign v_1 the color b and v_2 and v_4 the color a. Then there is at least one color in $L(v_3)$ that is not a. Assigning v_3 that color gives C_4 a list coloring for this collection of lists.

Case 2. *The color a belongs to exactly one of $L(v_2)$ and $L(v_4)$, say $a \in L(v_2) - L(v_4)$.* If there is some color $x \in L(v_2) \cap L(v_4)$, then assign v_2 and v_4 the color x and v_1 the color a. There is at least one color in $L(v_3)$ different from x. Assign v_3 that color. Hence, there is a list coloring of C_4 for this collection of lists. Next, suppose that there is no color belonging to both $L(v_2)$ and $L(v_4)$. If $a \in L(v_3)$, then assign a to both v_1 and v_3. There is a color available for both v_2 and v_4. If $a \notin L(v_3)$, then assign v_1 the color a, assign v_2 the color y in $L(v_2)$ different from a, assign v_3 any color z in $L(v_3)$ different from y, and assign v_4 any color in $L(v_4)$ different from z. This is a list coloring for C_4.

Case 3. $a \notin L(v_2) \cup L(v_4)$. Then assign v_1 the color a and v_3 any color from $L(v_3)$. Hence, there is an available color from $L(v_2)$ and $L(v_4)$ to assign to v_2 and v_4, respectively. Therefore, there is a list coloring of C_4 for this collection of lists.

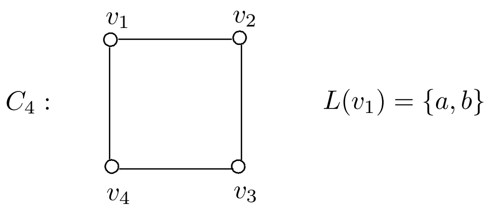


Figure 15.9: The graph C_4 is 2-list-colorable

Actually, $\chi_\ell(C_n) = 2$ for every even integer $n \geq 4$. Before showing this, however, it is useful to show that $\chi_\ell(T) = 2$ for every nontrivial tree T.

Theorem 15.14 *Every tree is 2-choosable. Furthermore, for every tree T, for a vertex u of T and for a collection $\mathfrak{L} = \{L(v) :\ v \in V(T)\}$ of color lists of size 2, where $a \in L(u)$, there exists an $\mathfrak{L}$-list-coloring of T in which u is assigned color a.*

Proof. We proceed by induction on the order of the tree. The result is obvious for a tree of order 1 or 2. Assume that the statement is true for all trees of order k, where $k \geq 2$. Let T be a tree of order $k+1$ and let

$$\mathfrak{L} = \{L(v) :\ v \in V(T)\}$$

be a collection of color lists of size 2. Let $u \in V(T)$ and suppose that $a \in L(u)$. Let x be an end-vertex of T such that $x \neq u$ and let

$$\mathfrak{L}' = \{L(v) :\ v \in V(T - x)\}.$$

Let y be the neighbor of x in T. By the induction hypothesis, there exists an $\mathfrak{L}'$-list-coloring c' of $T - x$ in which u is colored a. Now let $b \in L(x)$ such that $b \neq c'(y)$. Then the coloring c defined by

$$c(v) = \begin{cases} b & \text{if } v = x \\ c'(v) & \text{if } v \neq x \end{cases}$$

is an $\mathfrak{L}$-list-coloring of T in which u is colored a. ■

Corollary 15.15 *For every nontrivial tree T, $\chi_\ell(T) = 2$.*

Theorem 15.16 *Every even cycle is 2-choosable.*

Proof. We already know that C_4 is 2-choosable. Let C_n be an n-cycle, where $n \geq 6$ is even. Suppose that $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Let there be given a collection $\mathfrak{L} = \{L(v_i) :\ 1 \leq i \leq n\}$ of color lists of size 2 for the vertices of C_n. We show that C_n is $\mathfrak{L}$-list-colorable. We consider two cases.

Case 1. *All of the color lists are the same, say $L(v_i) = \{1, 2\}$ for $1 \leq i \leq n$.* If we assign the color 1 to v_i for odd i and the color 2 to v_i for even i, then C_n is $\mathfrak{L}$-list-colorable.

Case 2. *The color lists in* $\mathfrak{L}$ *are not all the same.* Then there are adjacent vertices v_i and v_{i+1} in G such that $L(v_i) \neq L(v_{i+1})$. Thus there exists a color $a \in L(v_{i+1}) - L(v_i)$. The graph $C_n - v_i$ is a path of order $n-1$. Let $\mathfrak{L}' = \{L(v) : v \in V(C_n - v_i)\}$. By Theorem 15.14, there exists an $\mathfrak{L}'$-list-coloring c' of $C_n - v_i$ in which $c'(v_{i+1}) = a$. Let $b \in L(v_i)$ such that $b \neq c'(v_{i-1})$. Then the coloring c defined by

$$c(v) = \begin{cases} b & \text{if } v = v_i \\ c'(v) & \text{if } v \neq v_i \end{cases}$$

is an $\mathfrak{L}$-list-coloring of G. ■

Corollary 15.17 *For every even integer* $n \geq 4$, $\chi_\ell(C_n) = 2$.

Since the chromatic number of every odd cycle is 3, the list chromatic number of every odd cycle is at least 3. Indeed, every odd cycle is 3-choosable (see Exercise 28).

We have seen that all trees and even cycles are 2-choosable. Of course, these are both classes of bipartite graphs. Not every bipartite graph is 2-choosable, however. To illustrate this, we consider $\chi_\ell(K_{3,3})$, where $K_{3,3}$ is shown in Figures 15.10(a). By Theorem 15.13, $\chi_\ell(K_{3,3}) \leq 1 + \Delta(K_{3,3}) = 4$. In fact, $\chi_\ell(K_{3,3}) \leq 3$, as we next show. Let there be given lists $L(v_i)$, $1 \leq i \leq 6$, where $|L(v_i)| = 3$. We consider two cases.

Case 1. *Some color occurs in two or more of the lists* $L(v_1)$, $L(v_2)$, $L(v_3)$ *or in two or more of the lists* $L(v_4)$, $L(v_5)$, $L(v_6)$, *say color* a *occurs in* $L(v_1)$ *and* $L(v_2)$. Then assign v_1 and v_2 the color a and assign v_3 any color in $L(v_3)$. Then there is an available color in $L(v_i)$ for v_i $(i = 4, 5, 6)$.

Case 2. *The lists* $L(v_1)$, $L(v_2)$, $L(v_3)$ *are pairwise disjoint as are the lists* $L(v_4)$, $L(v_5)$, $L(v_6)$. Let $a_1 \in L(v_1)$ and $a_2 \in L(v_2)$. If none of the lists $L(v_4)$, $L(v_5)$, $L(v_6)$ contain both a_1 and a_2, then let a_3 be any color in $L(v_3)$. Then there is an available color for each of v_4, v_5, v_6 to construct a proper coloring of $K_{3,3}$. If exactly one of the lists $L(v_4)$, $L(v_5)$, $L(v_6)$ contains both a_1 and a_2, then select a color $a_3 \in L(v_3)$ so that none of $L(v_4)$, $L(v_5)$, $L(v_6)$ contains all of a_1, a_2, a_3. By assigning v_3 the color a_3, we see that there is an available color for each of v_4, v_5, and v_6.

Hence, as claimed, $\chi_\ell(K_{3,3}) \leq 3$. We show, in fact, that $\chi_\ell(K_{3,3}) = 3$. Consider the sets $L(v_i)$, $1 \leq i \leq 6$, shown in Figures 15.10(b). Assume, without loss of generality, that v_1 is colored 1. Then v_4 must be colored 2 and v_5 must be colored 3. Whichever color is chosen for v_3 is the same color as that of either v_4 or v_5. This produces a contradiction. Hence, $K_{3,3}$ is not 2-choosable and so $\chi_\ell(K_{3,3}) = 3$. The graph $G = K_{3,3}$ shows that it is possible for $\chi_\ell(G) > \chi(G)$. In fact, $\chi_\ell(G)$ can be considerably larger than $\chi(G)$.

Theorem 15.18 *If* r *and* k *are positive integers such that* $r \geq \binom{2k-1}{k}$, *then*

$$\chi_\ell(K_{r,r}) \geq k + 1.$$

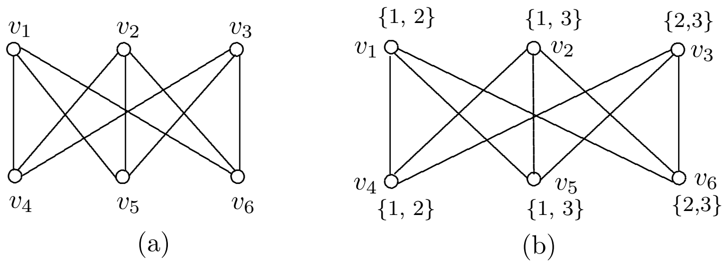


Figure 15.10: The graph $K_{3,3}$ is 3-choosable

Proof. Assume, to the contrary, that $\chi_\ell(K_{r,r}) \le k$. Then $K_{r,r}$ is k-choosable. Let U and W be the partite sets of $K_{r,r}$, where

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } W = \{w_1, w_2, \ldots, w_r\}.$$

Let $S = \{1, 2, \ldots, 2k-1\}$. There are $\binom{2k-1}{k}$ distinct k-element subsets of S. Assign these color lists to $\binom{2k-1}{k}$ vertices of U and to $\binom{2k-1}{k}$ vertices of W. Any remaining vertices of U and W are assigned any of the k-element subsets of S. For $i = 1, 2, \ldots, r$, choose a color $a_i \in L(u_i)$ and let $T = \{a_i : 1 \le i \le r\}$. We consider two cases.

Case 1. $|T| \le k-1$. Then there exists a k-element subset S' of S that is disjoint from T. However, $L(u_j) = S'$ for some j with $1 \le j \le r$. This is a contradiction.

Case 2. $|T| \ge k$. Hence, there exists a k-element subset T' of T. Thus, $L(w_j) = T'$ for some j with $1 \le j \le r$. Whichever color from $L(w_j)$ is assigned for w_j, this color has been assigned to some vertex u_i. Thus, u_i and w_j have been assigned the same color and u_iw_j is an edge of $K_{r,r}$. This is a contradiction. ■

For positive integers r and k such that $r \ge \binom{2k-1}{k}$, Theorem 15.18 provides a lower bound on $\chi_\ell(K_{r,r})$ for r sufficiently large with respect to k. The following 1992 result of Noga Alon [3] gives an upper bound for $\chi_\ell(K_{r,r})$ for all $r \ge 3$.

Theorem 15.19 *For every integer $r \ge 3$,*

$$\chi_\ell(K_{r,r}) \le \lceil 2 \log_2 r \rceil .$$

Theorem 15.19 will be discussed again in Chapter 21 where a proof will be given.

Exercises for Chapter 15

Section 15.1. Perfect Graphs

1. Determine G if, in the proof of Theorem 15.1,

   (a) $H = K_2$; (b) $H = C_5$.

2. Show, for every two integers ℓ and k with $2 \leq \ell \leq k$, that there exists a graph G with $\omega(G) = \ell$ and $\chi(G) = k$.

3. Prove or disprove: For every integer $k \geq 3$, there exists a triangle-free, k-critical graph.

4. Give an example of a regular triangle-free 4-chromatic graph.

5. Show that the graph of the octahedron (which is the graph $K_{2,2,2}$) is perfect.

6. Let $n \geq 2$ be an integer. Prove that every $(n-1)$-chromatic graph of order n has clique number $n-1$.

7. Prove or disprove the following:

   (a) If G is a graph of order $n \geq 3$ with $\chi(G) = n-2$, then $\omega(G) = n-2$.

   (b) If G is a graph of sufficiently large order n with $\chi(G) = n-2$, then $\omega(G) = n-2$.

8. Prove that a critically k-chromatic graph G is perfect if and only if $G = K_k$.

9. Let A be a set and let S be a collection of nonempty subsets of A. The **intersection graph** of S is that graph whose vertices are the elements of S and where two vertices are adjacent if the subsets have a nonempty intersection.

   (a) Prove that every graph is an intersection graph.

   (b) Let G be a nonempty graph. Show that a set $\mathcal{F}$ can be associated with G so that the intersection graph of $\mathcal{F}$ is the line graph of G.

10. For the graph $G = K_4 - e$, determine the smallest positive integer k for which there is a collection S of subsets of $A = \{1, 2, \ldots, k\}$ such that the intersection graph (see the definition in Exercise 9) of S is isomorphic to G.

11. Show that every induced subgraph of an interval graph is an interval graph.

12. Determine which connected graphs of order 4 are interval graphs.

13. (a) For the four open intervals $I_1 = (0,1)$, $I_2 = (1,2)$, $I_3 = (2,3)$, $I_4 = (3,4)$ of length 1 and the five open intervals

$$J_1 = \left(0, \tfrac{4}{5}\right),\ J_2 = \left(\tfrac{4}{5}, 1\tfrac{3}{5}\right),\ J_3 = \left(1\tfrac{3}{5}, 2\tfrac{2}{5}\right),$$
$$J_4 = \left(2\tfrac{2}{5}, 3\tfrac{1}{5}\right),\ J_5 = \left(3\tfrac{1}{5}, 4\right)$$

of length $\frac{4}{5}$, let $S = \{I_i :\ 1 \le i \le 4\} \cup \{J_j :\ 1 \le j \le 5\}$. Draw the interval graph G with $V(G) = S$ and determine the size of G.

(b) Repeat the problem in part (a) for the open intervals $I_1 = (0,1)$, $I_2 = (1,2)$, $I_3 = (2,3)$ and the open intervals $J_j = \left(\frac{3}{5}(j-1), \frac{3}{5}j\right)$ for $j = 1, 2, \ldots, 5$.

(c) Repeat the problem in part (a) for the open intervals $I_1 = (0,1)$, $I_2 = (1,2)$, $I_3 = (2,3)$, $I_4 = (3,4)$ and the open intervals $J_j = \left(\frac{4}{6}(j-1), \frac{4}{6}j\right)$ for $j = 1, 2, \ldots, 6$.

14. Let a and b be integers with $2 \le a < b$. For $i = 1, 2, \ldots, a$, let $I_i = (i-1, i)$ and for $j = 1, 2, \ldots, b$, let $J_j = \left(\frac{a}{b}(j-1), \frac{a}{b}j\right)$. Let G be the interval graph with $V(G) = S$.

(a) Show that G is acyclic.

(b) Determine the size of G.

15. (a) Determine the interval graph G with vertex set $S = \{I_1, I_2, \ldots, I_5\}$ consisting of the five closed intervals $I_1 = [1,7], I_2 = [6,8], I_3 = [6,9], I_4 = [7,20]$ and $I_5 = [20,22]$.

(b) Show that the same graph G can be obtained from a set $T = \{J_1, J_2, \ldots, J_5)$ of five open intervals where $I_i \subseteq J_i$ for $i = 1, 2, \ldots, 5$.

16. Show, for every integer $k \ge 3$, that there exists a k-chromatic graph that is not chordal.

17. Prove that every interval graph is a chordal graph.

18. Give an example of a chordal graph that is not an interval graph.

Section 15.2. The Perfect and Strong Perfect Graph Theorems

19. For each integer $n \ge 7$, give an example of a graph G_n of order n such that no induced subgraph of G_n is an odd cycle of length at least 5 but G_n is not perfect.

20. Let $G = C_{2k+1}$ for some integer $k \ge 2$. Show that $\overline{G}$ is not perfect by determining $\chi(\overline{G})$ and $\omega(\overline{G})$.

21. Determine whether $\overline{C}_8$ is perfect.

22. Let $\mathcal{G}$ be the set of all connected graphs of order 25, say $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, and let $G = G_1 + G_2 + \cdots + G_k$ be the union of these graphs. Prove or disprove the following:

(a) For the graph G, $\chi(G) = \omega(G)$.

(b) The graph G is perfect.

23. Let G be a graph where $v \in V(G)$. According to Theorem 15.9, if G is perfect, then the replication graph $R_v(G)$ is perfect. Is the converse true?

24. A **transitive orientation** of a graph G is an orientation D of G such that whenever (u, v) and (v, w) are arcs of D, then so too is (u, w). If G has a transitive orientation, then G is called a **comparability graph**.

    (a) Show that every bipartite graph is a comparability graph.

    (b) Show that every complete graph is a comparability graph.

    (c) Show that every comparability graph is perfect.

    (d) Use (a)–(c) to show that every bipartite graph and every complete graph is perfect.

    (e) Give an example of a comparability graph that is neither complete nor bipartite.

25. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **shadow graph** $Shad(G)$ of G is that graph with vertex set $V(G) \cup \{u_1, u_2, \ldots, u_n\}$, where u_i is called the **shadow vertex** of v_i and where u_i is adjacent to both u_j and v_j if v_i is adjacent to v_j for $1 \leq i, j \leq n$. How are $\omega(G)$ and $\omega(Shad(G))$ related and how are $\chi(G)$ and $\chi(Shad(G))$ related?

26. (a) Show that if G is perfect, then $|V(H)| \leq \alpha(H) \cdot \omega(H)$ for every induced subgraph H of G (see Theorem 15.11).

    (b) Use Theorem 15.11 to prove the Perfect Graph Theorem (Theorem 15.10).

Section 15.3. List Colorings

27. Use the greedy coloring algorithm to show that $\chi_\ell(G) \leq 1 + \Delta(G)$ for every graph G.

28. Prove, without using Exercise 27, that every odd cycle is 3-choosable.

29. Prove that $\chi_\ell(K_{2,3}) = 2$.

30. Prove that $\chi_\ell(K_{2,4}) = 3$.

31. Prove that $\chi_\ell(K_{3,27}) > 3$.

32. We have seen that $\chi_\ell(K_{3,3}) = 3$. By Theorem 15.18, $\chi_\ell(K_{10,10}) > 3$. Give a precise and detailed proof of this special case of Theorem 15.18.

33. Prove that the list-chromatic number of $P_n \,\square\, K_2$ is 3 for every integer $n \geq 4$.

34. Suppose for $G = K_{3,3}$ that a set $\mathfrak{L} = \{L(v) : v \in V(G)\}$ of color lists is given for the vertices v in G, where $|L(u)| = 3$ for one vertex u in G and $|L(w)| = 2$ for all other vertices w in G. Does there exist a list coloring of G for these lists?

Chapter 16

Map Colorings

For well over a century, the most famous unsolved problem in graph theory, and one of the most famous unsolved problems in mathematics, was a map coloring problem called the Four Color Problem. This problem, dealing with determining whether a conjecture called the Four Color Conjecture was true or false, had a major impact on the development of graph theory. This problem will be discussed in the current chapter as will related coloring problems dealing with graphs that are embeddable on certain surfaces.

16.1 The Four Color Problem

Vertex colorings (discussed in Chapters 14 and 15), edge colorings (to be discussed in Chapters 17 and 18) and map colorings (to be discussed in this chapter) can all be traced to a single problem.

The Four Color Problem *Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary are colored differently?*

It is not only the subject of graph colorings that owes its origin to the Four Color Problem. Much of the early development of graph theory can be traced to the study of this problem.

The Guthrie Brothers

The origin of the Four Color Problem is rather well documented. On October 23, 1852, Frederick Guthrie (1833–1886), a student at University College London, visited his mathematics professor, the famous Augustus De Morgan (1806–1871), to describe an apparent mathematical discovery of his older brother Francis. While coloring the counties of a map of England, Francis observed that he could color them with four colors, which led him to conjecture that no more than four colors would be needed to color the regions of any map.

The Four Color Conjecture *The regions of every map can be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently.*

Francis Guthrie (1831–1899) attempted to prove this Four Color Conjecture and although he thought he may have been successful, he was not completely satisfied with his proof. Francis discussed his discovery with Frederick. With Francis' approval, Frederick mentioned this apparent theorem to Professor De Morgan, who expressed pleasure with it and believed it to be a new result. Evidently, Frederick asked Professor De Morgan if he was aware of an argument that would establish the truth of the theorem. De Morgan became very interested in this problem himself.

The first statement in print of the Four Color Problem evidently occurred in an anonymous review written in the April 14, 1860 issue of the literary journal *Athenaeum*. Although the author of the review was not identified, De Morgan was quite clearly the writer. This review led to the Four Color Problem becoming known in the United States.

De Morgan had made little progress towards a solution of the Four Color Problem at the time of his death in 1871 and overall interest in this problem had faded. During a meeting of the London Mathematical Society on June 13, 1878, the famous mathematician Arthur Cayley (1821–1895) raised a question about the Four Color Problem that brought renewed attention to the problem:

> *Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?*

Cayley observed that if a map with a certain number of regions has been colored with four colors and a new map is obtained by adding a new region, then there is no guarantee that the new map can be colored with four colors – without first recoloring the original map. This showed that any attempted proof of the Four Color Conjecture by mathematical induction would not be straightforward.

Alfred Bray Kempe

Among those who studied under Arthur Cayley was Alfred Bray Kempe (1849–1922), who also attempted to solve the Four Color Problem. Kempe's approach for solving this problem involved locating a region R in a map M such that R is surrounded by five or fewer neighboring regions. The existence of such a region R is a consequence of Corollary 10.5. Once R was found, Kempe then showed that for every coloring of M (minus the region R) with four colors, there is a coloring of the entire map M with four colors. Such an argument would show that M could not be a minimum counterexample (that is, a map with a minimum number of regions that cannot be colored with four colors).

For example, no minimum counterexample M could possibly contain a region R surrounded by three regions R_1, R_2 and R_3 as shown in Figure 16.1(a). In this case, we could shrink the region R to a point, producing a new map M' with one less region. The map M' can then be colored with four colors, only three of which are used to color R_1, R_2 and R_3 as in Figure 16.1(b). Returning to the original map M, we see that there is now an available color for R as shown in Figure 16.1(c), implying that M could be colored with four colors after all, thereby producing a contradiction. Certainly, if the map M contains a region surrounded by fewer than three regions, a contradiction can be obtained in a similar manner.

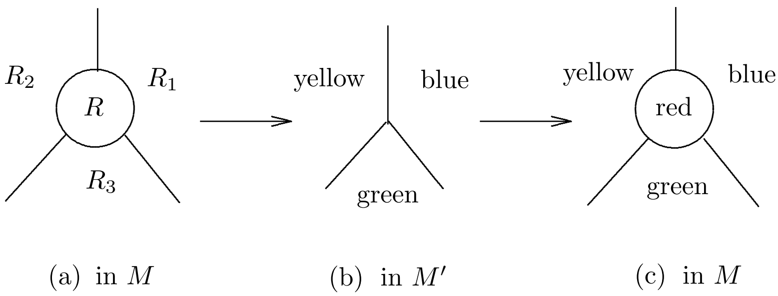


Figure 16.1: A region surrounded by three regions in a map

Kempe Chains

Suppose, however, that the map M contained no region surrounded by three or fewer regions but did contain a region R surrounded by four regions, say R_1, R_2, R_3 and R_4, as shown in Figure 16.2(a). If, once again, we shrink the region R to a point, producing a map M' with one less region, then we know that M' can be colored with four colors. If two or three colors are used to color R_1, R_2, R_3 and R_4, then we can return to M and there is a color available for R. However, this technique does not work if the regions R_1, R_2, R_3 and R_4 are colored with four distinct colors, as shown in Figure 16.2(b).

What we can do in this case, however, is to determine whether the map M' has a chain of regions, beginning at R_1 and ending at R_3, all of which are colored red or green. If no such chain exists, then the two colors of every red-green chain of regions beginning at R_1 can be interchanged. We can then return to the map M, where the color red is now available for R. That is, the map M can be colored with four colors, producing a contradiction. But what

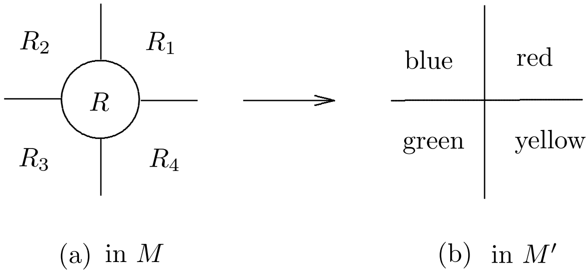


Figure 16.2: A region surrounded by four regions in a map

if the map contains a red-green chain of regions beginning at R_1 and ending at R_3? (See Figure 16.3, where r, b, g, y denote the colors red, blue, green, yellow.) Then interchanging the colors red and green offers no benefit to us. However, in this case, there can be no blue-yellow chain of regions, beginning at R_2 and ending at R_4. In such a situation, the colors of every blue-yellow chain of regions beginning at R_2 can be interchanged. Returning to M, we see that the color blue is now available for R, which once again says that M can be colored with four colors and produces a contradiction.

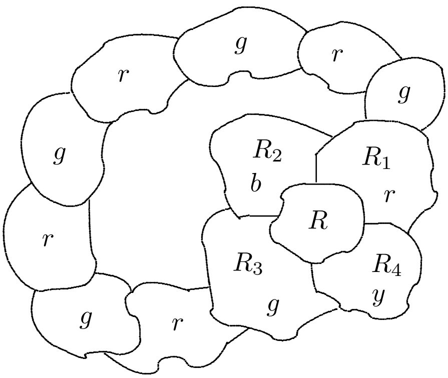


Figure 16.3: A red-green chain of regions from R_1 to R_3

So, if R were surrounded by four or fewer neighboring regions, Kempe's proof involved using chains of regions whose colors alternate between two colors and then interchanging these colors, if appropriate, to arrive at a coloring of the regions of M (minus R) with four colors so that the neighboring regions of R used at most three of these colors and thereby left a color available for R. In fact, these chains of regions became known as **Kempe chains**.

There was one case, however, that still needed to be resolved, namely the case where no region in the map is surrounded by four or fewer neighboring regions. As we noted, the map must then contain some region R surrounded by exactly five neighboring regions. At least three of the four colors must be used to color the five neighboring regions of R. If only three colors are used to color these five regions, then a color is available for R. Hence, we are left with the single situation in which all four colors are used to color the five neighboring regions surrounding R (see Figure 16.4), where once again r, b, g, y indicate the colors red, blue, green, yellow. In this case, two neighboring regions of R are colored blue.

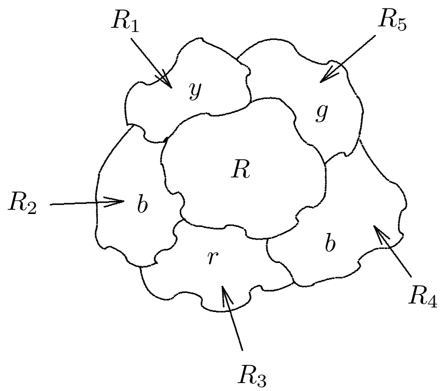


Figure 16.4: The final case in Kempe's solution of the Four Color Problem

Kempe proceeded as follows with this final case. Among the regions adjacent to R, only the region R_1 is colored yellow. Consider all the regions of the map M that are colored either yellow or red and that, beginning at R_1, can be reached by an alternating sequence of neighboring yellow and red regions, that is, by a yellow-red Kempe chain. If the region R_3 (which is the neighboring region of R that is colored red) cannot be reached by a yellow-red Kempe chain, then the colors yellow and red can be interchanged for all regions in M that can be reached by a yellow-red Kempe chain beginning at R_1. This results in a coloring of all regions in M (except R) in which neighboring regions are colored differently and such that each neighboring region of R is colored red, blue or green. We can then color R yellow to arrive at a 4-coloring of the entire map M. From this, we may assume that the region R_3 can be reached by a yellow-red Kempe chain beginning at R_1. (See Figure 16.5.)

Consider the region R_5, which is colored green. We consider all regions of M colored green or red that, beginning at R_5, can be reached by a green-red Kempe chain. If the region R_3 cannot be reached by a green-red Kempe chain that begins at R_5, then the colors green and red can be interchanged for all regions in M that can be reached by a green-red Kempe chain beginning at R_5. Upon doing this, a 4-coloring of all regions in M (except R) is obtained, in which each neighboring region of R is colored red, blue or yellow. We can then color R green to produce a 4-coloring of the entire map M. We may therefore assume that R_3 can be reached by a green-red Kempe chain that begins at R_5.

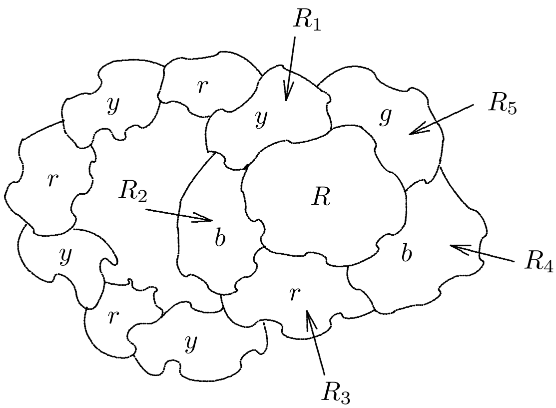


Figure 16.5: A yellow-red Kempe chain in the map M

(See Figure 16.6.)

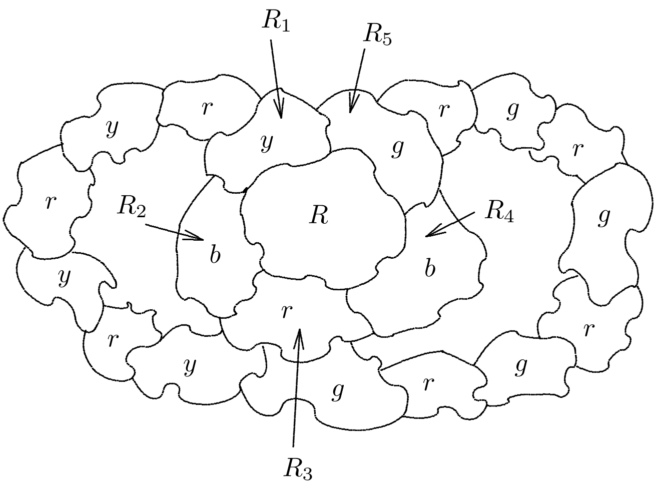


Figure 16.6: Yellow-red and green-red Kempe chains in the map M

Because there is a ring of regions consisting of R and a green-red Kempe chain, there cannot be a blue-yellow Kempe chain in M beginning at R_4 and ending at R_1. In addition, because there is a ring of regions consisting of R and a yellow-red Kempe chain, there is no blue-green Kempe chain in M beginning at R_2 and ending at R_5. Hence, we interchange the colors blue and yellow for all regions in M that can be reached by a blue-yellow Kempe chain beginning at R_4 and interchange the colors blue and green for all regions in M that can be reached by a blue-green Kempe chain beginning at R_2. Once these two color interchanges have been performed, each of the five neighboring regions of R is colored red, yellow or green. Then R can be colored blue and a 4-coloring of the map M has been obtained, completing the proof.

As it turned out, this proof given by Kempe contained a fatal flaw, but one that would go unnoticed for a decade. Despite the fact that Kempe's attempted proof of the Four Color Problem was erroneous, he made a number of interesting observations in his article. He noticed that if a piece of tracing paper was placed over a map and a point was marked on the tracing paper over each region of the map and two points were joined by a line segment whenever the corresponding regions had a common boundary, then a diagram of a "linkage" was produced. Furthermore, the problem of determining whether the regions of the map can be colored with four colors so that neighboring regions are colored differently is the same problem as determining whether the points in the linkage can be colored with four colors so that every two points joined by a line segment are colored differently. (See Figure 16.7.)

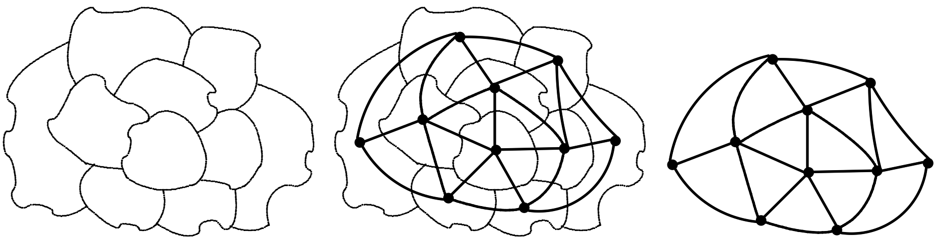


Figure 16.7: A map and corresponding plane graph

In 1878, James Joseph Sylvester referred to a linkage as a graph and it is this terminology that became standard. Later it became commonplace to refer to the points and lines of a linkage as the vertices and edges of the graph. Since the graphs constructed from maps in this manner (referred to as the **planar dual** or the **dual graph** of the map) can themselves be drawn in the plane without two edges (line segments) intersecting, the graphs produced were plane graphs. In terms of graphs, the Four Color Conjecture can then be restated.

The Four Color Conjecture *Every planar graph is* 4*-colorable.*

Percy John Heawood

The next important figure in the history of the Four Color Problem was Percy John Heawood (1861–1955), who spent the period 1887–1939 as a lecturer, professor and vice-chancellor at Durham College in England. When Heawood was a student at Oxford University in 1880, one of his teachers was Professor Henry Smith, who spoke often of the Four Color Problem. Heawood read Kempe's paper and it was he who discovered the serious error in the proof.

In 1889 Heawood [125] wrote a paper of his own, published in 1890, in which he presented the map shown in Figure 16.8.

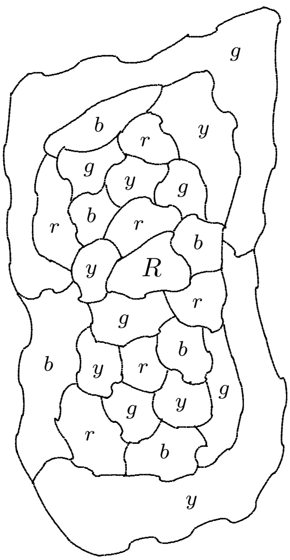
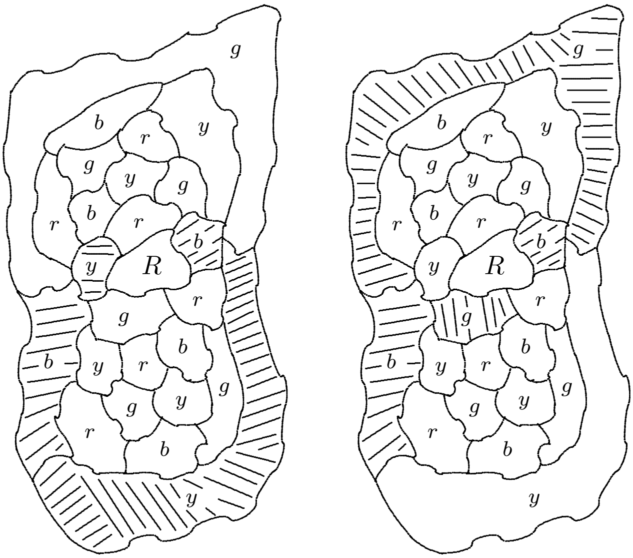


Figure 16.8: The Heawood map: A counterexample to Kempe's proof

In the **Heawood map**, two of the five neighboring regions surrounding the uncolored region R are colored red; while for each of the colors blue, yellow and green, there is exactly one neighboring region of R with that color. According to Kempe's argument, since blue is the color of the region that shares a boundary with R as well as with the two neighboring regions of R colored red, we are concerned with whether this map contains a blue-yellow Kempe chain between two neighboring regions of R as well as a blue-green Kempe chain between two neighboring regions of R. It does. These Kempe chains are shown in Figures 16.9(a) and 16.9(b), respectively.

Because the Heawood map contains these two Kempe chains, it follows by Kempe's proof that this map does not contain a red-yellow Kempe chain between the two neighboring regions of R that are colored red and yellow *and* does not contain a red-green Kempe chain between the two neighboring regions of R that are colored red and green. This is, in fact, the case. Figure 16.10(a) indicates all regions that can be reached by a red-yellow Kempe chain beginning at the red region that borders R and that is not adjacent to the yellow region bordering R. Furthermore, Figure 16.10(b) indicates all regions that can be



(a) (b)

Figure 16.9: Blue-yellow and blue-green Kempe chains in the Heawood map

reached by a red-green Kempe chain beginning at the red region that borders R and that is not adjacent to the green region bordering R.

In the final step of Kempe's proof, the two colors within each Kempe chain are interchanged, resulting in a coloring of the Heawood map with four colors. This double interchange of colors is shown in Figure 16.10(c). However, as Figure 16.10(c) shows, this results in neighboring regions with the same color, namely red. Consequently, Kempe's proof is unsuccessful when applied to the Heawood map, as colored in Figure 16.8. What Heawood had shown was that Kempe's method of proof was incorrect. That is, Heawood had discovered a counterexample to Kempe's technique, not to the Four Color Conjecture itself. Indeed, it is not particularly difficult to give a 4-coloring of the regions of the Heawood map so that every two neighboring regions are colored differently.

Unavoidable Sets of Reducible Configurations

Many mathematicians, using a variety of approaches, would attack the Four Color Problem during the 20th century. It was known that if the Four Color Conjecture could be verified for **cubic maps** (maps where three boundary lines meet at each meeting point), then the Four Color Conjecture would be true for all maps. In fact, every map M that has no region completely surrounded by

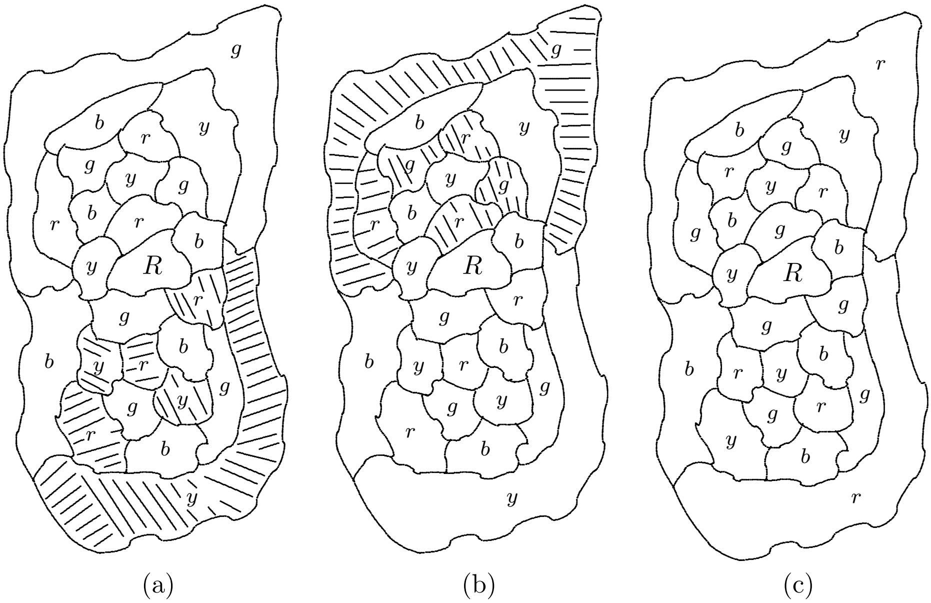


Figure 16.10: Steps in illustrating Kempe's technique

another region can be converted into a cubic map M' by drawing a circle about each meeting point in M and creating new meeting points and one new region (see Figure 16.11). If the map M' can be colored with four colors, then so can M.

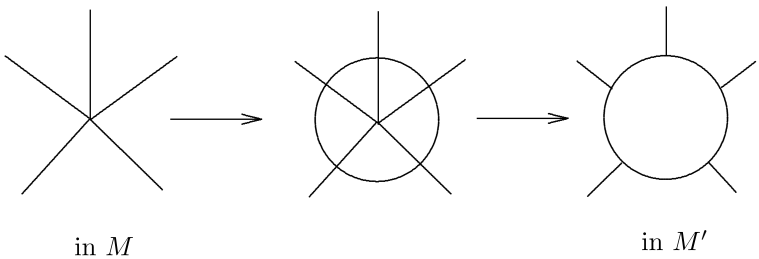


Figure 16.11: Converting a map into a cubic map

Every cubic map must contain a region surrounded by two, three, four or five neighboring regions. These four kinds of **configurations** (arrangements of regions) were called **unavoidable** because every cubic map had to contain at least one of them. Thus, the arrangements of regions shown in Figure 16.12 make up an unavoidable set of configurations.

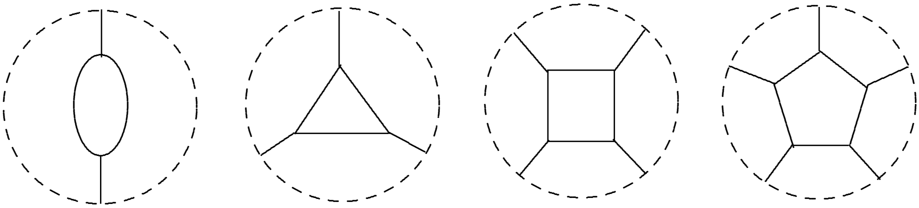


Figure 16.12: An unavoidable set of configurations in a cubic map

A **reducible configuration** is any configuration of regions that cannot occur in a minimum counterexample of the Four Color Conjecture. Many mathematicians who attempted to solve the Four Color Problem attempted to do so by trying to find an unavoidable set S of reducible configurations. Since S is unavoidable, this means that every cubic map must contain at least one configuration in S. Because each configuration in S is reducible, this means that it cannot occur in a minimum counterexample. Essentially then, a proof of the Four Color Conjecture by this approach would be a proof by minimum counterexample resulting in a number of cases (one case for each configuration in the unavoidable set S) where each case then leads to a contradiction (that is, each configuration is shown to be reducible).

Solution of Four Color Problem

Among the many mathematicians who attempted to solve the Four Color Problem were Kenneth Appel and Wolfgang Haken, who were faculty members at the University of Illinois during the 1970s. Appel and Haken devised an algorithm that tested for "reduction obstacles". The work of Appel and Haken was greatly aided by Appel's doctoral student John Koch, who wrote an efficient program that tested certain kinds of configurations for reducibility.

The partnership in the developing proof concerned the active involvement of a team of three, namely Appel, Haken and a computer. As their work progressed, Appel and Haken needed ever-increasing amounts of time on a computer. Because of Appel's political skills, he was able to get time on the IBM 370-168 located in the university's administration building. Eventually, everything paid off. In June of 1976, Appel and Haken had constructed an unavoidable set of 1936 reducible configurations, which was later reduced to 1482. The proof was finally announced at the 1976 Summer Meeting of the American Mathematical Society and the Mathematical Association of America at the University of Toronto.

A second proof of the Four Color Theorem, using the same overall approach, was announced and described by Frank Allaire of Lakehead University, Canada in 1977, although the complete details were never published. Due in part to the history of incorrect proofs of the Four Color Conjecture and the controversy surrounding a proof that made use of computers so substantially, in 1996 Neil

Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas constructed their own proof of the Four Color Theorem, but this too was heavily computer-aided.

16.2 Colorings of Planar Graphs

While Heawood [125] discovered an error in Kempe's proposed solution of the Four Color Problem and neither he nor Kempe was able to correct the error, Heawood was able to use Kempe's technique to prove that every map could be colored with five or fewer colors. This led to the following theorem.

Theorem 16.1 **(The Five Color Theorem)** *Every planar graph is* 5-*colorable.*

Proof. We proceed by induction on the order n of the graph. Clearly, the result is true if $1 \leq n \leq 5$. Assume that every planar graph of order $n-1$ is 5-colorable, where $n \geq 6$, and let G be planar graph of order n. We show that G is 5-colorable.

By Corollary 10.5, G contains a vertex v with $\deg v \leq 5$. Since $G - v$ is a planar graph of order $n-1$, it follows by the induction hypothesis that $G-v$ is 5-colorable. Let there be given a 5-coloring of $G - v$, where the colors used are denoted by 1, 2, 3, 4, 5. If one of these colors is not used to color the neighbors of v, then this color can be assigned to v, producing a 5-coloring of G. Hence, we may assume that $\deg v = 5$ and that all five colors are used to color the neighbors of v.

Let there be a planar embedding of G and suppose that v_1, v_2, v_3, v_4, v_5 are the neighbors of v arranged cyclically about v. We may assume that v_i has been assigned the color i for $1 \leq i \leq 5$. Let H be the subgraph of $G-v$ induced by the set of vertices colored 1 or 3. Thus, $v_1, v_3 \in V(H)$. If v_1 and v_3 should belong to different components of H, then by interchanging the colors of the vertices belonging to the component H_1 of H containing v_1, a 5-coloring of G can be produced by assigning the color 1 to v.

Suppose then that v_1 and v_3 belong to the same component of H. This implies that $G - v$ contains a $v_1 - v_3$ path P, every vertex of which is colored 1 or 3. The path P and the path (v_1, v, v_3) in G produce a cycle, enclosing either v_2 or both v_4 and v_5. In particular, this implies that there is no $v_2 - v_4$ path in $G - v$, every vertex of which is colored 2 or 4.

Let F be the subgraph of $G - v$ induced by the set of vertices colored 2 or 4, and let F_2 be the component of F containing v_2. Necessarily, $v_4 \notin V(F_2)$. By interchanging the colors of the vertices of F_2, a 5-coloring of G can be produced by assigning the color 2 to v. ■

Thus, after the publication of Heawood's famous paper in 1890, it was known that every planar graph could be colored with five or fewer colors but it was not known if any planar graph actually required five colors.

As we saw, some 86 years after the Five Color Theorem was proved, Kenneth Appel and Wolfgang Haken [9] proved the Four Color Theorem.

Theorem 16.2 **(The Four Color Theorem)** *Every planar graph is* 4-*colorable.*

Of course, every maximal planar graph is also 4-colorable. Since every maximal planar graph of order 3 or more contains triangles, the chromatic number of each such graph is at least 3. That is, if G is a maximal planar graph of order 3 or more, then $\chi(G) = 3$ or $\chi(G) = 4$. It is possible for a maximal planar graph of order 4 or more to have chromatic number 3. Both maximal planar graphs in Figure 16.13 not only have chromatic number 3, they also have something else in common. The following was first observed by Heawood [126] in 1898.

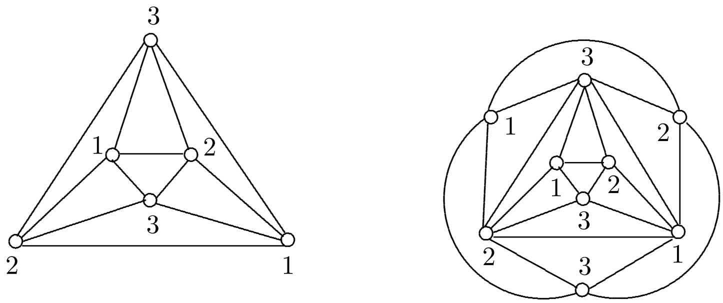


Figure 16.13: Two 3-colorable maximal planar graphs

Theorem 16.3 *A maximal planar graph G of order* 3 *or more has chromatic number* 3 *if and only if G is Eulerian.*

Proof. Let there be given a planar embedding of G. Suppose first that G is not Eulerian. Then G contains a vertex v of odd degree $k \geq 3$. Since G is maximal planar, G contains an odd cycle C consisting of the neighbors of v in G. Because v is adjacent to every vertex of C, it follows that $\chi(G) = 4$.

We verify the converse by induction on the order of maximal planar Eulerian graphs. If the order of G is 3, then $G = K_3$ and $\chi(G) = 3$. Assume that every maximal planar Eulerian graph of order k has chromatic number 3 for an integer $k \geq 3$ and let G be a maximal planar Eulerian graph of order $k+1$. Let there be given a planar embedding of G and let uw be an edge of G. Then uw is on the boundary of two (triangular) regions of G. Let x be the third vertex on the boundary of one of these regions and y the third vertex on the boundary of the other region. Suppose that

$$N(x) = \{u = x_1, x_2, \ldots, x_s = w\} \text{ and } N(y) = \{u = y_1, y_2, \ldots, y_t = w\},$$

where s and t are even such that

$$C = (x_1, x_2, \ldots, x_s, x_1) \text{ and } C' = (y_1, y_2, \ldots, y_t, y_1).$$

Then C and C' are even cycles. Let G' be the graph obtained from G by (1) deleting x, y and uw from G and (2) adding a new vertex z and joining z to every vertex of C and C'. Thus, z is adjacent to every vertex of the $(s+t-2)$-cycle

$$C'' = (u = x_1, x_2, \ldots, x_s = y_t, y_{t-1}, \cdots, y_1 = u).$$

The graph G' is thus a maximal planar Eulerian graph of order k. By the induction hypothesis, $\chi(G') = 3$. Let a 3-coloring of G' be given using the colors 1, 2 and 3. Since z is adjacent to every vertex of C and C', we may assume that z is colored 1 and that the vertices of C and C' alternate in the colors 2 and 3. Necessarily, one of u and w is colored 2 in G' and the other is colored 3. We may then remove z from G', add x and y back in together with uw and all edges of G incident with x or y. Assigning x and y the color 1 produces a 3-coloring of G. ■

16.3 List Colorings of Planar Graphs

In 1976 Vadim Vizing [248] and in 1980 Paul Erdős, Arthur L. Rubin and Herbert Taylor [82] conjectured that the maximum list chromatic number of a planar graph is 5. In 1993, Margit Voigt [249] gave an example of a planar graph of order 238 that is not 4-choosable and in 1994 Carsten Thomassen [234] completed the verification of this conjecture by showing that every planar graph is 5-choosable. To show that every planar graph is 5-choosable, it suffices to verify this result for maximal planar graphs. In fact, it suffices to verify this result for a slightly more general class of graphs.

Recall that a planar graph G is nearly maximal planar if there exists a planar embedding of G such that the boundary of every region is a cycle, at most one of which is not a triangle. If G is a nearly maximal planar graph, then we may assume that there is a planar embedding of G such that the boundary of every interior region is a triangle, while the boundary of the exterior region is a cycle of length 3 or more. Therefore, every maximal planar graph is nearly maximal planar.

5-Choosable Planar Graphs

Theorem 16.4 *Every planar graph is* 5*-choosable.*

Proof. It suffices to verify the theorem for nearly maximal planar graphs. In fact, we verify the following somewhat stronger statement by induction on the order of nearly maximal planar graphs:

> *Let G be a nearly maximal planar graph of order $n \geq 3$ such that the boundary of its exterior region is a cycle C (of length 3 or more) and*

such that $\mathfrak{L} = \{L(v) : v \in V(G)\}$ *is a collection of prescribed color lists for* G *with* $|L(v)| \geq 3$ *for each* $v \in V(C)$ *and* $|L(v)| \geq 5$ *for each* $v \in V(G) - V(C)$. *If* x *and* y *are any two consecutive vertices on* C *with* $a \in L(x)$ *and* $b \in L(y)$ *where* $a \neq b$, *then there exists an* $\mathfrak{L}$*-list-coloring of* G *with* x *and* y *colored* a *and* b, *respectively.*

The statement is certainly true for $n = 3$. Assume for an integer $n \geq 4$ that the statement is true for all nearly maximal planar graphs of order less than n satisfying the conditions in the statement. Let G be a nearly maximal planar graph of order n, the boundary of whose exterior region is the cycle C and such that

$$\mathfrak{L} = \{L(v) : v \in V(G)\}$$

is a collection of color lists for G for which $|L(v)| \geq 3$ for each $v \in V(C)$ and $|L(v)| \geq 5$ for each $v \in V(G) - V(C)$. Let x and y be any two consecutive vertices on C and suppose that $a \in L(x)$ and $b \in L(y)$ where $a \neq b$. We show that there exists an $\mathfrak{L}$-list-coloring c of G in which x and y are colored a and b, respectively. We consider two cases, according to whether C has a chord.

Case 1. *The cycle* C *has a chord* uw. The cycle C contains two $u - w$ paths P' and P'', exactly one of which, say P', contains both x and y. Let C' be the cycle determined by P' and uw and let G' be the nearly maximal planar subgraph of G induced by those vertices lying on or interior to C'. Let

$$\mathfrak{L}' = \{L(v) : v \in V(G')\}.$$

By the induction hypothesis, there is an $\mathfrak{L}'$-list-coloring c' of G' in which x and y are colored a and b, respectively. Suppose that $c'(u) = a'$ and $c'(w) = b'$.

Let C'' be the cycle determined by P'' and uw and let G'' be the nearly maximal planar subgraph of G induced by those vertices lying on or interior to C''. Furthermore, let

$$\mathfrak{L}'' = \{L(v) : v \in V(G'')\}.$$

Again, by the induction hypothesis, there is an $\mathfrak{L}''$-list-coloring c'' of G'' such that $c''(u) = c'(u) = a'$ and $c''(w) = c'(w) = b'$. Now the coloring c of G defined by

$$c(v) = \begin{cases} c'(v) & \text{if } v \in V(G') \\ c''(v) & \text{if } v \in V(G'') \end{cases}$$

is an $\mathfrak{L}$-list-coloring of G.

Case 2. *The cycle* C *has no chord.* Let v_0 be the vertex on C that is adjacent to x such that $v_0 \neq y$ and let

$$N(v_0) = \{x, v_1, v_2, \ldots, v_k, z\}$$

where z is on C. Since G is nearly maximal planar, we may assume that $xv_1, v_kz \in E(G)$ and $v_iv_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$ (see Figure 16.14).

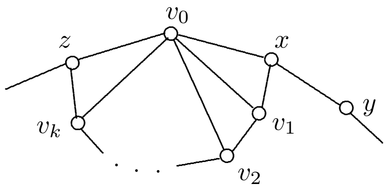


Figure 16.14: A step in the proof of Theorem 16.4

Let P be the $x - z$ path on C that does not contain v_0 and let

$$P^* = (x, v_1, v_2, \ldots, v_k, z).$$

Furthermore, let C^* be the cycle determined by P and P^*. Then $G - v_0$ is a nearly maximal planar graph of order $n-1$ in which C^* is the boundary of the exterior region. Since $|L(v_0)| \geq 3$, there are (at least) two colors a^* and b^* in $L(v_0)$ different from a. We now define a collection $\mathfrak{L}^*$ of color lists $L^*(v)$ for the vertices v of $G - v_0$ by

$$L^*(v) = L(v) \text{ if } v \neq v_i \ (1 \leq i \leq k)$$

and

$$L^*(v_i) = L(v_i) - \{a^*, b^*\} \ (1 \leq i \leq k)$$

and let

$$\mathfrak{L}^* = \{L^*(v) : \ v \in V(G - v_0)\}.$$

Hence, $|L^*(v)| \geq 3$ for $v \in V(C^*)$ and $|L^*(v)| \geq 5$ for $v \in V(G^*) - V(C^*)$. By the induction hypothesis, there is an $\mathfrak{L}^*$-list-coloring of $G - v_0$ with x and y colored a and b, respectively. Since at least one of the colors a^* and b^* has not been assigned to z, one of these colors is available for v_0, producing an $\mathfrak{L}$-list-coloring of G. Thus, G is $\mathfrak{L}$-choosable and is therefore 5-choosable. ■

The Mirzakhani Graph

As we mentioned, in 1993 Margit Voigt gave an example of a planar graph of order 238 that is not 4-choosable. In 1996 Maryam Mirzakhani [171] gave an even simpler example, namely a planar graph of order 63 that is not 4-choosable. In 2014, Mirzakhani was a recipient of the Fields Medal, awarded for achievement by mathematicians under the age of 40. The Fields Medal is considered the mathematical equivalent of a Nobel Prize. Mirzakhani was the first woman to be so honored. We now describe the Mirzakhani graph and verify that it is, in fact, not 4-choosable.

First, let H be the planar graph of order 17 shown in Figure 16.15(a). For each vertex u of H, a list $L(u)$ of three or four colors is given in Figure 16.15(b),

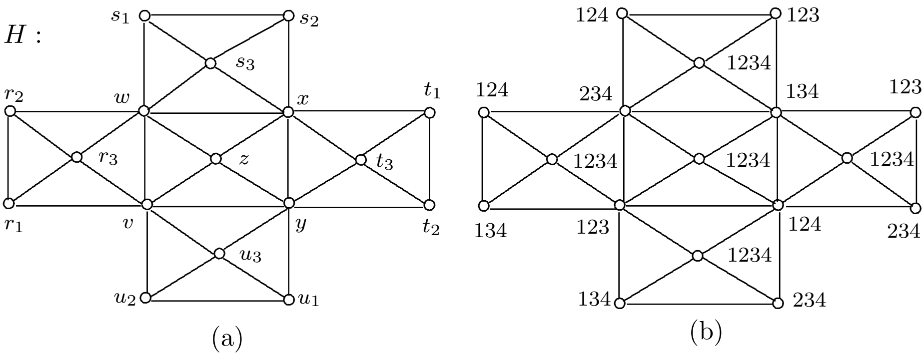


Figure 16.15: A planar graph of order 17

where $L(u) \subseteq \{1,2,3,4\}$. In fact, if $\deg_H u = 4$, then $L(u) = \{1,2,3,4\}$, while if $\deg_H v \neq 4$, then $|L(u)| = 3$. Let $\mathfrak{L} = \{L(u) : u \in V(H)\}$. We claim that H is not $\mathfrak{L}$-choosable.

Theorem 16.5 *The planar graph H of* Figure 16.15(a) *with the set $\mathfrak{L}$ of color lists in* Figure 16.15(b) *is not $\mathfrak{L}$-choosable.*

Proof. Assume, to the contrary, that H is $\mathfrak{L}$-choosable. Then there exists a 4-coloring c of H such that $c(u) \in L(u)$ for each vertex u in H. Since each vertex of degree 4 in H is adjacent to vertices assigned either two or three distinct colors, it follows that each vertex of degree 4 in H is adjacent to two (nonadjacent) vertices assigned the same color. We claim that $c(x) = 1$ or $c(w) = 2$. Suppose that $c(x) \neq 1$ and $c(w) \neq 2$. Then there are two possibilities. Suppose first that $c(x) = 3$ and $c(w) = 4$. Then either $c(s_1) = 3$ or $c(s_2) = 4$. This is impossible, however, since $3 \notin L(s_1)$ and $4 \notin L(s_2)$. Next, suppose that $c(x) = 4$ and $c(w) = 3$. Then either $c(v) = 4$ or $c(y) = 3$. This is impossible as well since $4 \notin L(v)$ and $3 \notin L(y)$. Hence, as claimed, $c(x) = 1$ or $c(w) = 2$. We consider these two cases.

Case 1. $c(x) = 1$. Since none of the vertices t_1, t_2 and y can be assigned the color 1, it follows that $c(t_1) = c(y) = 2$. Since none of the vertices u_1, u_2 and v can be assigned the color 2, it follows that $c(u_1) = c(v) = 3$. Therefore, none of the vertices r_1, r_2 and w can be assigned the color 3. Thus, $c(r_1) = c(w) = 4$. Hence, $c(x) = 1$, $c(y) = 2$, $c(v) = 3$ and $c(w) = 4$, which is impossible.

Case 2. $c(w) = 2$. Proceeding as in Case 1, we first see that $c(r_2) = c(v) = 1$. From this, it follows that $c(u_2) = c(y) = 4$. Next, we find that $c(t_2) = c(x) = 3$. Hence, $c(v) = 1$, $c(w) = 2$, $c(x) = 3$ and $c(y) = 4$, again an impossibility.

Therefore, as claimed, the graph H is not $\mathfrak{L}$-choosable for the set $\mathfrak{L}$ of lists described in Figure 16.15(b). ∎

Continuing with the description of the Mirzakhani graph, let H_1, H_2, H_3 and H_4 be four copies of the graph H in Figure 16.15(a). For $i = 1, 2, 3, 4$, the color i in the color list of every vertex of H_i in Figure 16.15(b) is replaced by 5 and the color i is then added to the color list of each vertex not having degree 4. The graphs H_i ($i = 1, 2, 3, 4$) and the color lists of their vertices are shown in Figure 16.16.

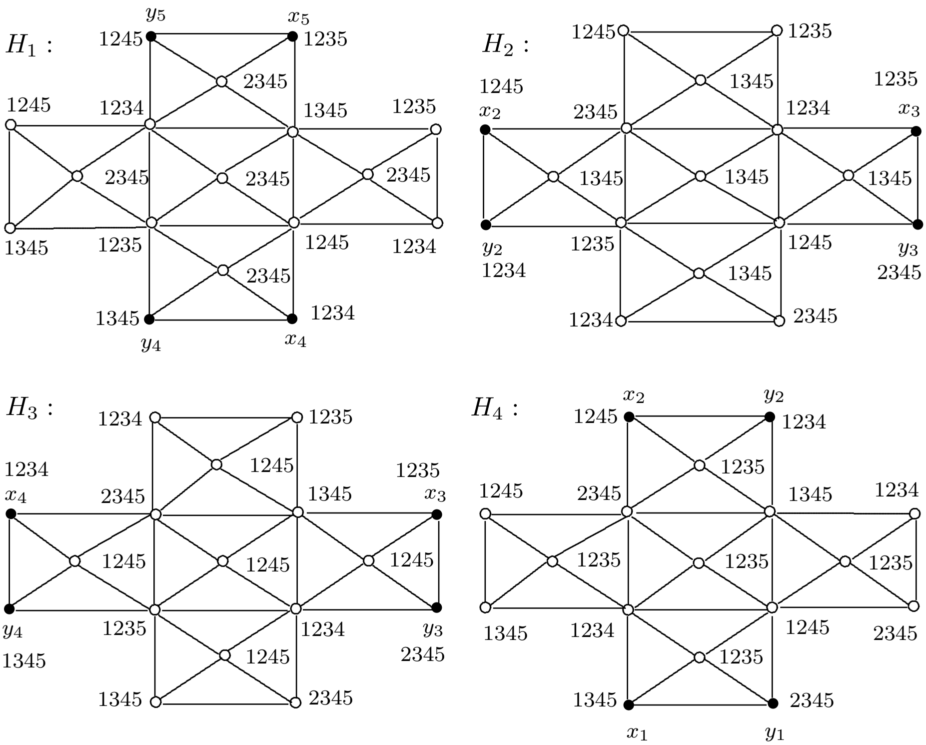


Figure 16.16: The graphs H_i ($i = 1, 2, 3, 4$)

The **Mirzakhani graph** G (a planar graph of order 63) is now constructed from the graphs H_i ($i = 1, 2, 3, 4$) of Figure 16.16 by identifying the two vertices labeled x_i and the two vertices labeled y_i for $i = 2, 3, 4$ and adding a new vertex p with $L'(p) = \{1, 2, 3, 4\}$ and joining p to each vertex of each copy H_i of H whose degree is not 4. The Mirzakhani graph is shown in Figure 16.17 along with the resulting color lists for each vertex. Let $\mathfrak{L}' = \{L'(u) : u \in V(G)\}$. We show that G is not $\mathfrak{L}'$-choosable.

Theorem 16.6 *The Mirzakhani graph is not* 4*-choosable.*

Proof. Let $L'(u)$ be the color list for each vertex u in G shown in Figure 16.17 and let $\mathfrak{L}' = \{L'(u) : u \in V(G)\}$. We claim that G is not $\mathfrak{L}'$-choosable.

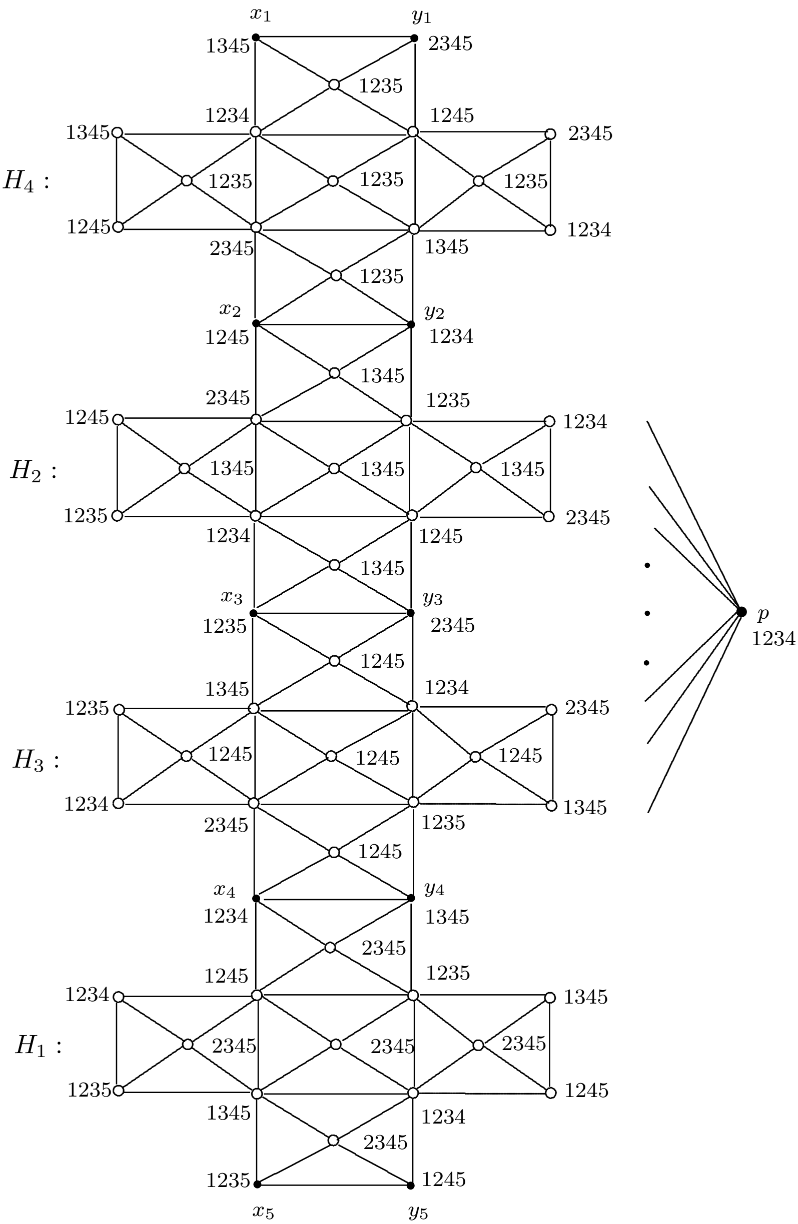


Figure 16.17: The Mirzakhani graph:
A non-4-choosable planar graph of order 63

Suppose, to the contrary, that G is $\mathfrak{L}'$-choosable. Then there is a coloring c' such that $c'(u) \in L'(u)$ for each $u \in V(G)$. Since the graph H of Figure 16.15(a) is not $\mathfrak{L}$-choosable for the set $\mathfrak{L}$ of lists in Figure 16.15(b), the only way for G to be $\mathfrak{L}'$-choosable is that $c'(v_i) = i$ for some $v_i \in V(H_i)$ for $i = 1, 2, 3, 4$. However then, regardless of the value of $c'(p)$, the vertex p is adjacent to a vertex in G having the same color as p, producing a contradiction. Thus, as claimed, G is not $\mathfrak{L}'$-choosable. Because $|L'(u)| = 4$ for each $u \in V(G)$, it follows that G is not 4-choosable. ■

Since the Mirzakhani graph has chromatic number 3, it follows that a 3-colorable planar graph need not be 4-choosable.

16.4 The Conjectures of Hajós and Hadwiger

We have often mentioned that $\chi(G) \geq \omega(G)$ for every graph G and that this inequality can be strict. Indeed, Theorem 15.1 implies that for every integer $k \geq 3$, there is a graph G such that $\chi(G) = k$ and $\omega(G) = 2$. Even though K_k need not be present in a k-chromatic graph G, it has been thought (and conjectured over the years) that K_k may be indirectly present in G. Of course, K_k is present in a k-chromatic graph for $k = 1$ and $k = 2$. This is not true for $k = 3$, however. Indeed, every odd cycle of order at least 5 is 3-chromatic but none of these graphs contains K_3 as a subgraph. All of these do contain a subdivision of K_3, however. Since every 3-chromatic graph contains an odd cycle, it follows that if G is a graph with $\chi(G) \geq 3$, then G must contain a subdivision of K_3 as a subgraph. In 1952 Gabriel Dirac [71] showed that the corresponding result is also true for graphs having chromatic number 4.

Theorem 16.7 *Every 4-chromatic graph contains a subdivision of K_4.*

Proof. We proceed by induction on the order $n \geq 4$ of a 4-chromatic graph G. The basis step of the induction follows since K_4 is the only graph of order 4 having chromatic number 4. For an integer $n \geq 5$, assume that every graph of order n' with $4 \leq n' < n$ having chromatic number 4 contains a subdivision of K_4. Let G be a graph of order n such that $\chi(G) = 4$. We show that G contains a subdivision of K_4.

Let H be a critically 4-chromatic subgraph of G. If the order of H is less than n, then it follows by the induction hypothesis that H (and G as well) contains a subdivision of K_4. Hence, we may assume that H has order n. Therefore, H is 2-connected.

Suppose first that $\kappa(H) = 2$ and $S = \{u, v\}$ is a vertex-cut of H. Let $F_1, F_2, \ldots, F_s$, $s \geq 2$, be the components of $H - S$ and for $1 \leq i \leq s$, let $H_i = H[V(F_i) \cup S]$ be the resulting S-branches of H. Since each S-branch is a proper subgraph of H, it follows that $\chi(H_i) \leq 3$ for each i $(1 \leq i \leq s)$. If there is a 3-coloring of each S-branch H_i (using the colors $1, 2, 3$, say) where u and v are assigned different colors, then by permuting colors, if necessary, u

may be assigned the color 1 and v the color 2, resulting in a 3-coloring of H, which is impossible. Hence, there is some S-branch, say H', where u and v are colored the same, which implies that $uv \notin E(H')$. Since there is no 3-coloring of $H' + uv$, it follows that $\chi(H' + uv) = 4$. Because the order of $H' + uv$ is less than n, it follows by the induction hypothesis that $H' + uv$ contains a subdivision F of K_4. If F does not contain the edge uv, then H' and therefore G contains F. Hence, we may assume that F contains the edge uv. In this case, let H'' be an S-branch of H distinct from H'. Because S is a minimum vertex-cut, both u and v are adjacent to vertices in each component of $H - S$. Hence, H'' contains a $u - v$ path P. Replacing the edge uv in F by P produces a subdivision of K_4 in H.

We may now assume that H is 3-connected. Let $w \in V(H)$. Then $H - w$ is 2-connected and so contains a cycle C. With the aid of Corollary 4.13, there are three distinct vertices w_1, w_2, w_3 on C such that H contains internally disjoint $w - w_i$ paths P_i $(1 \le i \le 3)$ for which every two of these paths have only w in common and w_i is the only vertex of P_i on C. Then C and the paths P_i $(1 \le i \le 3)$ produce a subdivision of K_4 in H.

Since H contains a subdivision of K_4, the graph G does as well. ∎

Consequently, for $2 \le k \le 4$, every k-chromatic graph contains a subdivision of K_k.

Hajós' Conjecture

In 1961 György Hajós [116] conjectured that for every integer $k \ge 2$, all k-chromatic graphs contain a subdivision of K_k.

Hajós' Conjecture *If G is a k-chromatic graph, where $k \ge 2$, then G contains a subdivision of K_k.*

In 1979 Paul Catlin [42] constructed a family of graphs that showed that Hajós' Conjecture is false for every integer $k \ge 7$. For example, the graph G of Figure 16.18 of order 15 consists of five mutually vertex-disjoint triangles T_i $(1 \le i \le 5)$ where every vertex of T_i is adjacent to every vertex of T_j if either $|i - j| = 1$ or if $\{i, j\} = \{1, 5\}$. Here, $\omega(G) = 6$ and $\chi(G) = 8$. Since $\omega(G) = 6$, the graph G does not contain K_8 as a subgraph (or even K_7 as a subgraph). We claim, in fact, that G does not contain a subdivision of K_8 either, for suppose that H is such a subgraph of G. Then H contains eight vertices of degree 7 and all other vertices of H have degree 2.

First, we show that no triangle T_i $(1 \le i \le 5)$ can contain exactly one vertex of degree 7 in H. Suppose that the triangle T_1, say, contains exactly one such vertex v. Then v is the initial vertex of seven paths to the remaining seven vertices of degree 7 in H, where every two of these paths have only v in common. Since v is adjacent only to six vertices outside of T_1, this is impossible.

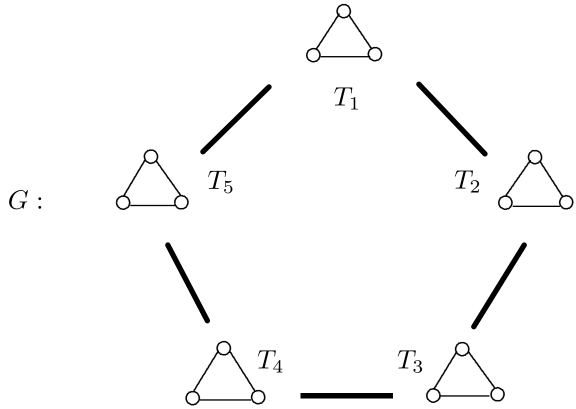


Figure 16.18: A counterexample to Hajós' Conjecture for $k = 8$

Next, we show that no triangle can contain exactly two vertices of degree 7 in H. Suppose that the triangle T_1, say, contains exactly two such vertices, namely u and v. Then u is the initial vertex of six paths to six vertices outside of T_1. Necessarily, these six paths must contain all six vertices in T_2 and T_5. However, this is true of the vertex v as well. This implies that the six vertices of T_2 and T_5 are the remaining vertices of degree 7 in H. Therefore, there are two vertices u_5 and v_5 in T_5 and two vertices u_2 and v_2 in T_2 such that the interior vertices of some $u_5 - u_2$ path, $u_5 - v_2$ path, $v_5 - u_2$ path and $v_5 - v_2$ path contain only vertices of T_3 and T_4. Since these four paths must be internally disjoint and since T_3 (and T_4) contains only three vertices, this is impossible.

Therefore, no triangle T_i $(1 \le i \le 5)$ can contain exactly one or exactly two vertices of degree 7 in H. This, however, says that no triangle T_i $(1 \le i \le 5)$ can contain exactly three vertices of degree 7 in H either. Thus, as claimed, G does not contain a subdivision of K_8. Hence, the graph G is a counterexample to Hajós' Conjecture for $k = 8$.

Let k be an integer such that $k \ge 9$ and consider the graph $F = G \vee K_{k-8}$. Then $\chi(F) = k$. We claim that F does not contain a subdivision of K_k, for suppose that H is such a subgraph in F. Delete all vertices of K_{k-8} that belong to H, arriving at a subgraph H' of G. This says that H' contains a subdivision of K_8, which is impossible. Hence, Hajós' Conjecture is false for all integers $k \ge 8$. As we noted, Catlin showed that this conjecture is false for every integer $k \ge 7$ (see Exercise 9).

Recall that a graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Furthermore, for a graph G and a vertex v of G, the replication graph $R_v(G)$ of G is that graph obtained from G by adding a new vertex v' to G and joining v' to every vertex in the closed neighborhood $N[v]$ of v. We saw in Theorem 15.9 that if G is perfect, then $R_v(G)$ is perfect for every $v \in V(G)$. Carsten Thomassen [235] showed that there is a connection between perfect graphs and Hajós' Conjecture.

Theorem 16.8 *A graph G is perfect if and only if every replication graph of G satisfies Hajós' Conjecture.*

Hadwiger's Conjecture

Recall also that a graph H is a minor of a graph G if H can be obtained from G by a succession of contractions, edge deletions and vertex deletions (in any order). Furthermore, from Theorem 10.19, if a graph G contains a subdivision of a graph H, then H is a minor of G. In particular, if a k-chromatic graph G contains a subdivision of K_k, then K_k is a minor of G. Of course, we have seen that for $k \geq 7$, a k-chromatic graph need not contain a subdivision of K_k. This does not imply, however, that a k-chromatic graph cannot contain K_k as a minor. Indeed, years before Hajós' Conjecture, on December 15, 1942, Hugo Hadwiger made the following conjecture during a lecture he gave at the University of Zürich in Switzerland.

Hadwiger's Conjecture *Every k-chromatic graph contains K_k as a minor.*

A published version [113] of Hadwiger's lecture appeared in 1943. Hadwiger's paper not only contained this conjecture, it contained three theorems of interest:

(1) Hadwiger's Conjecture is true for $1 \leq k \leq 4$.

(2) If G is a graph with $\delta(G) \geq k-1$ where $1 \leq k \leq 4$, then G contains K_k as a minor.

(3) If G is a connected graph of order n and size m that has K_k as a minor, then $m \geq n + \binom{k}{2} - k$ (see Exercise 10).

In 1937 Klaus Wagner [251] proved that every planar graph is 4-colorable if and only if every 5-chromatic graph contains K_5 as a minor. That is, Wagner showed the equivalence of the Four Color Conjecture and Hadwiger's Conjecture for $k = 5$ six years before Hadwiger stated his conjecture. In his 1943 paper [113], Hadwiger mentioned that the truth of his conjecture for $k = 5$ implies the truth of the Four Color Conjecture and referenced Wagner's paper but he did not refer to the equivalence. Hadwiger's Conjecture can therefore be considered as a generalization of the Four Color Conjecture.

Using the Four Color Theorem, Neil Robertson, Paul Seymour and Robin Thomas [207] verified Hadwiger's Conjecture for $k = 6$. While Hadwiger's Conjecture is true for $k \leq 6$, its truth is in question for every integer $k > 6$.

16.5 Chromatic Polynomials

During the time that the Four Color Problem was being investigated, a period that spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution. In 1912 George David Birkhoff [27] defined a function $P(M, \lambda)$ that gives the number of proper λ-colorings of a map M for a positive integer λ. As we will see, $P(M, \lambda)$ is a polynomial in λ for every map M and is called the chromatic polynomial of M. Consequently, if it could be verified that $P(M, 4) > 0$ for every map M, then this would have established the truth of the Four Color Conjecture.

In 1932 Hassler Whitney [256] extended the study of chromatic polynomials from maps to graphs. While Whitney and others obtained a number of results on chromatic polynomials of graphs, this did not contribute to a proof of the Four Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred in 1968, however, when Ronald C. Read [196] wrote a survey paper on chromatic polynomials.

For a graph G and a positive integer λ, the number of different proper λ-colorings of G is denoted by $P(G, \lambda)$ and is called the **chromatic polynomial** of G. Two λ-colorings c and c' of G from the same set $\{1, 2, \ldots, \lambda\}$ of λ colors are considered different if $c(v) \neq c'(v)$ for some vertex v of G. Obviously, if $\lambda < \chi(G)$, then $P(G, \lambda) = 0$. By convention, $P(G, 0) = 0$. Indeed, we have the following:

Let G be a graph. Then $\chi(G) = k$ if and only if k is the smallest positive integer for which $P(G, k) > 0$.

As an example, we determine the number of ways that the vertices of the graph G of Figure 16.19 can be colored from the set $\{1, 2, 3, 4, 5\}$. The vertex v can be assigned any of these five colors, while w can be assigned any color other than the color assigned to v. That is, w can be assigned any of the four remaining colors. Both u and t can be assigned any of the three colors not used to color v and w. Therefore, the number $P(G, 5)$ of 5-colorings of G is $5 \cdot 4 \cdot 3^2 = 180$. More generally, $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ for every positive integer λ. In particular, $P(G, 1) = P(G, 2) = 0$, $P(G, 3) = 6$ and $P(G, 4) = 48$. As we saw, $P(G, 5) = 180$.

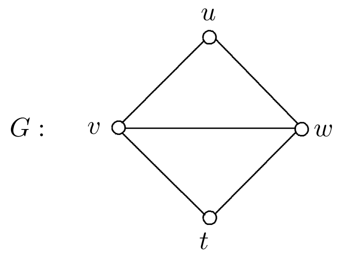


Figure 16.19: A graph G with $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$

There are some classes of graphs G for which $P(G,\lambda)$ can be easily computed.

Theorem 16.9 *For every positive integer λ,*

(a) $P(K_n,\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) = \lambda^{(n)}$,

(b) $P(\overline{K}_n,\lambda) = \lambda^n$.

In particular, if $\lambda \geq n$ in Theorem 16.9(a), then

$$P(K_n,\lambda) = \lambda^{(n)} = \frac{\lambda!}{(\lambda-n)!}.$$

A Technique for Constructing Chromatic Polynomials

We now determine the chromatic polynomial of the graph C_4, shown in Figure 16.20. There are λ choices for the color of v_1. The vertices v_2 and v_4 must be assigned colors different from that assigned to v_1. The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda-1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4. Hence, the number of distinct λ-colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda(\lambda-1)^2$.

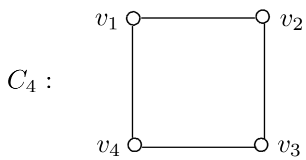


Figure 16.20: Computing the chromatic polynomial of C_4

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda-1$ choices for v_2, say, and $\lambda-2$ choices for v_4. Since v_3 can be assigned any color except the two colors assigned to v_2 and v_4, the number of λ-colorings of C_4 in which v_2 and v_4 are colored differently is $\lambda(\lambda-1)(\lambda-2)^2$. Hence, the number of distinct λ-colorings of C_4 is

$$\begin{aligned} P(C_4,\lambda) &= \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2 \\ &= \lambda(\lambda-1)(\lambda^2-3\lambda+3) \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \\ &= (\lambda-1)^4 + (\lambda-1). \end{aligned}$$

The preceding example illustrates an important observation. Suppose that u and v are nonadjacent vertices in a graph G. The number of λ-colorings of G

equals the number of λ-colorings of G in which u and v are colored differently plus the number of λ-colorings of G in which u and v are colored the same. Since the number of λ-colorings of G in which u and v are colored differently is the number of λ-colorings of $G + uv$ while the number of λ-colorings of G in which u and v are colored the same is the number of λ-colorings of the graph H obtained by identifying u and v, it follows that

$$P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda).$$

This observation is summarized below.

Theorem 16.10 *Let G be a graph containing nonadjacent vertices u and v and let H be the graph obtained from G by identifying u and v. Then*

$$P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda).$$

Note that if G is a graph of order $n \geq 2$ and size $m \geq 1$, then $G + uv$ has order n and size $m + 1$ while H has order $n - 1$ and size at most m.

Of course, the equation stated in Theorem 16.10 can also be expressed as

$$P(G + uv, \lambda) = P(G, \lambda) - P(H, \lambda).$$

In this context, Theorem 16.10 can be rephrased in terms of an edge deletion and a contraction.

Corollary 16.11 *Let G be a graph containing adjacent vertices u and v and let F be the graph obtained from G by identifying u and v. Then*

$$P(G, \lambda) = P(G - uv, \lambda) - P(F, \lambda).$$

By systematically applying Theorem 16.10 to pairs of nonadjacent vertices in a graph G, we eventually arrive at a collection of complete graphs whose chromatic polynomials are known. We now illustrate this. Suppose that we wish to compute the chromatic polynomial of the graph G of Figure 16.21. For the nonadjacent vertices u and v of G and the graph H obtained by identifying u and v, it follows by Theorem 16.10 that the chromatic polynomial of G is the sum of the chromatic polynomials of $G + uv$ and H.

At this point, it is useful to adopt a convention introduced by Alexander Zykov [263] and utilized later by Read [196]. Rather than repeatedly writing the equation that appears in the statement of Theorem 16.10, we represent the chromatic polynomial of a graph by a drawing of the graph and indicate in the drawing which pair u, v of nonadjacent vertices will be separately joined by an edge and identified. So, for the graph G of Figure 16.21, we have the “equation” also shown in Figure 16.21.

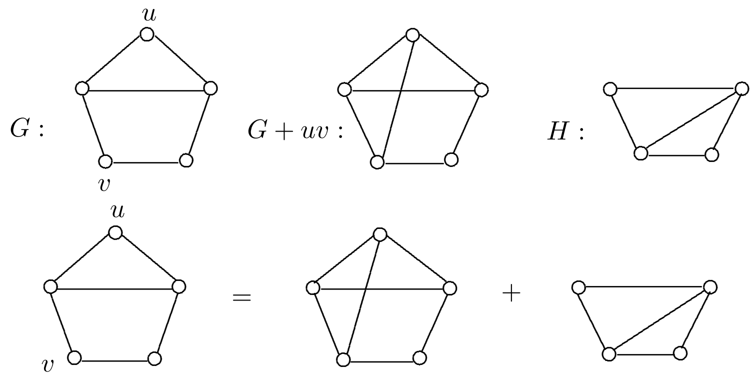


Figure 16.21: $P(G,\lambda) = P(G+uv,\lambda) + P(H,\lambda)$

Continuing in this manner, as shown in Figure 16.22, we obtain

$$P(G,\lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.$$

Using this approach, we see that the chromatic polynomial of every graph is the sum of chromatic polynomials of complete graphs. A consequence of this observation is the following:

Theorem 16.12 *The chromatic polynomial $P(G,\lambda)$ of a graph G is a polynomial in λ.*

Properties of Chromatic Polynomials

There are some interesting properties possessed by the chromatic polynomial of every graph. In fact, if G is a graph of order n and size m, then the chromatic polynomial $P(G,\lambda)$ of G can be expressed as

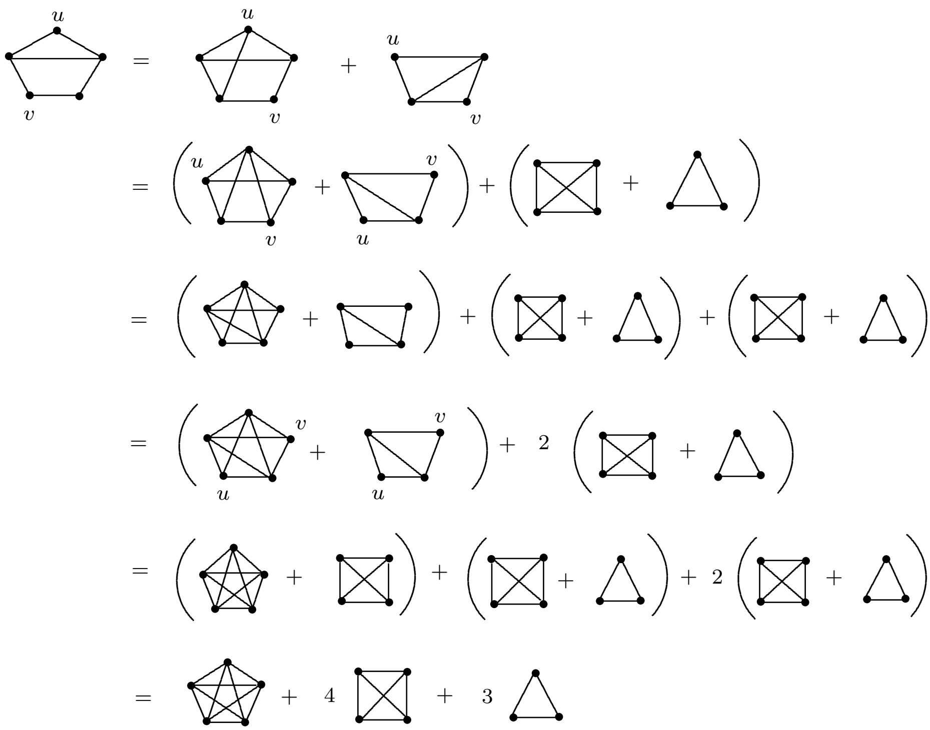
$$P(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n,$$

where $c_0 = 1$ (and so $P(G,\lambda)$ is a polynomial of degree n with leading coefficient 1), $c_1 = -m$, $c_i \geq 0$ if i is even with $0 \leq i \leq n$, and $c_i \leq 0$ if i is odd with $1 \leq i \leq n$. Since $P(G,0) = 0$, it follows that $c_n = 0$.

The following theorem is due to Whitney [256].

Theorem 16.13 *Let G be a graph of order n and size m. Then $P(G,\lambda)$ is a polynomial of degree n with leading coefficient 1 such that the coefficient of λ^{n-1} is $-m$ and whose coefficients alternate in sign.*

Proof. We proceed by induction on m. If $m = 0$, then $G = \overline{K}_n$ and $P(G,\lambda) = \lambda^n$, as we have seen. Then $P(\overline{K}_n,\lambda) = \lambda^n$ has the desired properties.



$$= \lambda^{(5)} + 4\lambda^{(4)} + 3\lambda^{(3)} = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

Figure 16.22: $P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$

Assume that the result holds for all graphs whose size is less than m, where $m \geq 1$. Let G be a graph of size m and let $e = uv$ be an edge of G. By Corollary 16.11,

$$P(G, \lambda) = P(G - e, \lambda) - P(F, \lambda),$$

where F is the graph obtained from G by identifying u and v. Since $G - e$ has order n and size $m - 1$, it follows by the induction hypothesis that

$$P(G - e, \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n,$$

where $a_0 = 1$, $a_1 = -(m-1)$, $a_i \geq 0$ if i is even with $0 \leq i \leq n$, and $a_i \leq 0$ if i is odd with $1 \leq i \leq n$. Furthermore, since F has order $n - 1$ and size m', where $m' \leq m - 1$, it follows that

$$P(F, \lambda) = b_0\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \cdots + b_{n-2}\lambda + b_{n-1},$$

where $b_0 = 1$, $b_1 = -m'$, $b_i \geq 0$ if i is even with $0 \leq i \leq n-1$, and $b_i \leq 0$ if i is odd with $1 \leq i \leq n-1$. By Corollary 16.11,

$$\begin{aligned} P(G,\lambda) &= P(G-e,\lambda) - P(F,\lambda) \\ &= (a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n) - \\ &\quad (b_0\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \cdots + b_{n-2}\lambda + b_{n-1}) \\ &= a_0\lambda^n + (a_1 - b_0)\lambda^{n-1} + (a_2 - b_1)\lambda^{n-2} + \cdots \\ &\qquad + (a_{n-1} - b_{n-2})\lambda + (a_n - b_{n-1}). \end{aligned}$$

Since $a_0 = 1$, $a_1 - b_0 = -(m-1) - 1 = -m$, $a_i - b_{i-1} \geq 0$ if i is even with $2 \leq i \leq n$, and $a_i - b_{i-1} \leq 0$ if i is odd with $1 \leq i \leq n$, $P(G,\lambda)$ has the desired properties and the theorem follows by mathematical induction. ■

Suppose that a graph G contains an end-vertex v whose only neighbor is u. Then, of course, $P(G-v,\lambda)$ is the number of λ-colorings of $G-v$. The vertex v can then be assigned any of the λ colors except the color assigned to u. This observation gives the following.

Theorem 16.14 *If G is a graph containing an end-vertex v, then*

$$P(G,\lambda) = (\lambda - 1)P(G - v, \lambda).$$

One consequence of Theorem 16.14 is the following result, the converse of which is true as well (see Exercise 22).

Corollary 16.15 *If T is a tree of order $n \geq 1$, then*

$$P(T,\lambda) = \lambda(\lambda-1)^{n-1}.$$

Proof. We proceed by induction on n. For $n = 1$, $T = K_1$ and certainly $P(K_1,\lambda) = \lambda$. Thus, the basis step of the induction is true. Suppose that $P(T',\lambda) = \lambda(\lambda-1)^{n-2}$ for every tree T' of order $n-1 \geq 1$ and let T be a tree of order n. Let v be an end-vertex of T. Thus, $T - v$ is a tree of order $n-1$. By Theorem 16.14 and the induction hypothesis,

$$P(T,\lambda) = (\lambda-1)P(T-v,\lambda) = (\lambda-1)\left[\lambda(\lambda-1)^{n-2}\right] = \lambda(\lambda-1)^{n-1},$$

as desired. ■

Two graphs are **chromatically equivalent** if they have the same chromatic polynomial. By Theorem 16.13, two chromatically equivalent graphs must have the same order, the same size and the same chromatic number. By Corollary 16.15, every two trees of the same order are chromatically equivalent. It is not known under what conditions two graphs are chromatically equivalent in general. A graph G is **chromatically unique** if $P(H,\lambda) = P(G,\lambda)$ implies that $H \cong G$. Here too, it is not known under what conditions a graph is chromatically unique.

16.6 The Heawood Map-Coloring Problem

In addition to the counterexample to Kempe's proof and a proof of the Five Color Theorem, Heawood's paper [125] contained several other interesting results, observations and comments. For example, Heawood also considered the problem of coloring maps that can be drawn on other surfaces. As we have observed, maps that can be drawn in the plane are precisely those maps that can be drawn on the surface of a sphere. Heawood proved that the regions of every map drawn on the surface of a torus can be colored with seven or fewer colors and that there is, in fact, a map on the torus that requires seven colors. The map M shown in Figure 16.23 is drawn on the torus. Considered as a graph embedded on the torus, this is a 3-regular graph of order 14, containing seven regions colored with $1, 2, \ldots, 7$, each of whose boundaries is a 6-cycle. Coloring the regions of this map requires seven colors. The **toroidal dual** of this map is the graph K_7 drawn on the torus (see Figure 11.17).

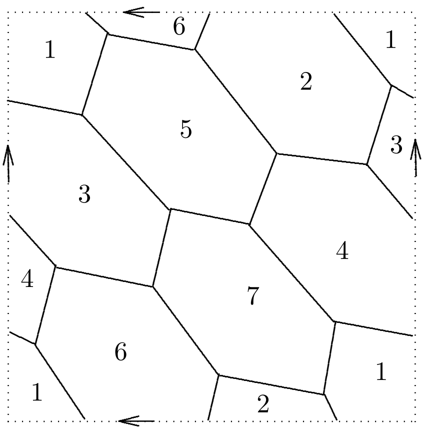


Figure 16.23: A map on the torus

More generally, Heawood showed that the regions of every map drawn on a surface of genus k ($k > 0$) can be colored with $\left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$ colors. In addition, he stated that such maps requiring this number of colors exist. He never proved this latter statement, however. In fact, it would take another 78 years to verify this statement.

Not surprisingly, Heawood became interested in the largest chromatic number of a graph that could be embedded on certain surfaces. The **chromatic number of a surface** S is defined by

$$\chi(S) = \max\{\chi(G)\},$$

where the maximum is taken over all graphs G that can be embedded on S. That $\chi(S_0) = 4$ is the Four Color Theorem. Heawood was successful in determining the chromatic number of the torus.

Theorem 16.16 $\chi(S_1) = 7$.

Proof. In Figure 11.17, we saw that the complete graph K_7 can be embedded on the torus. Since $\chi(K_7) = 7$, it follows that $\chi(S_1) \geq 7$.

Now, let G be a graph that can be embedded on the torus. Among the subgraphs of G, let H be one having the largest minimum degree. We show that $\delta(H) \leq 6$. Suppose that H has order n and size m. If $n \leq 7$, then certainly $\delta(H) \leq 6$. Hence, we may assume that $n > 7$.

Since G is embeddable on the torus, so is H. Thus, $\gamma(H) \leq 1$. It therefore follows by Theorem 11.13 that

$$1 \geq \gamma(H) \geq \frac{m}{6} - \frac{n}{2} + 1$$

and so $m \leq 3n$. Hence,

$$n\delta(H) \leq \sum_{v \in V(H)} \deg_H v = 2m \leq 6n$$

and so $\delta(H) \leq 6$. Therefore, in any case, $\delta(H) \leq 6$. By Theorem 14.19,

$$\chi(G) \leq 1 + \delta(H) \leq 7.$$

Thus, $\chi(S_1) = 7$. ■

In his important paper, Heawood [125] obtained an upper bound for the chromatic number of S_k for every positive integer k.

Theorem 16.17 *For every nonnegative integer k,*

$$\chi(S_k) \leq \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor.$$

Proof. Let G be a graph that is embeddable on the surface S_k and let

$$h = \frac{7 + \sqrt{1 + 48k}}{2}.$$

Hence, $1 + 48k = (2h - 7)^2$. Solving for $h - 1$, we have

$$h - 1 = 6 + \frac{12(k-1)}{h}. \tag{16.1}$$

Among the subgraphs of G, let H be one having the largest minimum degree. We show that $\delta(H) \leq h - 1$. Suppose that H has order n and size m. If $n \leq h$, then $\delta(H) \leq h - 1$, as desired. Hence, we may assume that $n > h$. Since G is embeddable on S_k, so is H. Therefore, $\gamma(H) \leq k$. By Theorem 11.13,

$$k \geq \gamma(H) \geq \frac{m}{6} - \frac{n}{2} + 1.$$

Thus, $m \le 3n + 6(k-1)$. We therefore have

$$n\delta(H) \le \sum_{v \in V(H)} \deg_H v = 2m \le 6n + 12(k-1)$$

and so, by (16.1),

$$\delta(H) \le 6 + \frac{12(k-1)}{n} < 6 + \frac{12(k-1)}{h} = h - 1.$$

Hence, $\delta(H) < h - 1$. Thus, $\delta(H) + 1 < h$ and so $\delta(H) + 1 \le \lfloor h \rfloor$. By Theorem 14.19,

$$\chi(G) \le 1 + \delta(H) \le \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor,$$

giving the desired result. ■

In fact, Heawood was under the impression that he had shown that

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor \tag{16.2}$$

for every positive integer k but, in fact, all he had established was the upper bound for $\chi(S_k)$ given in Theorem 16.17. During the year following the publication of Heawood's paper, Lothar Heffter [127] wrote a paper in which he drew attention to the incomplete nature of Heawood's argument. Heffter was, in fact, able to show that equality held in (16.2) not only for $k = 1$ but for $1 \le k \le 6$ and some other values of k as well. To verify equality in (16.2) for every positive integer k, it would be necessary to show, for every positive integer k, that there is a graph G_k that is embeddable on S_k for which

$$\chi(G_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor.$$

The problem concerning whether equality held in (16.2) for every positive integer k would become known as:

The Heawood Map-Coloring Problem *Determine, for every positive integer k, whether*

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor.$$

There was a great deal of confusion surrounding this famous problem and the origin of this confusion is also unknown. For example, in their well-known book *What is Mathematics*? (published in 1941), Richard Courant and Herbert E. Robbins [62] reported that

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor$$

for every positive integer k. Whether it was the belief that this formula is true which led Courant and Robbins to include this premature statement in their book or whether writing this statement in their book led to mathematical folklore is not known. Indeed, this was not even known to Courant and Robbins. There were reports that the Heawood Map-Coloring Problem had been solved as early as the early 1930s in Göttingen, Germany.

Solving the Heawood Map-Coloring Problem required the work of many mathematicians over a period of 78 years. However, primarily through the efforts of Gerhard Ringel and J. W. T. (Ted) Youngs [204], this problem was finally solved when a formula for the genus of the complete graph was determined (see Theorem 11.18).

Theorem 16.18 (**The Heawood Map-Coloring Theorem**) *For every positive integer k,*

$$\chi(S_k) = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor.$$

Proof. By Theorem 16.17,

$$\chi(S_k) \le \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor.$$

Hence, it remains only to verify the reverse inequality. Define

$$n = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor.$$

Thus, $n \le (7+\sqrt{1+48k})/2$. Solving this inequality for k, we have

$$k \ge (n-3)(n-4)/12$$

and so by Theorem 11.18,

$$k \ge \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil = \gamma(K_n).$$

Therefore, $\gamma(K_n) \le k$, which implies that

$$\chi(S_{\gamma(K_n)}) \le \chi(S_k).$$

Since K_n is clearly embeddable on $S_{\gamma(K_n)}$, it follows that $\chi(S_{\gamma(K_n)}) \ge n$ and so

$$\chi(S_k) \ge n = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor,$$

giving the desired result. ■

Exercises for Chapter 16

Section 16.2: Colorings of Planar Graphs

1. Show that the countries in a map of South America (see Figure 16.24) cannot be colored with three or fewer colors so that every two neighboring countries are colored differently.

Figure 16.24: Map of South America

2. It was once thought that the regions of every map can be colored with four or fewer colors because no map contains five mutually adjacent regions. Show that there exist maps that do not contain four mutually adjacent regions but yet four colors are required to color the regions of these maps.

3. Prove that if G is a maximal outerplanar graph of order 3 or more, then $\chi(G) = 3$.

4. Use the result in Exercise 3 to prove the Art Gallery Theorem: *The walls of an art gallery form the n sides of an n-gon. It is possible to position $\lfloor n/3 \rfloor$ security guards in the art gallery so that every location in the gallery has a straight line view by one of the guards.*

Section 16.3: List Colorings of Planar Graphs

5. It is known that the minimum degree of every induced subgraph of an outerplanar graph is at most 2. Use this fact to prove that every outerplanar graph is 3-choosable.

6. Use the fact that every planar graph contains a vertex of degree 5 or less to prove that every planar graph is 6-choosable.

Section 16.4: The Conjectures of Hajós and Hadwiger

7. Show that $K_{4,4}$ contains no subdivision of K_5.

8. Show that $K_{4,5}$ contains a subdivision of K_5.

9. Show that there exists a 7-chromatic graph that does not contain a subdivision of K_7.

10. Prove that if G is a connected graph of order n and size m that has K_k as a minor, then $m \geq n + \binom{k}{2} - k$.

Section 16.5: Chromatic Polynomials

11. Prove that the chromatic polynomial of every graph can be expressed as the sum and difference of the chromatic polynomials of empty graphs.

12. (a) Determine $P(C_6, \lambda)$ by repeated application of Theorem 16.10.
    (b) Use the polynomial obtained in (a) to determine $P(C_6, 2)$. Explain why this answer is not surprising.

13. (a) Determine $P(K_{2,2,2}, \lambda)$ by repeated application of Theorem 16.10.
    (b) Use the polynomial obtained in (a) to determine $P(K_{2,2,2}, 3)$. Explain why this answer is not surprising.

14. Prove that $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$ for each integer $n \geq 3$.

15. We know that every two trees of the same order are chromatically equivalent.

    (a) Which unicyclic graphs of the same order are chromatically equivalent?

(b) How many distinct chromatic polynomials are there for unicyclic graphs of order $n \geq 3$?

16. Prove that if G is a graph with components $G_1, G_2, \ldots, G_k$, then

$$P(G,\lambda) = \prod_{i=1}^{k} P(G_i,\lambda).$$

17. (a) Prove that if G is a nontrivial connected graph, then $P(G,\lambda) = \lambda g(\lambda)$, where $g(0) \neq 0$.

    (b) Prove that a graph G has exactly k components if and only if

$$P(G,\lambda) = \lambda^k f(\lambda),$$

    where $f(\lambda)$ is a polynomial with $f(0) \neq 0$.

18. Show that if F is a forest of order n with k components, then

$$P(F,\lambda) = \lambda^k(\lambda-1)^{n-k}.$$

19. Prove that if G is a connected graph with blocks $B_1, B_2, \ldots, B_r$, then

$$P(G,\lambda) = \frac{\prod_{i=1}^{r} P(B_i,\lambda)}{\lambda^{r-1}}.$$

20. It has been stated that if G and H are two chromatically equivalent graphs, then G and H have the same order, the same size and the same chromatic number. Show that the converse of this statement is false.

21. Prove for each integer $r \geq 2$ that $K_{r,r}$ is chromatically unique.

22. Let G be a graph. Prove that if $P(G,\lambda) = \lambda(\lambda-1)^{n-1}$, then G is a tree of order n.

23. Prove that if G is a connected graph of order n, then

$$P(G,\lambda) \leq \lambda(\lambda-1)^{n-1}.$$

24. Prove or disprove: The polynomial $\lambda^4 - 3\lambda^3 + 3\lambda^2$ is the chromatic polynomial of some graph.

25. Prove or disprove: The graph G in Figure 16.25 is chromatically unique.

26. Let G be a maximal planar graph of order $n \geq 3$ (embedded in the plane) with chromatic polynomial $P(G,\lambda)$. A plane graph H is obtained from G by placing a new vertex v in each region of G and joining v to the vertices on the boundary of this region. Express $P(H,\lambda)$ in terms of $P(G,\lambda)$.

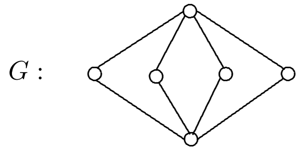


Figure 16.25: The graph G in Exercise 25

Section 16.6: The Heawood Map-Coloring Problem

27. It is known that the Petersen graph P is not planar. Thus, P cannot be embedded on the sphere.
    (a) Show that P can be embedded on the torus.
    (b) How many regions result from a toroidal embedding of P?
    (c) What is the minimum number of colors that can be assigned to the regions in (b) so that every two adjacent regions are colored differently?

28. Prove or disprove: If G is a graph such that $\chi(G) \leq \chi(S_k)$ for some positive integer k, then G can be embedded on S_k.

29. Give an example of a graph G with genus 2 and $\chi(G) = \chi(S_2)$. Verify that your example has these properties.

Chapter 17

Edge Colorings

The graph colorings that we have considered are vertex colorings in Chapters 14 and 15 and map colorings (region colorings of plane graphs) in Chapter 16. With the aid of matchings and 1-factors in Chapter 12, we now consider a third coloring, namely edge colorings. As with vertex colorings where the primary emphasis has been on proper vertex colorings, the customary requirement for edge colorings is that adjacent edges be colored differently, resulting in proper edge colorings. This will be our focus in the current chapter.

17.1 The Chromatic Index of a Graph

An **edge coloring** of a graph G is an assignment of colors to the edges of G, one color to each edge. If adjacent edges are assigned distinct colors, then the edge coloring is a **proper edge coloring**. Proper edge colorings are the most studied edge colorings and it is these colorings that we investigate in this chapter. When we refer to an edge coloring of a graph, we will mean a proper edge coloring unless stated otherwise.

Since a proper edge coloring of a nonempty graph G is a proper vertex coloring of its line graph $L(G)$, edge colorings of graphs is the same subject as vertex colorings of line graphs. Because investigating vertex colorings of line graphs provides no apparent advantage to investigating edge colorings of graphs, we will study this subject strictly in terms of edge colorings.

An edge coloring that uses colors from a set of k colors is a **k-edge coloring**. Thus, a k-edge coloring of a graph G can be described as a function $c : E(G) \to \{1, 2, \ldots, k\}$ such that $c(e) \neq c(f)$ for every two adjacent edges e and f in G. A graph G is **k-edge colorable** if there exists a k-edge coloring of G. In Figure 17.1(a), a 5-edge coloring of a graph H is shown; while in Figures 17.1(b) and 17.1(c), a 4-edge coloring and a 3-edge coloring of H are shown.

As with vertex colorings, we are often interested in edge colorings using a minimum number of colors. The **chromatic index** (or **edge chromatic**

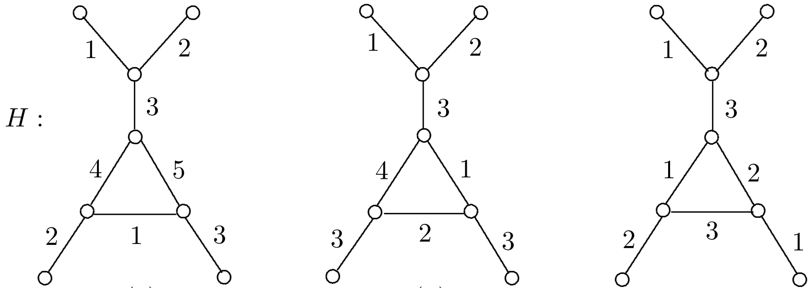


Figure 17.1: Edge colorings of a graph

number) $\chi'(G)$ of a graph G is the minimum positive integer k for which G is k-edge colorable. Furthermore, $\chi'(G) = \chi(L(G))$ for every nonempty graph G.

If a graph G is k-edge colorable for some positive integer k, then $\chi'(G) \leq k$. In particular, since the graph H of Figure 17.1 is 3-edge colorable, $\chi'(H) \leq 3$. On the other hand, since H contains three mutually adjacent edges (indeed, several such sets of three edges), at least three distinct colors are required in any edge coloring of H and so $\chi'(H) \geq 3$. Therefore, $\chi'(H) = 3$.

Let there be given a k-edge coloring of a nonempty graph G using the colors $1, 2, \ldots, k$ and let E_i $(1 \leq i \leq k)$ be the set of edges of G assigned the color i. Then the nonempty sets among $E_1, E_2, \ldots, E_k$ of $E(G)$ are the **edge color classes** of G for the given k-edge coloring. Thus, the nonempty sets in $\{E_1, E_2, \ldots, E_k\}$ produce a partition of $E(G)$ into edge color classes. Since no two adjacent edges of G are assigned the same color in a (proper) edge coloring of G, every nonempty edge color class consists of an independent set of edges of G. Indeed, the chromatic index of G is the minimum number of independent sets of edges into which $E(G)$ can be partitioned.

Recall that the edge independence number (matching number) $\alpha'(G)$ of a nonempty graph G is the maximum number of edges in an independent set of edges of G, that is, the size of a maximum matching in G. Furthermore, if the order of G is n, then $\alpha'(G) \leq n/2$. The following gives a simple yet useful lower bound for the chromatic index of a graph and is an analogue to the lower bound for the chromatic number of a graph presented in Theorem 14.9.

Theorem 17.1 *If G is a graph of size $m \geq 1$, then*

$$\chi'(G) \geq \frac{m}{\alpha'(G)}.$$

Proof. Suppose that $\chi'(G) = k$ and that $E_1, E_2, \ldots, E_k$ are the edge color classes in a k-edge coloring of G. Thus, $|E_i| \leq \alpha'(G)$ for each i $(1 \leq i \leq k)$. Hence,

$$m = |E(G)| = \sum_{i=1}^{k} |E_i| \leq k\alpha'(G)$$

and so $\chi'(G) = k \geq \frac{m}{\alpha'(G)}$. ■

Since every edge coloring of a graph G must assign distinct colors to adjacent edges, it follows for each vertex v of G that $\deg v$ colors must be used to color the edges incident with v. Therefore,

$$\chi'(G) \geq \Delta(G) \tag{17.1}$$

for every nonempty graph G.

In the graph G of order $n = 7$ and size $m = 10$ of Figure 17.2, $\Delta(G) = 3$. Hence, $\chi'(G) \geq 3$ by (17.1). Furthermore, since $X = \{uz, vx, wy\}$ is an independent set of three edges of G and $\alpha'(G) \leq n/2 = 7/2$, it follows that $\alpha'(G) = 3$. However, by Theorem 17.1, $\chi'(G) \geq m/\alpha'(G) = 10/3$ and so we arrive at an even larger lower bound for $\chi'(G)$, namely $\chi'(G) \geq 4$. The 4-edge coloring of G in Figure 17.2 shows that $\chi'(G) \leq 4$ and so $\chi'(G) = 4$.

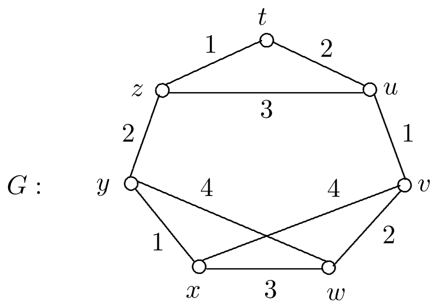


Figure 17.2: A graph with chromatic index 4

Vizing's Theorem

While $\Delta(G)$ is a rather obvious lower bound for the chromatic index of a nonempty graph G, the graph G of Figure 17.2 shows that it's possible that $\chi'(G) > \Delta(G)$. The Russian graph theorist Vadim G. Vizing [246] established a remarkable upper bound for the chromatic index of a graph. Vizing's theorem, published in 1964, must be considered the major theorem in the area of edge colorings. Vizing's theorem was rediscovered in 1966 by Ram Prakash Gupta [110].

Theorem 17.2 (**Vizing's Theorem**) *For every nonempty graph G,*

$$\chi'(G) \leq 1 + \Delta(G).$$

Proof. Suppose that the theorem is false. Then among all those graphs H for which $\chi'(H) \geq 2 + \Delta(H)$, let G be one of minimum size and let $\Delta = \Delta(G)$.

Thus, G is not $(1+\Delta)$-edge colorable. On the other hand, if $e = uv$ is an edge of G, then $G - e$ is $(1 + \Delta(G - e))$-edge colorable. Since $\Delta(G - e) \leq \Delta(G)$, the graph $G - e$ is $(1 + \Delta)$-edge colorable.

Let there be given a $(1+\Delta)$-edge coloring of $G-e$. Hence, with the exception of e, every edge of G is assigned one of $1 + \Delta$ colors such that adjacent edges are colored differently. For each edge $e' = uv'$ of G incident with u (including the edge e), we define the **dual color** of e' as any of the $1+\Delta$ colors that is not used to color the edges incident with v'. (See Figure 17.3.) Since $\deg v' \leq \Delta$, there is always at least one available color for the dual color of the edge uv'. It may occur that distinct edges have the same dual color.

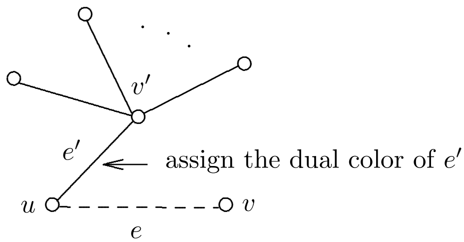


Figure 17.3: A step in the proof of Theorem 17.2

Denote the edge e by $e_0 = uv_0$ as well (where then $v_0 = v$) and suppose that e_0 has dual color α_1. (Thus, α_1 is not the color of any edge incident with v.) Necessarily, some edge $e_1 = uv_1$ incident with u is colored α_1, for otherwise the color α_1 could be assigned to e, producing a $(1 + \Delta)$-edge coloring of G.

Let α_2 be the dual color of e_1. (Thus, no edge incident with v_1 is colored α_2.) If there should be some edge incident with u that is colored α_2, then denote this edge by $e_2 = uv_2$ and let its dual color be denoted by α_3. (See Figure 17.4.) Proceeding in this manner, we then construct a sequence $e_0, e_1, \ldots, e_k$ $(k \geq 1)$ containing a maximum number of distinct edges, where $e_i = uv_i$ for $0 \leq i \leq k$. Consequently, the final edge e_k of this sequence is colored α_k and has dual color α_{k+1}. Therefore, each edge e_i $(0 \leq i \leq k)$ is colored α_i and has dual color α_{i+1}.

We claim that there is some edge incident with u that is colored α_{k+1}. Suppose that this is not the case. Then each of the edges $e_0, e_1, \ldots, e_k$ can be assigned its dual color, producing a $(1 + \Delta)$-edge coloring of G. This, however, is impossible. Thus, as claimed, there is an edge e_{k+1} incident with u that is colored α_{k+1}.

Since the sequence $e_0, e_1, \ldots, e_k$ contains the maximum number of distinct edges, it follows that $e_{k+1} = e_j$ for some j with $1 \leq j \leq k$ and so $\alpha_{k+1} = \alpha_j$. Since the color assigned to e_k is not the same as its dual color, it follows that $\alpha_{k+1} \neq \alpha_k$. Therefore, $1 \leq j < k$. Let $j = t + 1$ for $0 \leq t < k - 1$. Hence, $\alpha_{k+1} = \alpha_{t+1}$ and so e_k and e_t have the same dual color.

There must be a color β used to color an edge incident with v in $G - e$ that is not used to color any edge incident with u. If this were not the case, then there would be $\deg_G u - 1 \leq \Delta - 1$ colors used to color the edges incident with u

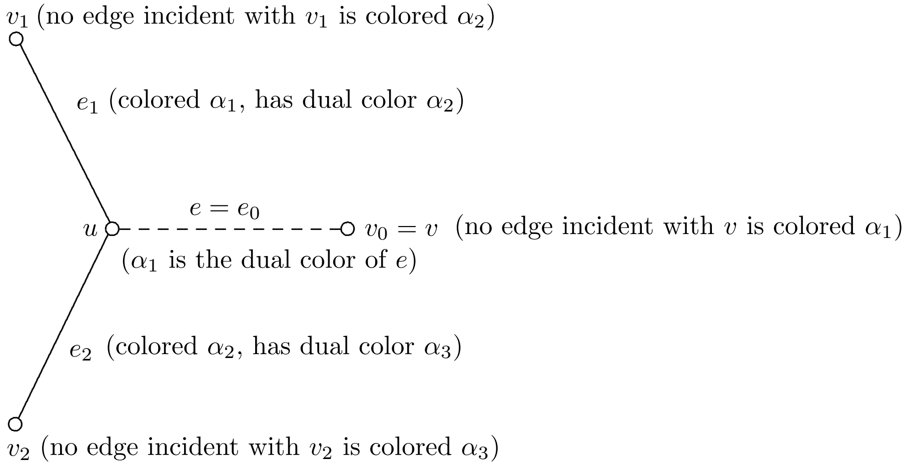


Figure 17.4: A step in the proof of Theorem 17.2

or v, leaving two or more colors available for e. Assigning e one of these colors produces a $(1 + \Delta)$-edge coloring of G, resulting in a contradiction.

The color β must also be assigned to some edge incident with v_i for each i with $1 \le i \le k$. If this were not the case, then there would exist a vertex v_r with $1 \le r \le k$ such that no edge incident with v_r is colored β. However, we could then change the color of e_r to β and color each edge e_i $(0 \le i < r)$ with its dual color to obtain a $(1 + \Delta)$-edge coloring of G, which is impossible.

Let P be a path of maximum length with initial vertex v_k whose edges are alternately colored β and α_{k+1}, and let Q be a path of maximum length with initial vertex v_t whose edges are alternately colored β and $\alpha_{t+1} = \alpha_{k+1}$. Suppose that P is a $v_k - x$ path and Q is a $v_t - y$ path. We now consider four cases depending on whether the vertices x and y belong to the set $\{v_0, v_1, \ldots, v_{k-1}, u\}$.

Case 1. $x = v_r$ *for some integer* r *with* $0 \le r \le k - 1$. Since α_{k+1} is the dual color of e_k, no edge incident with v_k is colored α_{k+1} and so the initial edge of P must be colored β. We have seen for every integer i with $0 \le i \le k$ that there is an edge incident with v_i colored β. Because of the defining property of P, the color of the terminal edge of P cannot be α_{k+1}. This implies that no edge incident with v_r is colored α_{k+1} and so both the initial and terminal edges of P are colored β. Unless $v_r = v_t$, the vertex v_t is not on P as no edge incident with v_t is colored α_{k+1}.

We now interchange the colors β and α_{k+1} of the edges of P. If $r = 0$, then e can be colored β; otherwise, $r > 0$ and no edge incident with v_r is colored β and the dual color of e_i with $0 \le i < r$ is not changed. Then the edge e_r can be colored β and each edge e_i with $0 \le i < r$ can be colored with its dual color. This, however, results in a $(1 + \Delta)$-edge coloring of G, which is impossible.

Case 2. *$y = v_r$ for some integer r with $0 \le r \le k$ where $r \ne t$.* As in Case 1, the initial and terminal edges of Q must also be colored β and no edge incident with v_r is colored α_{k+1}. Furthermore, Q does not contain the vertex v_k unless $v_r = v_k$. We now interchange the colors β and α_{k+1} of the edges of Q. If $r < t$, then we proceed as in Case 1. On the other hand, if $r > t$, we change the color of e to β if $t = 0$; while if $t > 0$, we change the color of e_t to β and color each edge e_i $(0 \le i < t)$ with its dual color. This implies that G is $(1 + \Delta)$-edge colorable, producing a contradiction.

Case 3. *Either* (1) *$x \ne v_r$ for $0 \le r \le k-1$ and $x \ne u$ or* (2) *$y \ne v_r$ for $r \ne t$ and $y \ne u$.* Since (1) and (2) are similar, we consider (1) only. Upon interchanging the colors β and α_{k+1} of the edges of P, the edge incident with v_k is colored β. Furthermore, the dual color of each edge e_i $(0 \le i < k)$ has not been altered. Thus, e_k is colored β and each edge e_i $(0 \le i < k)$ is colored with its dual color, producing a contradiction.

Case 4. *$x = y = u$.* Necessarily, the initial edges of P and Q are colored β and the terminal edges of P and Q are colored α_{k+1}. Since no edge incident with u is colored β, the paths P and Q cannot be edge-disjoint for this would imply that u is incident with two distinct edges having the same color (namely α_{k+1}), which is impossible. Thus, P and Q have the same terminal edge and so there is a first edge f that P and Q have in common. Since f is adjacent to another edge of P and another edge of Q, there are three mutually adjacent edges of G belonging to P or Q and so there are adjacent edges of $G - e$ that are colored the same. Since this is impossible, this case cannot occur. ∎

We now look at a few well-known graphs and classes of graphs and determine their chromatic index. We begin with the cycles. Since the cycle C_n $(n \ge 3)$ is 2-regular, $\chi'(C_n) = 2$ or $\chi'(C_n) = 3$. If n is even, then the edges may be alternately colored 1 and 2, producing a 2-edge coloring of C_n. If n is odd, then $\alpha'(C_n) = (n-1)/2$. Since the size of C_n is n, it follows by Theorem 17.1 that $\chi'(C_n) \ge n/\alpha'(C_n) = 2n/(n-1) > 2$ and so $\chi'(C_n) = 3$. Therefore,

$$\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

We now turn to complete graphs. Since K_n is $(n-1)$-regular, either $\chi'(K_n) = n-1$ or $\chi'(K_n) = n$. If n is even, then it follows by Theorem 13.2 that K_n is 1-factorable, that is, K_n can be factored into $n-1$ 1-factors $F_1, F_2, \ldots, F_{n-1}$. By assigning each edge of F_i $(1 \le i \le n-1)$ the color i, an $(n-1)$-edge coloring of K_n is produced. If n is odd, then $\alpha'(K_n) = (n-1)/2$. Since the size m of K_n is $n(n-1)/2$, it follows by Theorem 17.1 that $\chi'(K_n) \ge m/\alpha'(K_n) = n$. Thus, $\chi'(K_n) = n$. In summary,

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the chromatic index of every nonempty complete graph is an odd integer.

We now consider the complete bipartite graphs $K_{s,t}$ where $1 \le s \le t$. Since $\Delta(K_{s,t}) = t$, it follows that $\chi'(K_{s,t}) = t$ or $\chi'(K_{s,t}) = t + 1$. Suppose that the partite sets of $K_{s,t}$ are $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. For $1 \le i \le s$ and $1 \le j \le t$, assign the color $j - i + 1$ to the edge $u_i w_j$, where $j - i + 1$ is expressed modulo t (see Figure 17.5 for $s = 3$ and $t = 4$). Since this is a proper t-edge coloring of $K_{s,t}$, it follows that $\chi'(K_{s,t}) = t$.

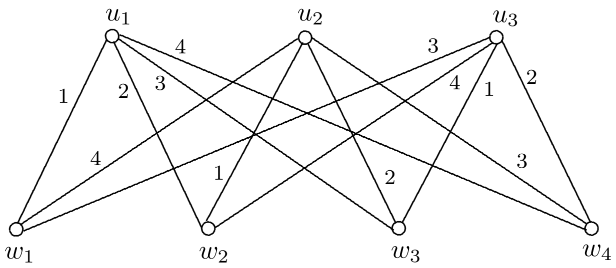


Figure 17.5: A 4-edge coloring of $K_{3,4}$

While the chromatic number of a multigraph is the same as the chromatic number of its underlying graph, quite obviously this is not true for the chromatic index. Theorem 17.1 holds for multigraphs as well as graphs, however. That is, if G is a multigraph of size $m \ge 1$, then

$$\chi'(G) \ge \frac{m}{\alpha'(G)}. \tag{17.2}$$

For a multigraph G, we write $\mu(G)$ for the **maximum multiplicity** or the **strength** of G, which is the maximum number of parallel edges joining the same pair of vertices of G. Vizing [246] and, independently, Gupta [110] found an upper bound for $\chi'(G)$ in terms of $\Delta(G)$ and $\mu(G)$.

Theorem 17.3 *For every nonempty multigraph G,*

$$\chi'(G) \le \Delta(G) + \mu(G).$$

Shannon's Theorem

For a graph G, Theorem 17.3 reduces to $\chi'(G) \le \Delta(G)+1$, which is Vizing's theorem (Theorem 17.2). Claude Elwood Shannon found an upper bound for the chromatic index of a multigraph G in terms of $\Delta(G)$ alone [221].

Theorem 17.4 (**Shannon's Theorem**) *If G is a multigraph, then*

$$\chi'(G) \le \frac{3\Delta(G)}{2}.$$

Proof. Suppose that the theorem is false. Among all multigraphs H with $\chi'(H) > 3\Delta(H)/2$, let G be one of minimum size. Let $\Delta(G) = \Delta$ and $\mu(G) = \mu$. Suppose that $\chi'(G) = k$. Hence, $\chi(G - f) = k - 1$ for every edge f of G. By Theorem 17.3, $k \leq \Delta + \mu$ and, by assumption, $k > 3\Delta/2$.

Suppose that μ edges join the vertices u and v. Let e be one of these edges. Since $\chi'(G - e) = k - 1$, there exists a $(k-1)$-edge coloring of $G - e$. The number of colors not used in coloring the edges incident with u is at least $(k-1) - (\Delta - 1) = k - \Delta$. Similarly, the number of colors not used in coloring the edges incident with v is at least $k - \Delta$ as well.

Each of these $k - \Delta$ or more colors not used to color an edge incident with u must be used to color an edge incident with v, for otherwise there is a color available for e, contradicting our assumption that $\chi'(G) = k$. Similarly, each of the $k - \Delta$ or more colors not used to color an edge incident with v must be used to color an edge incident with u. Hence, the total number of colors used to color the edges incident with u or v is at least $2(k - \Delta) + \mu - 1$ and so

$$2(k - \Delta) + \mu - 1 \leq k - 1.$$

Since $3\Delta/2 < k \leq \Delta + \mu$, it follows that $\mu > \Delta/2$ and so

$$2(k - \Delta) + (\Delta/2) - 1 < 2(k - \Delta) + \mu - 1 \leq k - 1.$$

Therefore,

$$2k - (3\Delta/2) - 1 < k - 1,$$

implying that $k < 3\Delta/2$, which is a contradiction. ■

There are occasions when the upper bound for the chromatic index of a multigraph given by Shannon's theorem (Theorem 17.4) is an improvement over that provided by Theorem 17.3. For the multigraph G of Figure 17.6(a), $\Delta(G) = 7$ and $\mu(G) = 4$. Thus, $\chi(G) \leq 11$ by Theorem 17.3. By Shannon's theorem, $\chi'(G) \leq (3 \cdot 7)/2$ and so $\chi'(G) \leq 10$. Since the size of G is 16 and at most two edges of G can be assigned the same color, it follows by Theorem 17.1 for multigraphs (17.2) that $\chi'(G) \geq 16/2 = 8$. In fact, $\chi'(G) = 8$ as the 8-edge coloring of G in Figure 17.6(b) shows.

17.2 Class One and Class Two Graphs

By Vizing's theorem (Theorem 17.2), it follows that for every nonempty graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = 1 + \Delta(G)$. A graph G is said to belong to or is of **Class one** if $\chi'(G) = \Delta(G)$ and is of **Class two** if $\chi'(G) = 1 + \Delta(G)$. Consequently, a major question in the area of edge colorings is that of determining to which of these two classes a given graph belongs. This is far from being an easy question to answer in general, however, as Ian Holyer [130] showed that the problem of determining the chromatic index of an arbitrary graph is **NP**-complete. In fact, the problem is **NP**-complete even for cubic graphs.

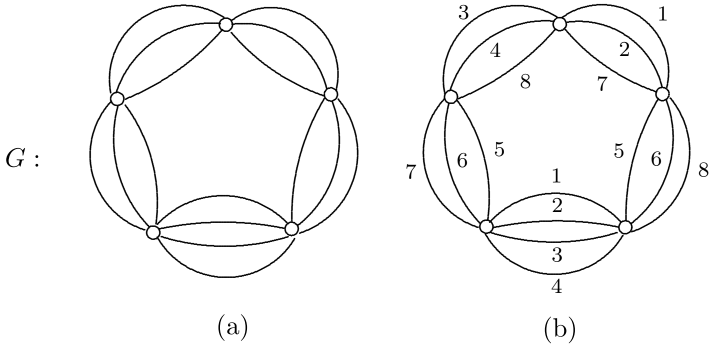


Figure 17.6: A multigraph G with $\Delta(G) = 7$, $\mu(G) = 4$ and $\chi'(G) = 8$

We saw in the preceding section that every complete bipartite graph is of Class one and that cycles and complete graphs are of Class one if their orders are even and are of Class two if their orders are odd. Of course, both the cycles and complete graphs are regular graphs. For an r-regular graph G, either $\chi'(G) = r$ or $\chi'(G) = r + 1$. If $\chi'(G) = r$, then there is an r-edge coloring of G, resulting in r color classes $E_1, E_2, \ldots, E_r$. Since every vertex v of G has degree r, the vertex v is incident with exactly one edge in each set E_i $(1 \leq i \leq r)$. Therefore, each color class E_i is a perfect matching and G is 1-factorable. Conversely, if G is 1-factorable, then $\chi'(G) = r$.

Theorem 17.5 *A regular graph G is of Class one if and only if G is* 1-*factorable.*

Since the Petersen graph P is not 1-factorable, it is of Class two and so $\chi'(P) = 4$. The formulas mentioned above for the chromatic index of cycles and complete graphs are immediate consequences of Theorem 17.5, as is the following.

Corollary 17.6 *Every regular graph of odd order is of Class two.*

König's Theorem

We have already observed that the even cycles are of Class one. The four graphs shown in Figure 17.7 are of Class one as well. The even cycles and the four graphs of Figure 17.7 are all bipartite. These graphs serve to illustrate a theorem of Denés König [146] (see Exercise 13).

Theorem 17.7 (**König's Theorem**) *Every nonempty bipartite graph is of Class one.*

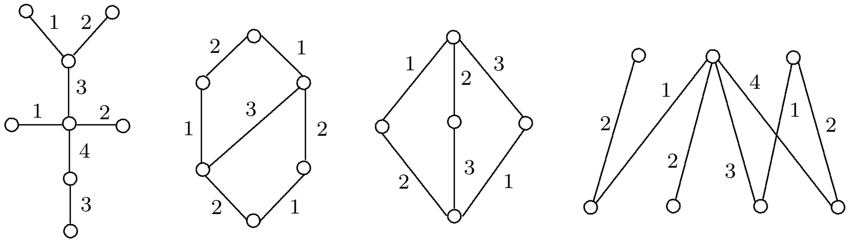


Figure 17.7: Some Class one graphs

Proof. Suppose that the theorem is false. Then among the bipartite graphs that are of Class two, let G be one of minimum size. Thus, G is a bipartite graph such that $\chi'(G) = \Delta(G) + 1$. Let $e \in E(G)$, where $e = uv$. Necessarily, u and v belong to different partite sets of G. Then $\chi'(G - e) = \Delta(G - e)$. Now $\Delta(G - e) = \Delta(G)$, for otherwise, G is $\Delta(G)$-edge colorable.

Let there be given a $\Delta(G)$-edge coloring of $G - e$. Each of the $\Delta(G)$ colors must be assigned to an edge incident either with u or with v in $G - e$; for otherwise, this color could be assigned to e producing a $\Delta(G)$-edge coloring of G. Because $\deg_{G-e} u < \Delta(G)$ and $\deg_{G-e} v < \Delta(G)$, there is a color α of the $\Delta(G)$ colors not used in coloring the edges of $G - e$ incident with u and a color β of the $\Delta(G)$ colors not used in coloring the edges of $G - e$ incident with v. Then $\alpha \neq \beta$ and, furthermore, some edge incident with v is colored α and some edge incident with u is colored β.

Let P be a path of maximum length having initial vertex v whose edges are alternately colored α and β. The path P cannot contain u, for otherwise, P must terminate at u and so P has odd length. This, however, implies that the initial and terminal edges of P are both colored α, which is impossible since u is incident with no edge colored α. Interchanging the colors α and β of the edges of P produces a new $\Delta(G)$-edge coloring of $G - e$ in which neither u nor v is incident with an edge colored α. Assigning the color α to e produces a $\Delta(G)$-edge coloring of G, which is a contradiction. ■

An immediate consequence of Theorem 17.7 is that for integers s and t with $1 \leq s \leq t$, $\chi'(K_{s,t}) = t$, as we observed in the preceding section. We have seen that if G is a graph of size m, then any partition of $E(G)$ into independent sets must contain at least $\frac{m}{\alpha'(G)}$ sets. If v is a vertex with $\deg v = \Delta(G)$, then each of the $\Delta(G)$ edges incident with v must belong to distinct independent sets. Thus, $\frac{m}{\alpha'(G)} \geq \Delta(G)$ and so $m \geq \alpha'(G) \cdot \Delta(G)$. If $m > \alpha'(G) \cdot \Delta(G)$, then we can say more.

Theorem 17.8 *If G is a graph of size m such that*

$$m > \alpha'(G) \cdot \Delta(G),$$

then G is of Class two.

Proof. By Theorem 17.1, $\chi'(G) \ge \frac{m}{\alpha'(G)}$. Thus,

$$\chi'(G) \ge \frac{m}{\alpha'(G)} > \frac{\alpha'(G) \cdot \Delta(G)}{\alpha'(G)} = \Delta(G),$$

which implies that $\chi'(G) = 1 + \Delta(G)$ and so G is of Class two. ■

If G is a graph of order n, then $\alpha'(G) \le \lfloor \frac{n}{2} \rfloor$. Therefore, the largest possible value of $\alpha'(G) \cdot \Delta(G)$ is $\lfloor \frac{n}{2} \rfloor \cdot \Delta(G)$. A graph G of order n and size m is called **overfull** if $m > \lfloor \frac{n}{2} \rfloor \cdot \Delta(G)$. If n is even, then $\lfloor n/2 \rfloor = n/2$ and

$$2m = \sum_{v \in V(G)} \deg v \le n\Delta(G).$$

Therefore, $m \le (n/2) \cdot \Delta(G) = \lfloor \frac{n}{2} \rfloor \cdot \Delta(G)$ and so G is not overfull. Thus, no graph of even order is overfull.

Since $\alpha'(G) \le \lfloor \frac{n}{2} \rfloor$ for every graph G of order n, Theorem 17.8 has an immediate corollary (see Exercise 15).

Corollary 17.9 *Every overfull graph is of Class two.*

Applications of Edge Colorings

We now look at two problems whose solutions involve edge colorings.

Example 17.10 *Seven countries, namely Argentina (A), Brazil (B), Chile (C), France (F), Germany (G), Italy (I) and Spain (S), have been invited to send their soccer teams to participate in a tournament. The schedule calls for Argentina to play all other teams except Brazil and Chile. France will play all other teams except Germany and Italy, while Spain plays all other teams except Italy. In general, all pairs of teams play each other except for the five pairs of exceptions noted above. If no team is to play two matches on the same day, what is the minimum number of days needed to schedule this tournament?*

Solution. We construct a graph H with $V(H) = \{A, B, C, F, G, I, S\}$ whose vertices are the seven soccer teams. Two vertices (soccer teams) x and y are adjacent in H if they play a soccer match against each other. The graph H is shown in Figure 17.8. The answer to the question is the chromatic index of H. The order of H is $n = 7$ and the degrees of its vertices are $5, 5, 5, 5, 4, 4, 4$. Thus, $\Delta(H) = 5$ and the size of H is $m = 16$. Since

$$16 = m > \left\lfloor \frac{n}{2} \right\rfloor \cdot \Delta(H) = 5 \cdot 3 = 15,$$

the graph H is overfull. By Corollary 17.9, H is of Class two and so $\chi(H) = 1 + \Delta(H) = 6$. A 6-edge coloring of H is also shown in Figure 17.8. This

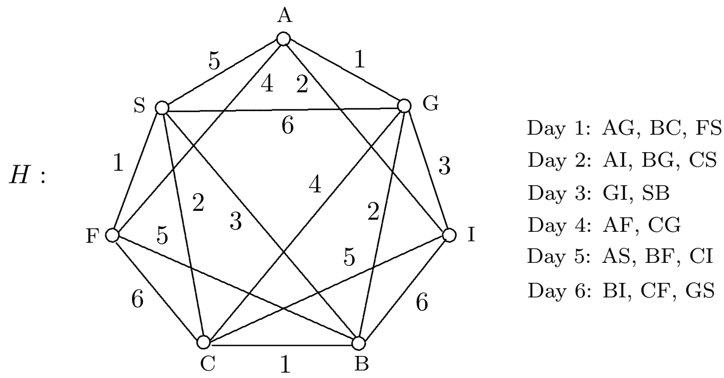


Figure 17.8: The graph H in Example 17.10 and a 6-edge coloring of H

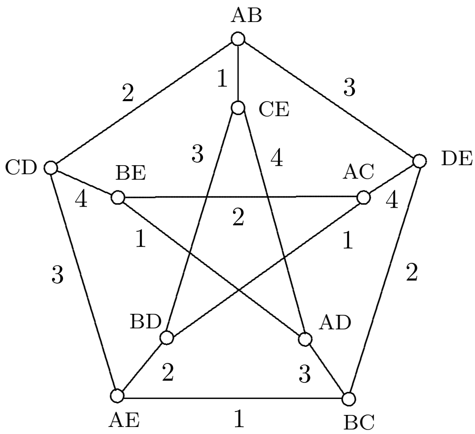
provides us with a possible schedule for the soccer tournament taking place over a minimum of six days. ♦

Example 17.11 *One year it is decided to have a charity tennis tournament consisting entirely of doubles matches. Five tennis players (denoted by A, B, C, D, E) have agreed to participate. Each pair $\{W, X\}$ of tennis players will play a match against every other pair $\{Y, Z\}$ of tennis players, where then $\{W, X\} \cap \{Y, Z\} = \emptyset$, but no 2-person team is to play two matches on the same day. What is the minimum number of days needed to schedule such a tournament? Give an example of such a tournament using a minimum number of days.*

Solution. We construct a graph G whose vertex set is the set of 2-element subsets of $\{A, B, C, D, E\}$. Thus, the order of G is $\binom{5}{2} = 10$. Two vertices $\{W, X\}$ and $\{Y, Z\}$ are adjacent if these sets are disjoint. The graph G is shown in Figure 17.9. Thus, G is the Petersen graph. To answer the question, we determine the chromatic index of G, which we have seen is 4. A 4-edge coloring of G is given in Figure 17.9 together with a possible schedule of tennis matches over a period of four days. ♦

Overfull Subgraphs

As we have seen, the size of a graph G of Class one and having order n cannot exceed $\lfloor \frac{n}{2} \rfloor \cdot \Delta(G)$. Since the size of any overfull graph of order n exceeds this number, every overfull graph is therefore of Class two. There are related subgraphs such that if a graph G should contain one of these, then G must also be of Class two.



Day 1: AB-CE, AC-BD, AE-BC, AD-BE
Day 2: AB-CD, AC-BE, AE-BD, BC-DE
Day 3: AB-DE, AD-BC, AE-CD, BD-CE
Day 4: AC-DE, AD-CE, BE-CD

Figure 17.9: The Petersen graph G in Example 17.11 and a 4-edge coloring of G

A subgraph H of odd order n' and size m' of a graph G is an **overfull subgraph** of G if

$$m' > \left\lfloor \frac{n'}{2} \right\rfloor \cdot \Delta(G) = \frac{n'-1}{2} \cdot \Delta(G).$$

Actually, if H is an overfull subgraph of G, then $\Delta(H) = \Delta(G)$ (see Exercise 16). This says that H is itself of Class two. Not only is an overfull subgraph of a graph G of Class two, G itself is of Class two.

Theorem 17.12 *Every graph having an overfull subgraph is of Class two.*

Proof. Let H be an overfull subgraph of a graph G. As we noted, $\Delta(H) = \Delta(G)$ and H is of Class two; so $\chi'(H) = 1 + \Delta(H)$. Thus,

$$\chi'(G) \geq \chi'(H) = 1 + \Delta(H) = 1 + \Delta(G)$$

and so $\chi'(G) = 1 + \Delta(G)$. ■

In the definition of an overfull subgraph H of order n' and size m' in a graph G, we have $m' > \frac{n'-1}{2} \cdot \Delta(G)$. As we saw in Theorem 17.12, this implies that G is of Class two. On the other hand, if G should contain a subgraph H of order n' and size m' such that $m' > \frac{n'-1}{2} \cdot \Delta(H)$, where $\Delta(H) < \Delta(G)$, then H is an overfull graph and so H is of Class two. This need not imply that G is of Class two, however. For example, $C_5 \square K_2$ is of Class one but C_5 is of Class two.

While every graph containing an overfull subgraph must be of Class two, a graph can be of Class two without containing any overfull subgraph. The Petersen graph P (which is 3-regular of order 10) contains no overfull subgraph; yet we saw that P is of Class two.

The following result provides a useful property of overfull subgraphs of a graph.

Theorem 17.13 *Let G be an r-regular graph of even order $n = 2k$, where $\{V_1, V_2\}$ is a partition of $V(G)$ such that $|V_1| = n_1$ and $|V_2| = n_2$ are odd. If $G_1 = G[V_1]$ is an overfull subgraph of G, then $G_2 = G[V_2]$ is also an overfull subgraph of G. Furthermore, if k is odd, then $r < k$; while if k is even, then $r < k - 1$.*

Proof. Let the size of G_i be m_i for $i = 1, 2$ and let m' denote the number of edges in the set $[V_1, V_2]$. Thus,

$$rn_1 = 2m_1 + m' \text{ and } rn_2 = 2m_2 + m'.$$

Since G_1 is overfull, $m_1 > \left(\frac{n_1-1}{2}\right) r$. We show that $m_2 > \left(\frac{n_2-1}{2}\right) r$. Now,

$$m_2 = \frac{rn_2 - m'}{2} = \frac{r(n - n_1) - (rn_1 - 2m_1)}{2} = \frac{rn}{2} - rn_1 + m_1.$$

Since $2m_1 > rn_1 - r$, it follows that

$$\begin{aligned} 2m_2 &= rn - 2rn_1 + 2m_1 > rn - 2rn_1 + rn_1 - r \\ &= r(n - n_1 - 1) = r(n_2 - 1). \end{aligned}$$

Hence, $m_2 > \left(\frac{n_2-1}{2}\right) r$ and G_2 is overfull. Therefore, both G_1 and G_2 are overfull.

Now, either $n_1 \le k$ or $n_2 \le k$, say the former. If k is even, then $n_1 \le k - 1$. Since G_1 is overfull,

$$m_1 > \left(\frac{n_1-1}{2}\right) r \text{ and so } 2m_1 > (n_1 - 1)r.$$

Since the size m_1 of G_1 is at most $\binom{n_1}{2}$, it follows that

$$2\binom{n_1}{2} \ge 2m_1 > (n_1 - 1)r \text{ and so } n_1(n_1 - 1) > (n_1 - 1)r.$$

Consequently, $r < n_1 \le k - 1$ and so $r < k - 1$. If k is odd, then $r < k$. ■

The Overfull Conjecture

Amanda G. Chetwynd and Anthony J. W. Hilton [52] conjectured that the only way that a graph G with sufficiently large minimum degree can be of Class two is for G to contain an overfull subgraph.

The Overfull Conjecture *Let G be a graph of order n such that $\Delta(G) > n/3$. Then G is of Class two if and only if G contains an overfull subgraph.*

Let v be a vertex of the Petersen graph P. Then $P - v$ has order $n = 9$ and $\Delta(P - v) = \frac{n}{3} = 3$. Even though $P - v$ is of Class two, it has no overfull subgraph. Hence, if the Overfull Conjecture is true, the resulting theorem cannot be improved in general.

We encountered the following conjecture in Chapter 13.

The 1-Factorization Conjecture *If G is an r-regular graph of even order n such that (1) $r \geq n/2$ if $n \equiv 2 \pmod 4$ or (2) $r \geq (n-2)/2$ if $n \equiv 0 \pmod 4$, then G is 1-factorable.*

In Theorem 13.3, we saw that this conjecture is true for sufficiently large even integers n. This theorem is restated below, this time in terms of edge colorings.

Theorem 17.14 *If G is an r-regular graph of sufficiently large even order n such that (1) $r \geq n/2$ if $n \equiv 2 \pmod 4$ or (2) $r \geq (n-2)/2$ if $n \equiv 0 \pmod 4$, then G is of Class one.*

If the Overfull Conjecture is true, then the 1-Factorization Conjecture is true for *all* positive even integers n.

Theorem 17.15 *If the Overfull Conjecture is true, then so too is the 1-Factorization Conjecture.*

Proof. Assume, to the contrary, that the Overfull Conjecture is true but the 1-Factorization Conjecture is false. Then either (1) there exists an r-regular graph G of order n where $n \equiv 2 \pmod 4$ with $r \geq n/2$ such that G is not 1-factorable or (2) there exists an r-regular graph G of order n where $n \equiv 0 \pmod 4$ with $r \geq (n-2)/2$ such that G is not 1-factorable. Thus, G is of Class two. Since $\Delta(G) > n/3$, it follows by the Overfull Conjecture that G contains an overfull subgraph G_1.

Let G_2 be the subgraph of G induced by $V(G) - V(G_1)$. By Theorem 17.13, G_2 is also overfull. At least one of G_1 and G_2 has order at most $n/2$. Suppose that G_1 has order at most $n/2$. Again, by Theorem 17.13, if $n/2$ is odd, then $r < n/2$; while if $n/2$ is even, then $r < (n/2) - 1$. Since (1) $r \geq n/2$ if $n/2$ is odd and (2) $r \geq (n-2)/2$ if $n/2$ is even, a contradiction is produced. ■

17.3 Tait Colorings

The Scottish physicist and mathematician Peter Guthrie Tait (1831–1901) was one of many individuals who played a role in the story of the Four Color Problem (see Chapter 16). Tait became acquainted with the Four Color Problem through Arthur Cayley and became interested in Alfred Bray Kempe's solution of the problem. In fact, Tait felt that Kempe's solution was too lengthy and came up with several solutions of his own. Unfortunately, as in the case of Kempe's solution, none of Tait's solutions proved to be correct. Despite this, one of his attempted proofs contained an interesting and useful idea. Tait's idea was to consider coloring the boundary lines of cubic maps. In fact, he stated as a lemma that:

> *The boundary lines of every cubic map can always be colored with three colors so that the three lines at each meeting point are colored differently.*

Tait also mentioned that this lemma could be easily proved and showed how the lemma could be used to prove the Four Color Theorem. Although Tait was correct that this lemma could be used to to prove the Four Color Theorem, he was incorrect when he said that the lemma could be easily proved. Indeed, as it turned out, this lemma is equivalent to the Four Color Theorem [232] and, of course, was equally difficult to prove.

Tait's Theorem

In the proof of the following theorem, we use the so-called Klein four-group which is the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$ of two copies of the cyclic group of order 2.

Theorem 17.16 (**Tait's Theorem**) *A bridgeless cubic plane graph G is 4-region colorable if and only if G is 3-edge colorable.*

Proof. Suppose first that G is 4-region colorable. Let a 4-region coloring of G be given, where the colors are the four elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, each color can be expressed as (a, b) or ab, where $a, b \in \{0, 1\}$. The four colors used are then $c_0 = 00$, $c_1 = 01$, $c_2 = 10$ and $c_3 = 11$. Addition of colors is defined as coordinate-wise addition in $\mathbb{Z}_2$. For example, $c_1 + c_3 = 01 + 11 = 10 = c_2$. Since every element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is self-inverse, the sum of two distinct elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is never $c_0 = 00$.

We now define an edge coloring of G. Since G is bridgeless, each edge e lies on the boundary of two distinct regions. Define the color of e as the sum of the colors of the two regions having e on their boundary. Thus, no edge of G is assigned the color c_0 and a 3-edge coloring of G is produced. It remains to show that this 3-edge coloring is proper. Let e_1 and e_2 be two adjacent edges of G and let v be the vertex incident with e_1 and e_2. Then v is incident with a third edge e_3 as well. For $1 \le i < j \le 3$, suppose that e_i and e_j are on the boundary of the region R_{ij}. Since the colors of R_{13} and R_{23} are different, the sum of the colors of R_{13} and R_{23} and the sum of the colors of R_{12} and R_{23} are different and so this 3-edge coloring is proper.

We now turn to the converse. Suppose that G is 3-edge colorable. Let a 3-edge coloring of G be given using the colors $c_1 = 01$, $c_2 = 10$ and $c_3 = 11$, as described above. This produces a partition of $E(G)$ into three perfect matchings E_1, E_2 and E_3, where E_i $(1 \le i \le 3)$ is the set of edges colored c_i.

Let G_1 be the spanning subgraph of G with edge set $E(G_1) = E_1 \cup E_2$ and let G_2 be the spanning subgraph of G with edge set $E(G_2) = E_2 \cup E_3$. Thus, both G_1 and G_2 are 2-regular spanning subgraphs of G and so each of G_1 and G_2 is the disjoint union of even cycles. For $i = 1, 2$, every region of G_i is the union of regions of G. For each cycle C in the graph G_i $(i = 1, 2)$, every region of G either lies interior or exterior to C. Furthermore, since G is bridgeless, each edge of G_i $(i = 1, 2)$ belongs to a cycle C' of G_i and is on the boundary of two distinct regions of G_i, one of which lies interior to C' and the other exterior to C'.

We now define a 4-region coloring of G. We assign to a region R of G the color a_1a_2 with $a_i \in \{0, 1\}$ for $i = 1, 2$, where $a_i = 0$ if R lies interior to an even number of cycles in G_i and $a_i = 1$ otherwise. Figure 17.10(a) shows a 3-edge coloring of a cubic graph G, Figure 17.10(b) shows the cycles of G_1, Figure 17.10(c) shows the cycles of G_2 and Figure 17.10(d) shows the resulting 4-region coloring of G as defined above.

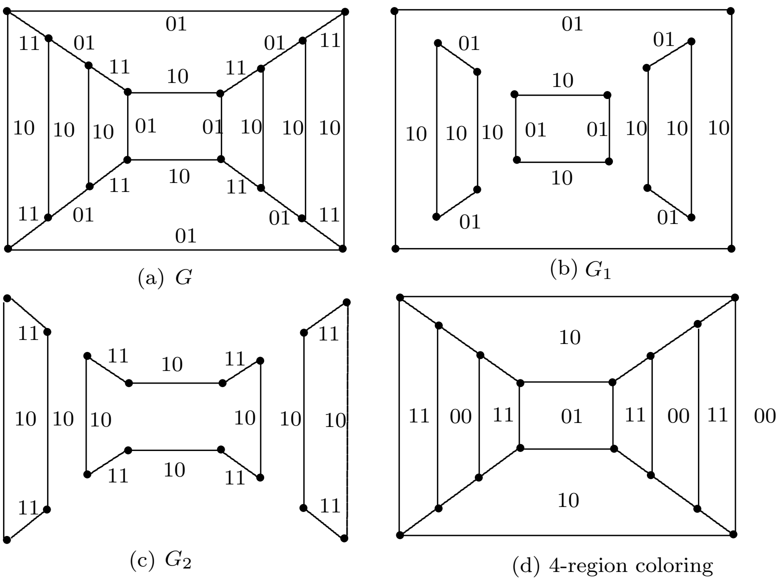


Figure 17.10: A step in the proof of Theorem 17.16

It remains to show that this 4-region coloring of G is proper, that is, that every two adjacent regions of G are assigned different colors. Let R_1 and R_2 be two adjacent regions of G. Thus, there is an edge e that lies on the boundary of both R_1 and R_2. If e is colored c_1 or c_2, then e lies on a cycle in G_1; while if e is colored c_2 or c_3, then e lies on a cycle in G_2. Thus, e lies on a cycle in G_1, on a cycle in G_2 or on a cycle in both G_1 and G_2. Let C be a cycle in G_1, say, containing the edge e. Exactly one of R_1 and R_2 lies interior to C; while for every other cycle C' of G_1, either R_1 and R_2 are both interior to C' or both exterior to C'. Hence, the first coordinates of the colors of R_1 and R_2 are different. Therefore, the colors of every two adjacent regions of G differ in the first coordinate or the second coordinate or both. Hence, this 4-region coloring of G is proper. ∎

Eventually, proper 3-edge colorings of cubic graphs became known as **Tait colorings**. Certainly, a cubic graph G has a Tait coloring if and only if G is 1-factorable. Every Hamiltonian cubic graph necessarily has a Tait coloring, for the edges of a Hamiltonian cycle can be alternately colored 1 and 2, with the remaining edges (constituting a perfect matching) colored 3. Tait believed that every 3-connected cubic planar graph is Hamiltonian. If Tait was correct, then this would mean that every 3-connected cubic planar graph is 3-edge colorable. However, as we are about to see, this implies that every 2-connected cubic planar graph is 3-edge colorable. But the 2-connected cubic graphs are precisely the connected bridgeless cubic graphs and so by Tait's theorem (Theorem 17.16), the Four Color Conjecture would be true.

Theorem 17.17 *If every 3-connected cubic planar graph is 3-edge colorable, then every 2-connected cubic planar graph is 3-edge colorable.*

Proof. Suppose that the statement is false. Then all 3-connected cubic planar graphs are 3-edge colorable but there exist cubic planar graphs having connectivity 2 that are not 3-edge colorable. Among the cubic planar graphs having connectivity 2 that are not 3-edge colorable, let G be one of minimum order n. Certainly n is even and since there is no such graph of order 4, it follows that $n \geq 6$. As we saw in Theorem 4.6, $\kappa(G) = \lambda(G) = 2$. This implies that every minimum edge-cut of G consists of two nonadjacent edges of G.

Let $\{u_1v_1, x_1y_1\}$ be a minimum edge-cut of G. Thus, the vertices u_1, v_1, x_1 and y_1 are distinct and G has the appearance shown in Figure 17.11, where F_1 and H_1 are the two components of $G - u_1v_1 - x_1y_1$.

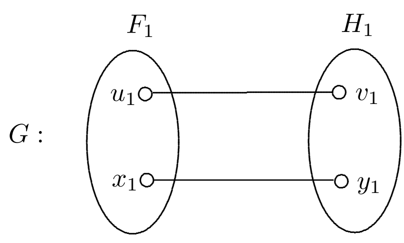


Figure 17.11: A step in the proof of Theorem 17.17

Suppose that $u_1x_1, v_1y_1 \notin E(G)$. Then $F_1 + u_1x_1$ and $H_1 + v_1y_1$ are 2-connected cubic planar graphs of order less than n and so are 3-colorable. Let 3-edge colorings of $F_1 + u_1x_1$ and $H_1 + v_1y_1$ be given using the colors 1, 2 and 3. Now permute the colors of the edges in both $F_1 + u_1x_1$ and $H_1 + v_1y_1$, if necessary, so that both u_1x_1 and v_1y_1 are assigned the color 1. Deleting the edges u_1x_1 and v_1y_1, adding the edges u_1v_1 and x_1y_1 and assigning both u_1v_1 and x_1y_1 the color 1 results in a 3-edge coloring of G, which is impossible.

Thus, we may assume that at least one of u_1x_1 and v_1y_1 is an edge of G, say $u_1x_1 \in E(G)$. Then u_1 is adjacent to a vertex u_2 in F_1 that is different from x_1, and x_1 is adjacent to a vertex x_2 in F_1 that is different from u_1. Since $\kappa(G) = 2$, it follows that $u_2 \neq x_2$. We then have the situation shown in Figure 17.12, where $\{u_2u_1, x_2x_1\}$ is a minimum edge-cut of G, and F_2 and H_1 are the two components of $G - u_1 - x_1$.

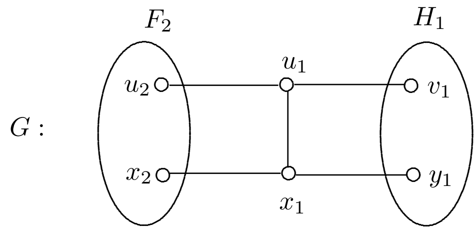


Figure 17.12: A step in the proof of Theorem 17.17

Suppose that $u_2x_2, v_1y_1 \notin E(G)$. Then $F_2 + u_2x_2$ and $H_1 + v_1y_1$ are 2-connected cubic planar graphs of order less than n and so are 3-edge colorable. Let 3-edge colorings of $F_2 + u_2x_2$ and $H_1 + v_1y_1$ be given using the colors 1, 2 and 3. Now permute the colors of the edges in both $F_2 + u_2v_2$ and $H_1 + v_1y_1$, if necessary, so that u_2x_2 is colored 2 and v_1y_1 is colored 1. Deleting the edges u_2x_2 and v_1y_1, assigning u_1u_2 and x_1x_2 the color 2, assigning u_1v_1 and x_1y_1 the color 1 and assigning u_1x_1 the color 3 produces a 3-edge coloring of G, which is a contradiction.

We may therefore assume that at least one of u_2x_2 and v_1y_1 is an edge of G. Continuing in this manner, we have a sequence

$$\{u_1, x_1\},\ \{u_2, x_2\},\ \ldots,\ \{u_k, x_k\}\ (k \geq 1)$$

of pairs of vertices of F_1 such that $u_kx_k \notin E(G)$ and $u_ix_i \in E(G)$ for $1 \leq i < k$ and a sequence

$$\{v_1, y_1\},\ \{v_2, y_2\},\ \ldots,\ \{v_\ell, y_\ell\}\ (\ell \geq 1)$$

of pairs of vertices of H_1 such that $v_\ell y_\ell \notin E(G)$ and $v_iy_i \in E(G)$ for $1 \leq i < \ell$, as shown in Figure 17.13, where F_k and H_ℓ are the two components of

$$G - (\{u_1, \ldots, u_{k-1}\} \cup \{x_1, \ldots, x_{k-1}\} \cup \{v_1, \ldots, v_{\ell-1}\} \cup \{y_1, \ldots, y_{\ell-1}\}).$$

Since $F_k + u_kx_k$ and $H_\ell + v_\ell y_\ell$ are 2-connected cubic planar graphs of order less than n, each is 3-edge colorable. Let 3-edge colorings of $F_k + u_kx_k$ and $H_\ell + v_\ell y_\ell$ be given using the colors 1, 2 and 3. Permute the colors 1, 2 and 3 of the edges in both $F_k + u_kx_k$ and $H_\ell + v_\ell y_\ell$, if necessary, so that both u_kx_k and $v_\ell y_\ell$ are colored 1 if k and ℓ are of the same parity and u_kx_k is colored 2 and $v_\ell y_\ell$ is colored 1 if k and ℓ are of the opposite parity. Deleting the edges u_kx_k and $v_\ell y_\ell$, alternating the colors 1 and 2 along the paths

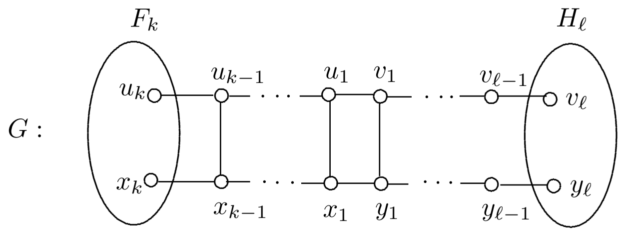


Figure 17.13: A step in the proof of Theorem 17.17

$$(v_\ell, v_{\ell-1}, \ldots, v_1, u_1, u_2, \ldots, u_k) \text{ and } (y_\ell, y_{\ell-1}, \ldots, y_1, x_1, x_2, \ldots, x_k),$$

and assigning the color 3 to the edges

$$u_1x_1,\ u_2x_2,\ \ldots,\ u_kx_k,\ v_1y_1,\ v_2y_2,\ \ldots,\ v_\ell y_\ell$$

produces a 3-edge coloring of G, again a contradiction. ■

Class One Planar Graphs

As a consequence of Theorems 17.16 and 17.17, every bridgeless cubic planar graph is 4-colorable – *provided* Tait was correct that every 3-connected cubic planar graph is Hamiltonian. As we mentioned in Section 10.4, the Tutte graph, shown again in Figure 17.14, is a 3-connected cubic graph that is not Hamiltonian.

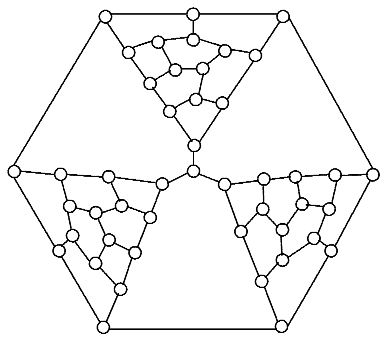


Figure 17.14: The Tutte graph

Despite the fact that Tait was wrong and that 3-edge colorability of bridgeless cubic planar graphs was never used to prove the Four Color Theorem, the eventual verification of the Four Color Theorem did instead show that every bridgeless cubic planar graph is 3-edge colorable.

Corollary 17.18 *Every bridgeless cubic planar graph is of Class one.*

An immediate consequence of Corollary 17.18 is that there are planar graphs G of Class one with $\Delta(G) = 3$. In fact, there are planar graphs G of Class two with $\Delta(G) = 3$. Indeed, one can say more. For every integer k with $2 \le k \le 5$, there is a planar graph of Class one and a planar graph of Class two, both having maximum degree k (see Exercise 20). This story does not go much further, however, for in 1965 Vadim Vizing [247] proved the following.

Theorem 17.19 *If G is a planar graph with $\Delta(G) \ge 8$, then G is of Class one.*

In 2001 Daniel Sanders and Yue Zhao [216] resolved one of the two missing cases.

Theorem 17.20 *If G is a planar graph with $\Delta(G) = 7$, then G is of Class one.*

Thus, only one case remains. Vizing conjectured that this remaining case also results in a planar graph of Class one.

Vizing's Planar Graph Conjecture Every planar graph with maximum degree 6 is of Class one.

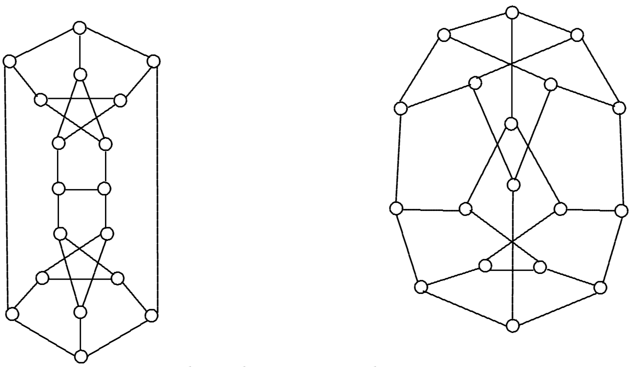
Snarks

While every bridgeless cubic planar graph is of Class one, there are many cubic graphs that are of Class two. First, every cubic graph containing a bridge is of Class two (see Exercise 12). Furthermore, every bridgeless cubic graph of Class two is necessarily non-Hamiltonian and nonplanar. The Petersen graph is an example of a bridgeless cubic graph of Class two.

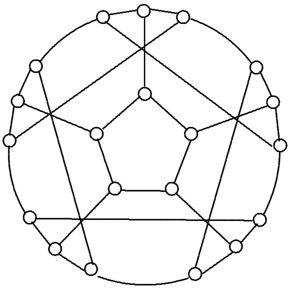
Recall that the girth of a graph G that is not a forest is the length of a smallest cycle in G and that the girth of the Petersen graph is 5. The **cyclic edge-connectivity** of a graph is the smallest number of edges whose removal results in a disconnected graph, each component of which contains a cycle. The cyclic edge-connectivity of the Petersen graph is 5.

There is a class of bridgeless cubic graphs of Class two that are of special interest. A **snark** is a cubic graph of Class two that has girth at least 5 and cyclic edge-connectivity at least 4. The Petersen graph is therefore a snark. The girth and cyclic edge-connectivity requirements are present in the definition to rule out trivial examples. The term "snark" was coined for these graphs in 1976 by Martin Gardner, a longtime popular writer for the magazine *Scientific American*. Gardner borrowed this name from Lewis Carroll (the pen-name of the mathematician Charles Lutwidge Dodgson), well known for his book *Alice's Adventures in Wonderland*. One century earlier, in 1876, Carroll wrote a non-sensical poem titled *The Hunting of the Snark* in which a group of adventurers are in pursuit of a legendary and elusive beast: the snark. Gardner chose to call these graphs "snarks" because just as Carroll's snarks were difficult to find, so too were these graphs difficult to find (at least for the first several years).

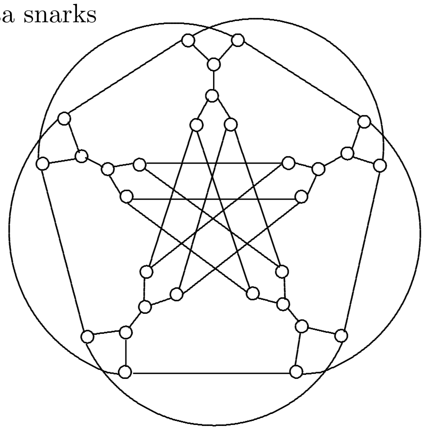
The oldest snark is the Petersen graph, discovered in 1891. In 1946 Danilo Blanuša [28] discovered two more snarks (the **Blanuša snarks**), both of order 18 (see Figure 17.15(a)).



(a) The Blanuša snarks



(b) A flower snark



(c) The double-star snark

Figure 17.15: Snarks

The **Descartes snark** (discovered by Tutte [67] in 1948) has order 210. George Szekeres [229] discovered the **Szekeres snark** of order 50 in 1973. Until 1973 these were the only known snarks. In 1975, however, Rufus Isaacs [132] described two infinite families of snarks, one of which essentially contained all previously known snarks while the second family was completely new. This second family contained the so-called **flower snarks**, one example of which is shown in Figure 17.15(b). In addition, Isaacs found a snark that belonged to neither family. This **double-star snark** is shown in Figure 17.15(c).

All of the snarks shown in Figure 17.15 appear to have a certain resemblance to the Petersen graph. In fact, William Tutte conjectured that every snark has the Petersen graph as a minor. Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas announced in 2001 that they had verified this conjecture. This was reported in [185].

Theorem 17.21 *Every snark has the Petersen graph as a minor.*

Exercises for Chapter 17

Section 17.1: The Chromatic Index of a Graph

1. Determine the chromatic index of the graph shown in Figure 17.16.

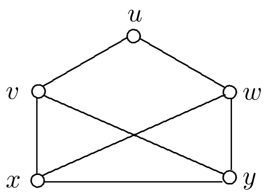


Figure 17.16: The graph in Exercise 1

2. What can be said about the chromatic index of a graph obtained by adding a pendant edge to a vertex of maximum degree in a graph?

3. Prove that $\chi'(G \,\square\, K_2) = \Delta(G \,\square\, K_2)$ for every graph G.

4. (a) Show that $\chi'(G \vee K_1) \geq \chi'(G)$ for every nonempty graph G.
   (b) Show that there are infinitely many graphs G such that $\chi'(G \vee K_1) = \chi'(G)$.

5. For $i = 1, 2$, let G_i be a graph of order n_i and chromatic index k_i and let

$$B = \max\{n_1, n_2\} + \max\{k_1, k_2\}.$$

   (a) Show that $\chi'(G_1 \vee G_2) \leq B$.
   (b) Give an example of graphs G_1 and G_2 such that $\chi'(G_1 \vee G_2) < B$.

6. (a) Determine the upper bounds for $\chi'(G)$ given by Theorems 17.3 and 17.4 for the multigraph G shown in Figure 17.17.
   (b) Determine $\chi'(G)$ for the multigraph G shown in Figure 17.17.

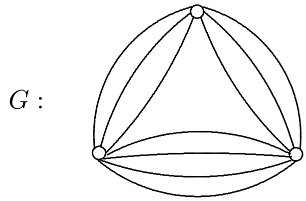


Figure 17.17: The multigraph G in Exercise 6

Section 17.2: Class One and Class Two Graphs

7. Let H be a nonempty regular graph of odd order and let G be a graph obtained from H by deleting $\frac{1}{2}(\Delta(H) - 1)$ or fewer edges. Show that G is of Class two.

8. Prove or disprove: If G_1 and G_2 are Class one graphs and H is a graph with $G_1 \subseteq H \subseteq G_2$, then H is of Class one.

9. Prove that every Hamiltonian cubic graph is of Class one.

10. Show that graphs G_1 and G_2 in Figure 17.18 are of Class two.

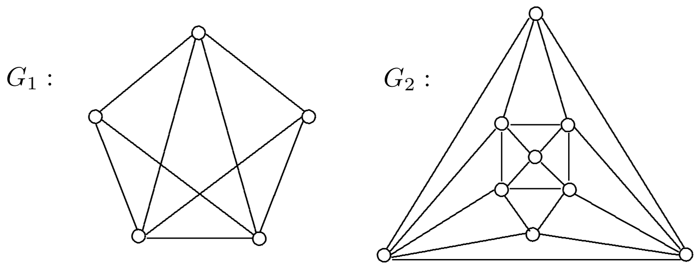


Figure 17.18: Graphs in Exercise 10

11. Determine the classes of the graphs of the five regular polyhedra.

12. Prove that every cubic graph having connectivity 1 is of Class two.

13. Use the fact that every r-regular bipartite graph is 1-factorable to give an alternative proof of Theorem 17.7: *If G is a nonempty bipartite graph, then $\chi'(G) = \Delta(G)$.*

14. Show that Corollary 17.6 (*Every regular graph of odd order is of Class two.*) is also a corollary of Theorem 17.8 (*If G is a graph of size m such that $m > \alpha'(G)\Delta(G)$, then G is of Class two.*).

15. Prove Corollary 17.9: *Every overfull graph is of Class two.*

16. Show that if H is an overfull subgraph of a graph G, then $\Delta(H) = \Delta(G)$.

17. Show that the condition that G_1 is an overfull subgraph of G is needed in Theorem 17.13.

18. A nonempty graph G is of Type one if $\chi'(L(G)) = \omega(L(G))$ and of Type two if $\chi'(L(G)) = 1+\omega(L(G))$. Prove or disprove: Every nonempty graph is either of Type one or of Type two.

19. The **total deficiency** of a graph G of order n and size m is the number $n \cdot \Delta(G) - 2m$. (Thus, every graph that is not regular has positive total deficiency.) Prove that if G is a graph of odd order whose total deficiency is less than $\Delta(G)$, then G is of Class two.

20. Show that for every integer k with $2 \leq k \leq 5$, there is a planar graph of Class one and a planar graph of Class two, both having maximum degree k.

21. For a positive integer k, let H be a $2k$-regular graph of order $4k + 1$. Let G be obtained from H by removing a set of $k - 1$ independent edges from H. Prove that G is of Class two.

22. For a positive integer k, let G be a $(2k + 2)$-regular graph of order $4k + 1$.

    (a) Show that G is Hamiltonian.

    (b) Let $C = (v_1, v_2, \ldots, v_{4k+1}, v_{4k+2} = v_1)$ be a Hamiltonian cycle of G and let $X = \{v_{2i-1}v_{2i} : 1 \leq i \leq 2k + 1\}$. Prove that $H = G - X$ is of Class two.

23. (a) In Figure 17.8, a solution of Example 17.10 is given by providing a 6-edge coloring of the graph H shown in that figure. Give a characteristic of the resulting soccer schedule which might not be considered ideal. Correct this deficiency by giving a different 6-edge coloring of H.

    (b) In Figure 17.9, a solution of Example 17.11 is given by providing a 4-edge coloring of the graph G shown in that figure. Give a characteristic of the resulting tennis schedule which might not be considered ideal. Correct this deficiency by giving a different 4-edge coloring of G.

24. Show that the bounds on r in Theorem 17.14 cannot be improved by verifying the following.

    (a) If G is an r-regular graph of sufficiently large even order n where $n \equiv 2 \pmod 4$ and $r \geq (n - 2)/2$, then G need not be of Class one.

    (b) If G is an r-regular graph of sufficiently large even order n where $n \equiv 0 \pmod 4$ and $r \geq (n - 4)/2$, then G need not be of Class one.

25. Show that every nontrivial self-complementary regular graph is of Class two.

26. Show for a sufficiently large even order n and every regular graph G of order n that at least one of G and $\overline{G}$ is of Class one.

Section 17.3: Tait Colorings

27. Let G be the graph shown in Figure 17.19. We have seen that every Hamiltonian cubic graph has a Tait coloring.

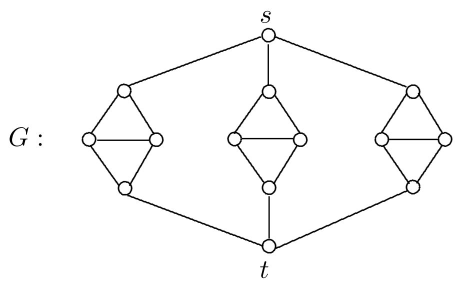


Figure 17.19: The graph in Exercise 27

(a) Is the graph G Hamiltonian?

(b) Does the graph G have a Tait coloring?

28. (a) Let G be a cubic graph and let H be the cubic graph obtained from G by replacing an edge uv of G by four new vertices w, x, y and z and the edges uw, vz, wx, wy, xy, xz and yz (see Figure 17.20(a)). Prove that G has a Tait coloring if and only if H has a Tait coloring.

(b) Use the result in (a) to determine whether the graph F shown in Figure 17.20(b) has a Tait coloring.

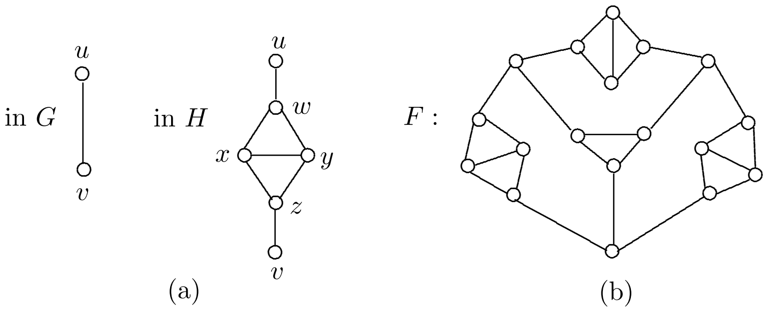


Figure 17.20: The graph in Exercise 28

29. Let G be the cubic graph shown in Figure 17.21.

(a) Is G planar?

(b) Is G Hamiltonian?

(c) Does G have a Tait coloring?

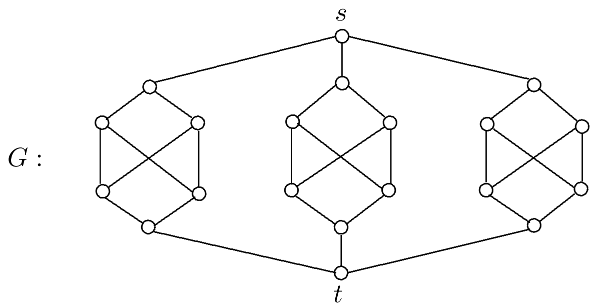


Figure 17.21: The graph in Exercise 29

30. Let G be a cubic graph and let H be the cubic graph obtained from G by replacing each edge uv of G by the structure shown in Figure 17.22. Prove or disprove: The graph G has a Tait coloring if and only if H has a Tait coloring.

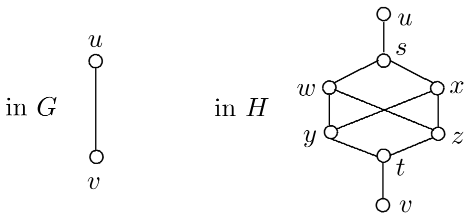


Figure 17.22: The graphs in Exercise 30

31. (a) Determine the chromatic index of the graph G shown in Figure 17.23.
    (b) Is this graph a snark?

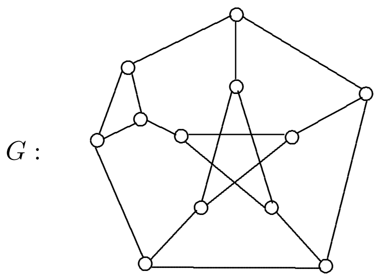


Figure 17.23: The graph in Exercise 31

Chapter 18

Nowhere-Zero Flows, List Edge Colorings

In Chapter 14, vertex colorings of graphs were introduced and in Chapter 17, edge colorings were introduced. In this chapter, colorings are described where both vertices and edges are assigned colors. The edge analogue of list colorings of vertices, introduced in Chapter 15, is also discussed in this chapter. As we saw in Chapter 17, Tait colorings are edge colorings that are intimately tied to colorings of the regions of bridgeless plane graphs. There are integer-valued labelings of the arcs of orientations of bridgeless plane graphs that also have connections to colorings of the regions of such graphs. It is this topic that is discussed first in this chapter.

18.1 Nowhere-Zero Flows

A **flow** on an oriented graph D is a function $\phi : E(D) \to \mathbb{Z}$ such that for each vertex v of D,

$$\sigma^+(v;\phi) = \sum_{(v,w)\in E(D)} \phi(v,w) = \sum_{(w,v)\in E(D)} \phi(w,v) = \sigma^-(v;\phi). \tag{18.1}$$

That is, for each vertex v of D, the sum of the flow values of the arcs directed away from v equals the sum of the flow values of the arcs directed toward v. The property (18.1) of ϕ is called the **conservation property**. Since $\sigma^+(v;\phi) - \sigma^-(v;\phi) = 0$ for every vertex v of D, the sum of the flow values of the arcs incident with v is even. This, in turn, implies that the number of arcs incident with v having an odd flow value is even.

For an integer $k \geq 2$, if a flow ϕ on an oriented graph D has the property that $|\phi(e)| < k$ for every arc e of D, then ϕ is called a k-**flow** on D. Furthermore, if $0 < |\phi(e)| < k$ for every arc e of D (that is, $\phi(e)$ is never 0), then ϕ is called a **nowhere-zero** k**-flow** on D. Hence, a nowhere-zero k-flow on D has the property that

$$\phi(e) \in \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$$

for every arc e of D. It is the nowhere-zero k-flows for particular values of k that will be of special interest to us.

Those graphs for which some orientation has a nowhere-zero 2-flow can be described quite easily.

Theorem 18.1 *A nontrivial connected graph G has an orientation with a nowhere-zero 2-flow if and only if G is Eulerian.*

Proof. Let G be an Eulerian graph and let C be an Eulerian circuit of G. Direct the edges of C so that C becomes a directed Eulerian circuit in the resulting Eulerian digraph D, where then $\text{od}\, v = \text{id}\, v$ for every vertex v of D. Since the function ϕ defined by $\phi(e) = 1$ for each arc e of D satisfies the conservation property, ϕ is a nowhere-zero 2-flow on D.

Conversely, suppose that G is a nontrivial connected graph that is not Eulerian. Then G contains a vertex u of odd degree. Let D be any orientation of G. Then any function ϕ defined on $E(D)$ for which $\phi(e) \in \{-1, 1\}$ has an odd number of arcs incident with v having an odd flow value. Thus, ϕ is not a nowhere-zero 2-flow. ■

The graph $G = C_4$ of Figure 18.1 is obviously Eulerian and therefore has an orientation D_1 with a nowhere-zero 2-flow. However, D_1 is not the only orientation of G with this property. The orientations D_2 and D_3 of G also have nowhere-zero 2-flows.

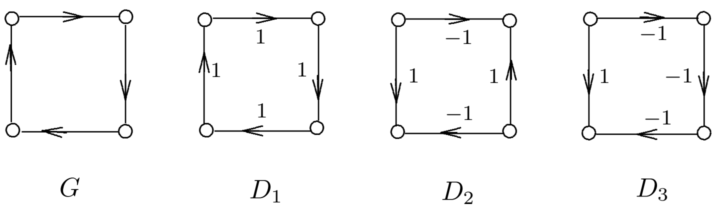


Figure 18.1: Oriented graphs with nowhere-zero 2-flows

The graph and digraphs shown in Figure 18.1 may suggest that the existence of a nowhere-zero k-flow for some integer $k \geq 2$ on all orientations of a graph G depends only on the existence of a nowhere-zero k-flow on a single orientation of G. This is, in fact, what happens.

Theorem 18.2 *Let G be a graph. If some orientation of G has a nowhere-zero k-flow, where $k \geq 2$, then every orientation of G has a nowhere-zero k-flow.*

Proof. Let D be an orientation of G having a nowhere-zero k-flow ϕ. Thus, $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for each vertex v of D. Let D' be the orientation of G obtained by reversing the direction of some arc $f = (x, y)$ of D, resulting in the arc $f' = (y, x)$ of D'. We now define the function

$$\phi' : E(D') \to \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$$

by

$$\phi'(e) \quad = \quad \begin{cases} \phi(e) & \text{if } e \neq f' \\ -\phi(f) & \text{if } e = f'. \end{cases}$$

For $v \in V(D) - \{x, y\}$, $\sigma^+(v;\phi') = \sigma^-(v;\phi')$. Also,

$$\begin{aligned} \sigma^+(x;\phi') &= \sigma^+(x;\phi) - \phi(f) = \sigma^-(x;\phi) + \phi'(f') = \sigma^-(x;\phi') \\ \sigma^+(y;\phi') &= \sigma^+(y;\phi) + \phi'(f') = \sigma^-(y;\phi) + (-\phi(f)) = \sigma^-(y;\phi'). \end{aligned}$$

Thus, ϕ' is a nowhere-zero k-flow of D'.

Now, if D'' is any orientation of G, then D'' can be obtained from D by a sequence of arc reversals in D. Since a nowhere-zero k-flow can be defined on each orientation, as described above, at each step of the sequence, a nowhere-zero k-flow can be defined on D''. ■

According to Theorem 18.2, the property of an orientation (indeed of *all* orientations) of a graph G having a nowhere-zero k-flow is a characteristic of G rather than a characteristic of its orientations. We therefore say that a **graph G has a nowhere-zero k-flow** if every orientation of G has a nowhere-zero k-flow. As a consequence of the proof of Theorem 18.2, we also have the following.

Theorem 18.3 *If G is a graph having a nowhere-zero k-flow for some $k \geq 2$, then there is an orientation D of G and a nowhere-zero k-flow on D all of whose flow values are positive.*

By Theorem 18.1, the graph G of Figure 18.2 does not have a nowhere-zero 2-flow. Trivially, it has an orientation D possessing a 2-flow. This orientation does have a nowhere-zero 3-flow, however. By Theorem 18.2, G itself has a nowhere-zero 3-flow.

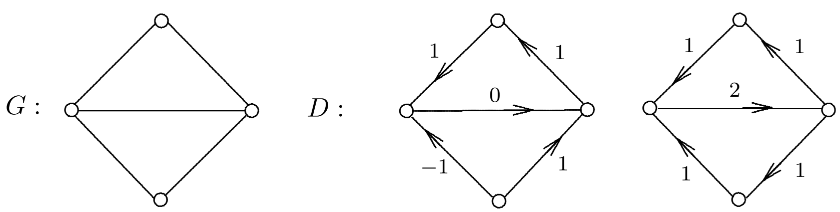


Figure 18.2: A graph with a nowhere-zero 3-flow

If D is an orientation of a graph G with a flow ϕ and a is an integer, then $a\phi$ is also a flow on D. In fact, if ϕ is a k-flow and $a \geq 1$, then $a\phi$ is an ak-flow. Indeed, if ϕ is a nowhere-zero k-flow, then $a\phi$ is a nowhere-zero ak-flow. More generally, we have the following (see Exercise 3).

Theorem 18.4 *If ϕ_1 and ϕ_2 are flows on an orientation D of a graph, then every linear combination of ϕ_1 and ϕ_2 is also a flow on D.*

Let G be a graph with a nowhere-zero k-flow for some integer $k \geq 2$ and let ϕ be a nowhere-zero k-flow defined on some orientation D of G. Then $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for every vertex v of D. Let $S = \{v_1, v_2, \ldots, v_t\}$ be a proper subset of $V(G)$ and let $T = V(G) - S$. Define

$$\sigma^+(S;\phi) = \sum_{(u,v)\in[S,T]} \phi(u,v)$$

and

$$\sigma^-(S;\phi) = \sum_{(v,u)\in[T,S]} \phi(v,u).$$

Since ϕ is a flow on D,

$$\sum_{i=1}^{t} \sigma^+(v_i;\phi) = \sum_{i=1}^{t} \sigma^-(v_i;\phi). \tag{18.2}$$

For every arc (v_a, v_b) with $v_a, v_b \in S$, the flow value $\phi(v_a, v_b)$ occurs in both the left and the right sums in (18.2). Cancelling all such terms in (18.2), we are left with $\sigma^+(S;\phi)$ on the left side of (18.2) and $\sigma^-(S;\phi)$ on the right side of (18.2). Thus, $\sigma^+(S;\phi) = \sigma^-(S;\phi)$. In summary, we have the following:

Theorem 18.5 *Let ϕ be a nowhere-zero k-flow defined on some orientation D of a graph G and let S be a nonempty proper subset of $V(G)$. Then the sum of the flow values of the arcs directed from S to $V(G) - S$ equals the sum of the flow values of the arcs directed from $V(G) - S$ to S.*

Bridgeless Graphs and Nowhere-Zero Flows

A fundamental problem in this area is that of determining those nontrivial connected graphs possessing a nowhere-zero k-flow for some integer $k \geq 2$. Suppose that ϕ is a nowhere-zero k-flow defined on some orientation D of a nontrivial connected graph G. If G should contain a bridge $e = uv$, where S and $V(G)-S$ are the vertex sets of the components of $G-e$, then the conclusion of Theorem 18.5 cannot occur. As a consequence of this observation, we have the following:

Corollary 18.6 *No graph with a bridge has a nowhere-zero k-flow for any integer $k \geq 2$.*

The nowhere-zero k-flows of bridgeless planar graphs will be of particular interest to us because of their connection with region colorings of these graphs. Suppose that D is an orientation of a bridgeless plane graph G possessing a k-region coloring c for some integer $k \geq 2$. Thus, for each region R of G, we may assume that the color $c(R)$ is one of the colors $1, 2, \ldots, k$. If uv is an edge of G belonging to the boundaries of two regions R_1 and R_2 of G such that $e = (u, v)$ is an arc of D, then define

$$\phi(e) = c(R_1) - c(R_2),$$

where R_1 is the region that lies to the right of e and R_2 is the region that lies to the left of e as indicated in Figure 18.3.

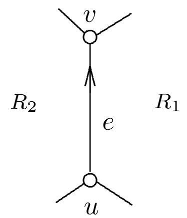
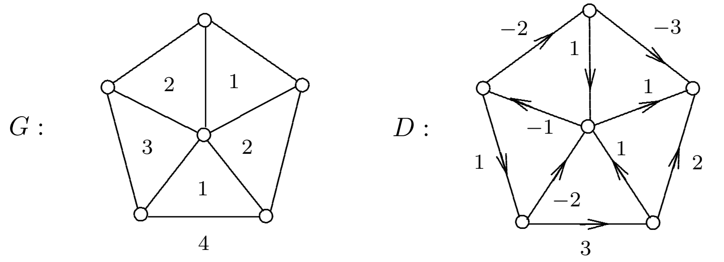


Figure 18.3: Defining $\phi(e)$ for $e = (u, v)$

A 4-region coloring of the plane graph of Figure 18.4(a) is given in that figure. A resulting nowhere-zero 4-flow of an orientation D of G is shown in Figure 18.4(b).



(a) a 4-region coloring of a plane graph

(b) a nowhere-zero 4-flow of an orientation D of G

Figure 18.4: Constructing a nowhere-zero flow from a region coloring of a plane graph

For bridgeless plane graphs, k-region colorability and the existence of nowhere-zero k-flows for an integer $k \geq 2$ are equivalent.

Theorem 18.7 *For an integer $k \geq 2$, a bridgeless plane graph G is k-region colorable if and only if G has a nowhere-zero k-flow.*

Proof. First, let there be given a k-region coloring c of G and let D be an orientation of G. For each arc $e = (u, v)$ of D, let R_1 be the region of G that lies to the right of e and R_2 the region of G that lies to the left of e. Define an integer-valued function ϕ of $E(D)$ by

$$\phi(e) = c(R_1) - c(R_2).$$

We show that ϕ is a nowhere-zero k-flow. Since uv is not a bridge of G, $c(R_1) \neq c(R_2)$ and since $1 \leq c(R) \leq k$ for each region R of G, it follows that

$$\phi(e) \in \{\pm 1, \pm 2, \ldots, \pm(k-1)\}.$$

It remains only to show that $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for each vertex v of D. Suppose that $\deg_G v = t$ and that $v_1, v_2, \ldots, v_t$ are the neighbors of v as we proceed cyclically about v in some direction. For $i = 1, 2, \ldots, t$, let R_i denote the region having vv_i and vv_{i+1} on its boundary where $v_{t+1} = v_1$. Thus, in D each edge vv_i $(1 \leq i \leq t)$ is either the arc (v, v_i) or the arc (v_i, v). Let $c(R_i) = c_i$ for $i = 1, 2, \ldots, t$. Let $j \in \{1, 2, \ldots, t\}$ and consider $c(R_j) = c_j$. If $(v, v_j), (v, v_{j+1}) \in E(D)$, then c_j and $-c_j$ occur in $\sigma^+(v;\phi)$ and c_j does not occur in $\sigma^-(v;\phi)$. The situation is reversed if $(v_j, v), (v_{j+1}, v) \in E(D)$. If $(v, v_{j+1}), (v_j, v) \in E(D)$, then the term c_j occurs in both $\sigma^+(v;\phi)$ and $\sigma^-(v;\phi)$; while if $(v_{j+1}, v), (v, v_j) \in E(D)$, then the term $-c_j$ occurs in both $\sigma^+(v;\phi)$ and $\sigma^-(v;\phi)$. Thus,

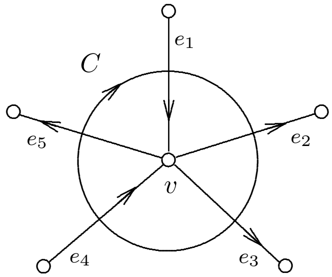
$$\sigma^+(v;\phi) = \sigma^-(v;\phi)$$

and ϕ is a nowhere-zero k-flow.

Next, suppose that G is a bridgeless plane graph having a nowhere-zero k-flow. This implies that for a given orientation D of G, there exists a nowhere-zero k-flow ϕ of D. By definition then, $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for every vertex v of D.

We now consider directed closed curves in the plane that do not pass through any vertex of D. Such closed curves may enclose none, one or several vertices of G. For a directed closed curve C, we define the number $\sigma(C;\phi)$ to be the sum of terms $\phi(e)$ or $-\phi(e)$ for each occurrence of an arc e crossed by C. In particular, as we proceed along C in the direction of C and cross an arc e, we contribute $\phi(e)$ to $\sigma(C;\phi)$ if e is directed to the right of C and contribute $-\phi(e)$ to $\sigma(C;\phi)$ if e is directed to the left of C.

If C is a directed simple closed curve in the plane that encloses no vertex of D, then for each occurrence of an arc e crossed by C that is directed to the right of C, there is an occurrence of e crossed by C that is directed to the left of C. Hence, in this case, $\sigma(C;\phi) = 0$. If C encloses a single vertex v, then because $\sigma^+(v;\phi) = \sigma^-(v;\phi)$, it follows that $\sigma(C;\phi) = 0$ here as well. (See Figure 18.5 for example.)



$$\sigma(C;\phi) = (\phi(e_1) + \phi(e_4)) - (\phi(e_2) + \phi(e_3) + \phi(e_5)) = 0$$

Figure 18.5: Computing $\sigma(C;\phi)$

Suppose now that C is a directed simple closed curve in the plane that encloses two or more vertices. Let $S = \{v_1, v_2, \cdots, v_s\}$, $s \geq 2$, be the set of vertices of D lying interior to C. Because, by Theorem 18.5, the sum of the flow values of the arcs directed from S to $V(G) - S$ equals the sum of the flow values of the arcs directed from $V(G) - S$ to S, it follows that $\sigma(C;\phi) = 0$ as well.

If C is a directed closed curve that is not a simple closed curve, then C is a union of directed simple closed curves $C_1, C_2, \ldots, C_r$ and so

$$\sigma(C;\phi) = \sum_{i=1}^{s} \sigma(C_i;\phi) = 0.$$

Consequently, $\sigma(C;\phi) = 0$ for *every* directed closed curve C in the plane.

We now show that there is a proper coloring of the regions of G using the colors $1, 2, \ldots, k$. Assign the color k to the exterior region of G. Let R be some interior region in G. Choose a point A in the exterior region and a point B in R and let P be an open curve directed from A to B so that P passes through no vertices of D. The number $\sigma(P;\phi)$ is defined as the sum (addition performed modulo k) of the numbers $\phi(e)$ or $-\phi(e)$, for each occurrence of an arc e crossed by P, where either $\phi(e)$ or $-\phi(e)$ is contributed to the sum $\sigma(P;\phi)$ according to whether e is directed to the right or to the left of P, respectively. The least positive integer in the equivalence class containing $\sigma(P;\phi)$ in the ring of integers $\mathbb{Z}_k$ is the color $c(R)$ assigned to R. Thus, $c(R) \in \{1, 2, \ldots, k\}$.

We now show that the color $c(R)$ assigned to R is well-defined. Suppose that the color assigned to R by the curve P above is $c(R) = a$. Let Q be another directed open curve from A to B and let $\sigma(Q;\phi) = b$. We claim that $a = b$. If $\tilde{Q}$ is the directed open curve from B to A obtained by reversing the direction of Q, then $\sigma(\tilde{Q};\phi) = -b$. Now let C be the directed closed curve obtained by following P by $\tilde{Q}$. Then, as we saw, $\sigma(C;\phi) = 0$. But

$$\sigma(C;\phi) = \sigma(P;\phi) + \sigma(\tilde{Q};\phi) = a - b.$$

So, $a - b = 0$ and $a = b$. Thus, the color $c(R)$ of R defined in this manner is, in fact, well-defined.

It remains to show that the region coloring c of G is proper. Let R' and R'' be two adjacent regions of D, where e' is an arc on the boundaries of both R' and R''. Let B' be a point in R' and let B'' be a point in R''. Furthermore, let P' be a directed open curve from A to B' that does not cross e' and suppose that P'' is a directed open curve from A to B'' that extends P' to B'' so that e' is the only additional arc crossed by P''. Therefore,

$$\sigma(P''; \phi) = \sigma(P'; \phi) + \phi(e')$$

or

$$\sigma(P''; \phi) = \sigma(P'; \phi) - \phi(e').$$

Since $\phi(e') \not\equiv 0 \pmod{k}$, it follows that the colors assigned to R' and R'' are distinct. Hence, the region coloring c of G is proper. ■

Letting $k = 4$ in Theorem 18.7, we have the following corollary of the Four Color Theorem.

Corollary 18.8 *Every bridgeless planar graph has a nowhere-zero* 4*-flow.*

While every bridgeless planar graph has a nowhere-zero 4-flow, it is not the case that every bridgeless nonplanar graph has a nowhere-zero 4-flow.

Theorem 18.9 *The Petersen graph does not have a nowhere-zero* 4*-flow.*

Proof. Suppose that the Petersen graph P has a nowhere-zero 4-flow. Then there exists an orientation D of P and a nowhere-zero 4-flow ϕ on D such that $\phi(e) \in \{1, 2, 3\}$ for every arc e of D (by Theorem 18.3). Since $\sigma^+(v; \phi) = \sigma^-(v; \phi)$ for every vertex v of D, the only possible flow values of the three arcs incident with v are $1, 1, 2$ and $1, 2, 3$. In particular, this implies that every vertex of D is incident with exactly one arc having flow value 2. Thus, the arcs of D with flow value 2 correspond to a 1-factor F of P. The remaining arcs of D then correspond to a 2-factor H of P. Because the Petersen graph is not Hamiltonian and has girth 5, the 2-factor H must consist of two disjoint 5-cycles. Every vertex v of H is incident with one or two arcs having flow value 1. Furthermore, if a vertex v of H is incident with two arcs having flow value 1, then these two arcs are either both directed toward v or both directed away from v. A vertex v of H is said to be of *type* I if there is an arc having flow value 1 directed toward v; while v is of *type* II if there is an arc having flow value 1 directed away from v. Consequently, each vertex v of H is either of type I or of type II, but not both. Moreover, the vertices in each of the 5-cycles of H must alternate between type I and type II, which is impossible for an odd cycle. ■

While the Petersen graph does not have a nowhere-zero 4-flow, it does have a nowhere-zero 5-flow, however (see Exercise 4). The Petersen graph plays an important role among cubic graphs as Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas verified in a series of papers [208, 210, 209, 214, 215].

Theorem 18.10 *Every bridgeless cubic graph not having the Petersen graph as a minor has a nowhere-zero* 4*-flow.*

There is a question of determining the smallest positive integer k such that every bridgeless graph has a nowhere-zero k-flow. In this connection, William Tutte [241] conjectured the following.

Conjecture 18.11 *Every bridgeless graph has a nowhere-zero* 5*-flow.*

While this conjecture is still open, Paul Seymour [220] did establish the following.

Theorem 18.12 *Every bridgeless graph has a nowhere-zero* 6*-flow.*

Tait colorings (introduced in Chapter 17) deal exclusively with cubic graphs, of course, and it is this class of graphs that has received added attention when studying nowhere-zero flows. By Theorem 18.1, no cubic graph has a nowhere-zero 2-flow. In the case of nowhere-zero 3-flows, it is quite easy to determine which cubic graphs have these.

Theorem 18.13 *A cubic graph G has a nowhere-zero* 3*-flow if and only if G is bipartite.*

Proof. Let G be a cubic bipartite graph with partite sets U and W. Since G is regular, G contains a 1-factor F. Orient the edges of F from U to W and assign each arc the value 2. Orient all other edges of G from W to U and assign each of these arcs the value 1. This is a nowhere-zero 3-flow.

For the converse, let G be a cubic graph having a nowhere-zero 3-flow. By Theorem 18.3, there is an orientation D on which is defined a nowhere-zero 3-flow having only the values 1 and 2. In particular, the values must be 1, 1, 2 for the three arcs incident with each vertex of D. The arcs having the value 2 form a subdigraph H of D and the underlying graph of H is a 1-factor of G. Let U be the set of vertices of G where each vertex of U has outdegree 1 in H and let W be the remaining vertices of G. Every arc of D not in H must then have a value of 1 and be directed from a vertex of W to a vertex of U. Thus, G is a bipartite graph with partite sets U and W. ■

Those cubic graphs having a nowhere-zero 4-flow depend only on their chromatic index.

Theorem 18.14 *A cubic graph G has a nowhere-zero* 4*-flow if and only if G is of Class one.*

Proof. Let G be a cubic graph that has a nowhere-zero 4-flow. By Theorem 18.3, there exists an orientation D of G and a nowhere-zero 4-flow ϕ on D such that $\phi(e) \in \{1, 2, 3\}$ for each arc e of D. Then each vertex is incident with arcs having either the values 1, 1, 2 or 1, 2, 3. In particular, each vertex of D is incident with exactly one arc having the value 2. The arcs having the value 2 produce a 1-factor in G and so the remaining arcs of D produce a 2-factor of G. If the 2-factor is a Hamiltonian cycle of G, then G is of Class one. Hence, we may assume that this 2-factor consists of two or more mutually disjoint cycles. Let C be one of these cycles and let H be the orientation of C in D.

Suppose that C is an r-cycle and let $S = V(C)$. Since each arc of D joining a vertex of S and a vertex of $V(D) - S$ has flow value 2 and since $\sigma^+(S, \phi) = \sigma^-(S; \phi)$ by Theorem 18.5, it follows that there is an even number t of arcs joining the vertices of S and the vertices of $V(D) - S$. Since this set of arcs is independent, it follows that $G[S]$ contains t vertices of degree 2 and $r - t$ vertices of degree 3. Because $G[S]$ has an even number of odd vertices, $r - t$ is even. However, since t is also even, r is even as well and C is an even cycle. Thus, the 2-factor of G is the union of even cycles and so G is of Class one.

We now verify the converse. Let G be a cubic graph of Class one. Therefore, G is 3-edge colorable and 1-factorable into factors F_1, F_2, F_3, where F_i is the 1-factor $(1 \le i \le 3)$ whose edges are colored i. Every two of these three 1-factors produce a 2-factor consisting of a union of disjoint cycles. Since the edges of each cycle alternate in colors, the cycles are even and each 2-factor is bipartite. Let G_1 be the 2-factor obtained from F_1 and F_3 and G_2 the 2-factor obtained from F_2 and F_3. Now, let D be an orientation of G and let D_i be the resulting orientation of G_i $(i = 1, 2)$. Since each component of each graph G_i is Eulerian, it follows by Theorem 18.1 that there is a nowhere-zero 2-flow ϕ_i on D_i. For $e \in E(D) - E(D_i)$, define $\phi_i(e) = 0$ for $i = 1, 2$. Then ϕ_i is a 2-flow on D. By Theorem 18.4, the function ϕ defined by $\phi = \phi_1 + 2\phi_2$ is also a flow on D. Because $\phi(e) \in \{\pm 1, \pm 2, \pm 3\}$, it follows that ϕ is a nowhere-zero 4-flow on D. ■

Since the Petersen graph is a cubic graph that is of Class two, it follows by Theorem 18.14 that it does not have a nowhere-zero 4-flow (which we also saw in Theorem 18.9). We noted that the Petersen graph does have a nowhere-zero 5-flow. Of course, if Conjecture 18.11 is true, then every bridgeless graph has a nowhere-zero 5-flow.

Cycle Double Covers

There is another concept and conjecture related to bridgeless graphs which ultimately returns us to snarks.

If G is a connected bridgeless plane graph, then the boundary of every region is a cycle and every edge of G lies on the boundaries of two regions. Thus, if S is the set of cycles of G that are the boundaries of the regions of G, then every edge of G belongs to exactly two elements of S. For which other graphs G is there a collection S of cycles of G such that every edge of G belongs to exactly two cycles of S?

A **cycle double cover** of a graph G is a set (actually a multiset) S of not necessarily distinct cycles of G such that every edge of G belongs to exactly two cycles of S. Certainly, no cycle of G can appear more than twice in S. Also, if G contains a bridge e, then e belongs to no cycle and G contains no cycle double cover. If G is Eulerian, then G has a cycle decomposition S'. If S is the set of cycles of G that contains each cycle of S' twice, then S is a cycle double cover of G.

That every bridgeless graph has a cycle double cover was conjectured by Paul Seymour [219] in 1979. George Szekeres [229] conjectured this for cubic graphs even earlier – in 1973.

The Cycle Double Cover Conjecture *Every nontrivial connected bridgeless graph has a cycle double cover.*

The following result of Bryant, Horsley and Pettersson [39], obtained in 2015, may be considered as an analogue of Theorem 13.16.

Theorem 18.15 *For every t integers $k_1, k_2, \ldots, k_t$ and an integer $n \geq 3$ such that $3 \leq k_i \leq n$ for each i $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} k_i = 2\binom{n}{2}$, the complete graph K_n has t cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_t}$ such that every edge of K_n belongs to exactly two of these cycles.*

As we have seen, the Cycle Double Cover Conjecture is true for all nontrivial connected bridgeless planar graphs and for all Eulerian graphs. Initially, it may seem apparent that this conjecture is true in general, for if we were to replace each edge of a bridgeless graph G by two parallel edges then the resulting multigraph H is Eulerian. This implies that there is a set S of cycles of H such that each edge of H belongs to exactly one cycle in S. This, in turn, implies that each edge of G belongs to exactly two cycles in S, completing the proof. This argument is faulty, however, for one or more cycles of H in S may be 2-cycles, which do not correspond to cycles of G. Nevertheless, no counterexample to the Cycle Double Cover Conjecture is known. If the Cycle Double Cover Conjecture is false, then there exists a minimum counterexample, namely, a connected bridgeless graph of minimum size having no cycle double cover. Francois Jaeger (1947–1997) proved that a minimum counterexample to the Cycle Double Cover Conjecture must be a snark [135]. However, all known snarks possess a cycle double cover.

18.2 List Edge Colorings

In Section 15.2 we encountered the topic of list colorings. In a list coloring of a graph G, there is a list (or set) of available colors for each vertex of G, with the goal being to select a color from each list so that a proper vertex coloring of G results. One of the major problems concerns the determination of the smallest positive integer k such that if every list contains k or more colors, then a proper vertex coloring of G can always be constructed. This smallest positive integer k is called the list chromatic number $\chi_\ell(G)$ of G. In this section we consider the edge analogue of this concept.

The List Chromatic Index of a Graph

Let G be a nonempty graph and for each edge e of G, let $L(e)$ be a list (or set) of colors. Furthermore, let $\mathfrak{L} = \{L(e) : e \in E(G)\}$. The graph G is **$\mathfrak{L}$-edge choosable** (or **$\mathfrak{L}$-list edge colorable**) if there exists a proper edge coloring c of G such that $c(e) \in L(e)$ for every edge e of G. For a positive integer k, a nonempty graph G is **k-edge choosable** (or **k-list edge colorable**) if for every set $\mathfrak{L} = \{L(e) : e \in E(G)\}$ where $|L(e)| \geq k$ for each edge e of G, the graph G is $\mathfrak{L}$-edge choosable. The **list chromatic index** $\chi'_\ell(G)$ is the minimum positive integer k for which G is k-edge choosable. Necessarily then,

$$\chi'(G) \leq \chi'_\ell(G)$$

for every nonempty graph G. Applying a greedy edge coloring to a nonempty graph G (see Exercise 9) gives

$$\chi'_\ell(G) \leq 2\Delta(G) - 1.$$

The List Coloring Conjecture

Since the graph K_3 is of Class two, $\chi'(K_3) = 1 + \Delta(K_3) = 3$. Thus, $\chi'_\ell(K_3) \geq 3$. However, if e_1, e_2 and e_3 are the three edges of K_3 and $\mathfrak{L}(e_1)$, $\mathfrak{L}(e_2)$ and $\mathfrak{L}(e_3)$ are three sets of three or more colors each, then three distinct colors $c(e_1), c(e_2)$ and $c(e_3)$ can be chosen such that $c(e_i) \in \mathfrak{L}(e_i)$ for $1 \leq i \leq 3$ and so K_3 is 3-edge choosable. Therefore, $\chi'_\ell(K_3) = 3$. Although it would be natural now to expect to see an example of a graph G with $\chi'(G) < \chi'_\ell(G)$, no graph is known to have this property.

While the following conjecture was made independently by Vadim Vizing, Ram Prakash Gupta, and Michael Albertson and Karen Collins (see [114]), it first appeared in print in a 1985 paper by Béla Bollobás and Andrew J. Harris [32].

The List Coloring Conjecture *For every nonempty graph G,*

$$\chi'(G) = \chi'_\ell(G).$$

The identical conjecture has been made for multigraphs as well. Since the list chromatic index of a graph equals the list chromatic number of its line graph, the List Coloring Conjecture can also be stated as $\chi_\ell(L(G)) = \chi(L(G))$ for every nonempty graph G.

In 1979 Jeffrey Howard Dinitz had already conjectured that $\chi'(G) = \chi'_\ell(G)$ when G is a regular complete bipartite graph. By Theorem 17.7, it is known that $\chi'(K_{r,r}) = r$.

Dinitz's Conjecture *For every positive integer r, $\chi'_\ell(K_{r,r}) = r$.*

Galvin's Theorem

Fred Galvin [101] not only verified Dinitz's Conjecture, he showed that the List Coloring Conjecture is true for all bipartite graphs (indeed, for all bipartite multigraphs). The proof of this result that we present is based on a proof of Tomaž Slivnik [222], which in turn is based on Galvin's proof. We begin with a lemma of Slivnik. First, we introduce some notation. Let G be a nonempty bipartite graph with partite sets U and W. For each edge e of G, let u_e denote the vertex of U incident with e and let w_e denote the vertex of W incident with e. For adjacent edges e and f, it therefore follows that $u_e = u_f$ or $w_e = w_f$, but not both.

Lemma 18.16 *Let G be a nonempty bipartite graph and let $c : E(G) \to \mathbb{N}$ be an edge coloring of G. For each edge e of G, let $\sigma_G(e)$ be the sum*

$$\begin{aligned}\sigma_G(e) \quad &= \quad 1 + |\{f \in E(G) : u_e = u_f \text{ and } c(f) > c(e)\}| \\ &\qquad + |\{f \in E(G) : w_e = w_f \text{ and } c(f) < c(e)\}|\end{aligned}$$

and let $L(e)$ be a set of $\sigma_G(e)$ colors. If

$$\mathfrak{L} = \{L(e) : e \in E(G)\},$$

then G is $\mathfrak{L}$-edge choosable.

Proof. We proceed by induction on the size m of G. Since the result holds if $m = 1$, the basis step for the induction is true. Assume that the statement of the theorem is true for all nonempty bipartite graphs of size less than m, where $m \geq 2$, and let G be a nonempty bipartite graph of size m on which is defined an edge coloring c and where the numbers $\sigma_G(e)$, the sets $L(e)$ and the set $\mathfrak{L}$ are defined in the statement of the theorem.

Let A be a set of edges of G. A matching $M \subseteq A$ is said to be **optimal** (in A) if the following is satisfied:

For every edge $e \in A - M$, there is an edge $f \in M$ such that either

(i) $u_e = u_f$ and $c(f) > c(e)$ or (ii) $w_e = w_f$ and $c(f) < c(e)$.

First, we show (by induction on the size of A) that for every $A \subseteq E(G)$, there is an optimal matching $M \subseteq A$.

Observe that if A itself is a matching, then $M = A$ is vacuously optimal. If $|A| = 1$, then $M = A$ is optimal and the basis step of this induction is satisfied.

Assume, for an integer k with $1 < k \leq m$, that for each set A' of edges of G with $|A'| = k - 1$, there is an optimal matching in A'. Let A be a set of edges of G with $|A| = k$. We show that A contains an optimal matching M.

An edge e belonging to A is **U-maximum** if there is no edge f in A for which $u_e = u_f$ and $c(f) > c(e)$, while e is **W-maximum** if there is no edge f in A for which $w_e = w_f$ and $c(f) > c(e)$. An edge $e \in A$ that is both U-maximum and W-maximum is called **c-maximum**. Consequently, an edge $e \in A$ is c-maximum if $c(f) < c(e)$ for every edge f adjacent to e. We consider two cases.

Case 1. *Every U-maximum edge in A is W-maximum.* Let

$$M = \{e \in A : \ e \text{ is } c\text{-maximum}\}.$$

We claim that M is optimal. Since no two edges of M can be adjacent, M is a matching. Let $e \in A - M$. Since e is not c-maximum, e is not U-maximum. So, there exists an edge $f \in A$ for which $c(f)$ is maximum and $u_e = u_f$. This implies that f is U-maximum and is consequently c-maximum as well. Therefore, $f \in M$ and $c(f) > c(e)$. Hence, M is optimal.

Case 2. *There exists an edge g in A that is U-maximum but not W-maximum.* Because g is not W-maximum, there exists an edge $h \in A$ such that $w_h = w_g$ and $c(h) > c(g)$. We consider the set $A - \{h\}$ which consists of $k - 1$ edges. By the induction hypothesis, there is an optimal matching M in $A - \{h\}$. Hence, for every edge e in the set $(A - \{h\}) - M = A - (M \cup \{h\})$, there is an edge $f \in M$ for which either (i) $u_e = u_f$ and $c(f) > c(e)$ or (ii) $w_e = w_f$ and $c(f) < c(e)$.

We show that M is optimal in the set A. First, we establish the existence of an edge f in A such that either (i) $u_f = u_h$ and $c(f) > c(h)$ or (ii) $w_f = w_h$ and $c(f) < c(h)$. We consider two subcases.

Subcase 2.1. $g \notin M$. Then $g \in A - (M \cup \{h\})$. By the induction hypothesis, there is an edge $f \in M$ for which either (i) $u_g = u_f$ and $c(f) > c(g)$ or (ii) $w_g = w_f$ and $c(f) < c(g)$. Since g is U-maximum, (i) cannot occur and so (ii) must hold. Thus, $c(f) < c(g) < c(h)$ and M is optimal.

Subcase 2.2. $g \in M$. Then g is the desired edge f and, again, M is optimal.

Therefore, M is optimal in either subcase and, consequently, for every set A of edges of G, there exists an optimal matching $M \subseteq A$.

It now remains to show that there is an $\mathfrak{L}$-list edge coloring of G. Select a color $a \in \cup_{e \in E(G)} L(e)$ and let

$$A = \{e \in E(G) : a \in L(e)\}.$$

As verified above, there is an optimal matching M in A. Let $G' = G - M$. For each edge e in G', let $L'(e) = L(e) - \{a\}$. If $e \in E(G) - A$, then $a \notin L(e)$ and

$$|L'(e)| = |L(e)| = \sigma_G(e) \geq \sigma_{G'}(e).$$

On the other hand, if $e \in A - M$, then $a \in L(e)$. Since M is optimal in A, there is an edge $f \in M$ such that either (i) $u_e = u_f$ and $c(f) > c(e)$ or (ii) $w_e = w_f$ and $c(f) < c(e)$. Thus,

$$|L'(e)| = |L(e)| - 1 = \sigma_G(e) - 1 \geq \sigma_{G'}(e).$$

Let $\mathfrak{L}' = \{L'(e) : e \in E(G')\}$.

Since the size of G' is less than that of G, it follows by the induction hypothesis that G' is $\mathfrak{L}'$-edge choosable. Thus, there exists a proper edge coloring $c' : E(G') \to \mathbb{N}$ of G' such that $c'(e) \in L'(e)$ for every edge e of G'. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} c'(e) & \text{if } e \in E(G') \\ a & \text{if } e \in M. \end{cases}$$

Then $c(e) \in L(e)$ for every edge e of G and $c(e) \neq c(f)$ for every two adjacent edges e and f of G. Hence, c is a proper edge coloring and G is $\mathfrak{L}$-edge choosable. ■

From Lemma 18.16, we can now present a proof of Galvin's theorem.

Theorem 18.17 (**Galvin's Theorem**) *If G is a bipartite graph, then*

$$\chi'_\ell(G) = \chi'(G).$$

Proof. Since G is bipartite, it follows by König's theorem (Theorem 17.7) that $\chi'(G) = \Delta(G) = \Delta$. Thus, there exists a proper edge coloring $c : E(G) \to \{1, 2, \ldots, \Delta\}$. For each edge e of G, let $L(e)$ be a list of colors such that

$$\begin{aligned} |L(e)| &= \sigma_G(e) = 1 + |\{f \in E(G) : u_e = u_f \text{ and } c(f) > c(e)\}| \\ &\qquad + |\{f \in E(G) : w_e = w_f \text{ and } c(f) < c(e)\}| \\ &\leq 1 + (\chi'(G) - c(e)) + (c(e) - 1) = \chi'(G). \end{aligned}$$

Let $\mathfrak{L} = \{L(e) : e \in E(G)\}$. By Lemma 18.16, G is $\mathfrak{L}$-edge choosable. Therefore, G is $\chi'(G)$-edge choosable and so $\chi'_\ell(G) \leq \chi'(G)$. Since $\chi'(G) \leq \chi'_\ell(G)$, we have the desired result. ■

Since every bipartite graph is of Class one, it follows that the list chromatic index of every bipartite graph G equals $\Delta(G)$.

18.3 Total Colorings

We now consider colorings that assign colors to both the vertices and the edges of a graph. A **total coloring** of a graph G is an assignment of colors to the vertices and edges of G such that distinct colors are assigned to (1) every two adjacent vertices, (2) every two adjacent edges and (3) every incident vertex and edge. A k**-total coloring** of a graph G is a total coloring of G from a set of k colors. A graph G is k**-total colorable** if there is a k-total coloring of G.

The **total chromatic number** $\chi''(G)$ of a graph G is the minimum positive integer k for which G is k-total colorable.

If c is a total coloring of a graph G and v is a vertex of G with $\deg v = \Delta(G)$, then c must assign distinct colors to the $\Delta(G)$ edges incident with v as well as to v itself. This implies that

$$\chi''(G) \geq 1 + \Delta(G) \textit{ for every graph } G.$$

Let $G = K_6 - e$ be the graph of order $n = 6$ and size $m = \binom{6}{2} - 1 = 14$ shown in Figure 18.6, where $e = xz$. Since $\Delta(G) = 5$, it follows that $\chi''(G) \geq 1 + \Delta(G) = 6$. The 7-total coloring of G indicated in the figure shows that $\chi''(G) \leq 7$. We show that $\chi''(G) = 7$. Assume, to the contrary, that $\chi''(G) = 6$. Thus, there exists a 6-total coloring of G. No color can be assigned to more than two vertices. Suppose that some color is assigned to two vertices. Then these vertices must be x and z. This color can be assigned to at most two edges and so this color can be used for at most four vertices and edges. If some color is assigned to one vertex, then this color can be assigned to at most two edges; while if some color is assigned to no vertices, then this color can be assigned to at most three edges. Therefore, the total number of vertices and edges that can be colored with six colors cannot exceed $4 + 5 \cdot 3 = 19$. However, since $n + m = 6 + 14 = 20$, this is a contradiction. Thus, as claimed, $\chi''(G) = 7$.

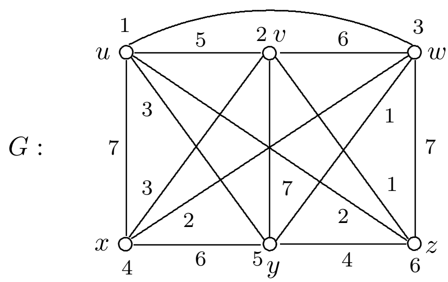


Figure 18.6: A graph with total chromatic number 7

The total chromatic number of complete graphs shows that both $\chi''(G) = 1 + \Delta(G)$ and $\chi''(G) > 1 + \Delta(G)$ are possible (see Exercise 14).

Theorem 18.18 *For an integer $n \geq 2$,*

$$\chi''(K_n) = \begin{cases} n & \textit{if } n \textit{ is odd} \\ n+1 & \textit{if } n \textit{ is even.} \end{cases}$$

The Total Coloring Conjecture

In the 1960s Mehdi Behzad [15] and Vadim Vizing [246] independently conjectured that, similar to the upper bound for the chromatic index established by Vizing, the total chromatic number of a graph G cannot exceed the lower bound $1 + \Delta(G)$ by more than 1. This conjecture has become known as the Total Coloring Conjecture.

The Total Coloring Conjecture For every graph G,

$$\chi''(G) \leq 2 + \Delta(G).$$

The sum of the chromatic number and chromatic index of a graph is always an upper bound for the total chromatic number of the graph.

Theorem 18.19 *If G is a nonempty graph, then*

$$\chi''(G) \leq \chi(G) + \chi'(G).$$

Proof. Suppose that $\chi(G) = k$ and $\chi'(G) = \ell$. Then there is a k-vertex coloring of G using the colors $1, 2, \ldots, k$ and an ℓ-edge coloring of G using the colors $k+1, k+2, \ldots, k+\ell$. Since such a coloring of $V(G) \cup E(G)$ also assigns distinct colors to every incident vertex and edge, this produces a $(k+\ell)$-total coloring of G and so $\chi''(G) \leq k + \ell = \chi(G) + \chi'(G)$. ■

Since $3 = \chi''(K_2) = \chi(K_2) + \chi'(K_2) = 2 + 1$, equality is possible in Theorem 18.19. On the other hand, if G is a graph with $\chi(G) \geq 3$ and $\chi''(G) = \chi(G) + \chi'(G)$, then $\chi''(G) = \chi(G) + \chi'(G) \geq 3 + \Delta(G)$, which would provide a counterexample to the Total Coloring Conjecture. No such counterexample is possible, as the following result of Mehdi Behzad, Gary Chartrand and John K. Cooper, Jr. [16] shows.

Theorem 18.20 *Let G be a nonempty graph. If $\chi''(G) = \chi(G) + \chi'(G)$, then G is bipartite.*

Proof. Suppose that G is a graph that is not bipartite. Then G is a graph of order at least 3 such that $\chi(G) = k$ for some integer $k \geq 3$ and so $V(G)$ can be partitioned into k vertex color classes (independent sets) $V_1, V_2, \ldots, V_k$. Suppose that $\chi'(G) = \ell$. Then $E(G)$ can be partitioned into ℓ edge color classes $E_1, E_2, \ldots, E_\ell$. For each edge $e = uv \in E_\ell$, there is a set V_i $(1 \leq i \leq k)$ containing neither u nor v. We move e from E_ℓ to V_i. After moving each edge of E_ℓ to one of the sets $V_1, V_2, \ldots, V_k$ in this manner, k new sets $X_1, X_2, \ldots, X_k$ are produced such that $V_i \subseteq X_i$ for all i $(1 \leq i \leq k)$ and $E_\ell \subseteq \cup_{i=1}^{k} X_i$, we now assign the color i $(1 \leq i \leq k)$ to each element of X_i and assign the color $k+j$ to each edge of E_j $(1 \leq j \leq \ell - 1)$. This produces a $(k+\ell-1)$-total coloring of G and so

$$\chi''(G) \leq k + \ell - 1 = \chi(G) + \chi'(G) - 1.$$

Thus, $\chi''(G) \neq \chi(G) + \chi'(G)$. ■

We now present an infinite class of (necessarily bipartite) graphs G for which $\chi''(G) = \chi(G) + \chi'(G)$.

Theorem 18.21 *For every positive integer r,*

$$\chi''(K_{r,r}) = \chi(K_{r,r}) + \chi'(K_{r,r}).$$

Proof. Let $G = K_{r,r}$. Then $|V(G) \cup E(G)| = r^2 + 2r$. Suppose that $\chi''(G) = k$. Then $V(G) \cup E(G)$ can be partitioned into sets $X_1, X_2, \ldots, X_k$ where each set X_i $(1 \le i \le k)$ consists of an independent set of vertices and an independent set of edges, where no vertex of X_i is incident with an edge of X_i. Necessarily, $|X_i| \le r$ for each i $(1 \le i \le k)$. Hence, $k \ge r + 2$. Since $\chi'(K_{r,r}) = r$ by König's theorem (Theorem 17.7) and $\chi(K_{r,r}) = 2$, it follows by Theorem 18.19 that $\chi''(G) \le r + 2$ and so

$$\chi''(G) = 2 + r = \chi(K_{r,r}) + \chi'(K_{r,r}),$$

as desired. ■

Even though it is not known if $2 + \Delta(G)$ is an upper bound for the total chromatic number of every graph, the number $2 + \chi'_\ell(G)$ *is* an upper bound.

Theorem 18.22 *For every graph G,*

$$\chi''(G) \le 2 + \chi'_\ell(G).$$

Proof. Suppose that $\chi'_\ell(G) = k$. Then

$$\begin{aligned} \chi(G) &\le 1 + \Delta(G) \le 1 + \chi'(G) \le 1 + \chi'_\ell(G) \\ &< 2 + \chi'_\ell(G) = 2 + k. \end{aligned}$$

Thus, G is $(k+2)$-colorable. Let a $(k+2)$-coloring c of G be given, using the colors $1, 2, \ldots, k+2$, say. For each edge $e = uv$ of G, let $L(e) = \{1, 2, \ldots, k+2\}$ and let

$$L'(e) = L(e) - \{c(u), c(v)\}.$$

Since $|L'(e)| \ge k$ for each edge e of G and $\chi'_\ell(G) = k$, it follows that there is a proper edge coloring c' of G such that $c'(e) \in L'(e)$ and so $c'(e) \notin \{c(u), c(v)\}$. Hence, the total coloring c'' of G defined by

$$c''(x) = \begin{cases} c(x) & \text{if } x \in V(G) \\ c'(x) & \text{if } x \in E(G) \end{cases}$$

is a $(k+2)$-total coloring of G and so

$$\chi''(G) \le 2 + k = 2 + \chi'_\ell(G),$$

as desired. ■

The List Coloring Conjecture states that $\chi'(G) = \chi'_\ell(G)$ for every nonempty graph G. If this conjecture is true, then $\chi'_\ell(G) \le 1 + \Delta(G)$ by Vizing's theorem (Theorem 17.2) and so $\chi''(G) \le 3 + \Delta(G)$ by Theorem 18.22. In 1998 Michael Molloy and Bruce Reed [172] established the existence of a constant c such that $c + \Delta(G)$ is an upper bound for $\chi''(G)$ for every graph G. In particular, they proved the following:

Theorem 18.23 *For every graph G,*

$$\chi''(G) \leq 10^{26} + \Delta(G).$$

Just as the chromatic index of a nonempty graph G equals the chromatic number of its line graph $L(G)$, the total chromatic number of G also equals the chromatic number of a related graph.

Recall that the total graph $T(G)$ of a graph G is that graph for which $V(T(G)) = V(G) \cup E(G)$ and such that two distinct vertices x and y of $T(G)$ are adjacent if x and y are adjacent vertices of G, adjacent edges of G or an incident vertex and edge. It therefore follows that

$$\chi''(G) = \chi(T(G)) \text{ for every graph } G.$$

A graph G and its total graph are shown in Figure 18.7, together with a total coloring of G and a vertex coloring of $T(G)$. In this case, $\chi''(G) = \chi(T(G)) = 4$.

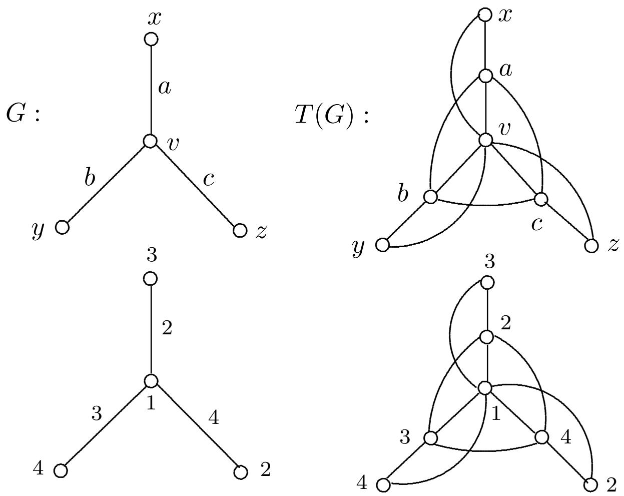


Figure 18.7: Total graphs and total colorings

Exercises for Chapter 18

Section 18.1: Nowhere-Zero Flows

1. Let G be the Eulerian graph shown in Figure 18.8.

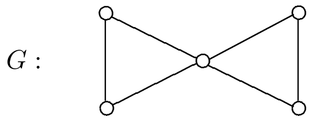


Figure 18.8: The graph G in Exercise 1

   (a) Give an example of an orientation D_1 of G and a nowhere-zero 2-flow ϕ_1 on D such that all flow values are -1.

   (b) Give an example of an orientation D_2 of G and a nowhere-zero 2-flow ϕ_2 on D such that exactly five flow values are $+1$.

   (c) Give an example of an orientation D_3 of G and a nowhere-zero 2-flow ϕ_3 on D such that half of the flow values are $+1$.

2. Let G be a graph having a nowhere-zero k-flow for some integer $k \geq 2$. Prove that for each partition $\{E_1, E_2\}$ of $E(G)$, there exists a nowhere-zero k-flow ϕ such that $\phi(e) > 0$ if and only if $e \in E_1$.

3. Prove Theorem 18.4: *If ϕ_1 and ϕ_2 are flows on an orientation D of a graph G, then every linear combination of ϕ_1 and ϕ_2 is also a flow on D.*

4. Show that the Petersen graph has a nowhere-zero 5-flow.

5. Prove that every bridgeless graph containing an Eulerian trail has a nowhere-zero 3-flow.

6. Can a graph have a cycle double cover consisting only of three Hamiltonian cycles?

7. Show that K_6 has a cycle double cover consisting only of Hamiltonian cycles.

8. (a) Does there exist a cycle double cover of K_4 consisting only of triangles?

   (b) Does there exist a cycle double cover of K_5 consisting only of triangles?

   (c) Does there exist a cycle double cover of K_5 consisting only of 4-cycles?

Section 18.2: List Edge Colorings

9. Let G be a nonempty graph. Show that $\chi'_\ell(G) \le 2\Delta(G) - 1$ by applying a greedy edge coloring to G.

10. The graph G in Figure 18.9 is a bipartite graph with $\Delta(G) = 3$. Therefore, $\chi'(G) = 3$. By Theorem 18.17, $\chi'_\ell(G) = 3$. Hence, for every integer i with $1 \le i \le 9$ and every set $L'(e_i)$ of three or more colors, G is $\mathfrak{L}'$-edge choosable where $\mathfrak{L}' = \{L'(e_i) : 1 \le i \le 9\}$. Determine the minimum positive integer k with $\sum_{i=1}^{9} |L(e_i)| = k$ such that G is $\mathfrak{L}$-edge choosable where $\mathfrak{L} = \{L(e_i) : 1 \le i \le 9\}$.

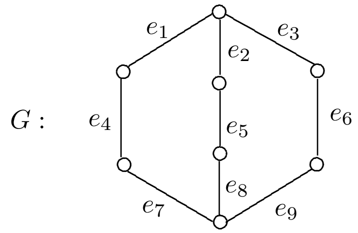


Figure 18.9: The graph in Exercise 10

Section 18.3: Total Colorings

11. For $G = K_{2,2,2}$, determine $\chi(G)$, $\chi'(G)$ and $\chi''(G)$.

12. Determine $\chi(G)$, $\chi'(G)$ and $\chi''(G)$ for the 4-regular graph $G = C_7^2$ shown in Figure 18.10.

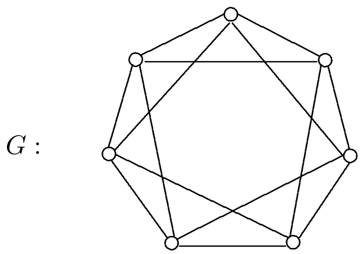


Figure 18.10: The graph in Exercise 12

13. Determine $\chi''(G)$ for the graph G of Figure 18.11.

14. Verify Theorem 18.18. *For an integer* $n \ge 2$*,*

    (*a*) $\chi''(K_n) = n$ *if* n *is odd;*

    (*b*) $\chi''(K_n) = n + 1$ *if* n *is even.*

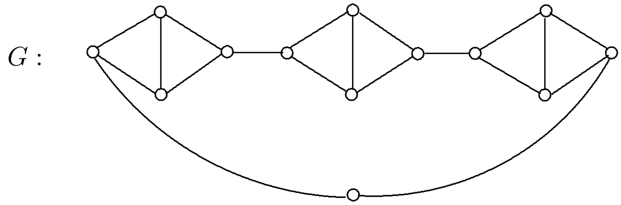


Figure 18.11: The graph in Exercise 13

15. Show for integers s and t with $1 \leq s < t$ that $\chi''(K_{s,t}) = t + 1$.

16. Verify the Total Coloring Conjecture for graphs G with $\Delta(G) \leq 2$.

Chapter 19

Extremal Graph Theory

In this chapter and the next we introduce some concepts and theorems from an area of graph theory known as *extremal graph theory.* Of the many problems encountered in this area, a common one asks for the minimum size of a graph G of a given order which guarantees that G contains a certain subgraph or possesses a certain property. More generally, an *extremal problem* asks for the maximum or minimum value of a graph theoretic parameter in a class of graphs with a given property.

19.1 Turán's Theorem

The first result in graph theory that might be considered belonging to extremal graph theory is a 1907 theorem due to Willem Mantel [161] who determined the minimum size of a graph of a given order which guarantees that the graph contains a triangle. This result was presented (and proved) in Chapter 1 as Theorem 1.9, which we state again below.

Theorem 19.1 *If the size of a graph G of order $n \geq 3$ is at least $\lfloor n^2/4 \rfloor + 1$, then G contains a triangle.*

We saw that this result is no longer true if $\lfloor n^2/4 \rfloor + 1$ is replaced by a smaller integer because the complete bipartite graph $K_{\lfloor n/4 \rfloor, \lceil n/4 \rceil}$ of order n has size $\lfloor n^2/4 \rfloor$ and, of course, contains no triangles.

Turán Graphs

A much more general result was obtained by Paul Turán in 1941 and it is this result that is often credited as beginning extremal graph theory. For an integer $n \geq 3$ and for each positive integer $k \leq n$, let $n_1, n_2, \ldots, n_k$ be k integers such that

$$n = n_1 + n_2 + \ldots + n_k,\ 1 \le n_1 \le n_2 \le \ldots \le n_k, \text{ and } n_k - n_1 \le 1.$$

For every two integers k and n with $1 \le k \le n$, the integers $n_1, n_2, \ldots, n_k$ are unique. For example, for $n = 11$ and $k = 3$, the integers n_1, n_2, n_3 are $3, 4, 4$; while for $n = 14$ and $k = 6$, the integers $n_1, n_2, \ldots, n_6$ are $2, 2, 2, 2, 3, 3$. The complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is called the **Turán graph** $T_{n,k}$. Thus, the Turán graph $T_{n,k}$ is the complete k-partite graph of order n, the cardinalities of whose partite sets differ by at most 1. The number of elements in each partite set of $T_{n,k}$ is either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. If n/k is an integer, then $\lfloor \frac{n}{k} \rfloor = \lceil \frac{n}{k} \rceil = \frac{n}{k}$; while if n/k is not an integer and r is the remainder when n is divided by k, then exactly r of the partite sets of $T_{n,k}$ have cardinality $\lceil \frac{n}{k} \rceil$.

Turán's Theorem

Since the Turán graph $T_{n,k}$ is a complete k-partite graph, $T_{n,k}$ contains no $(k+1)$-clique (a clique of order $k+1$). The size of $T_{n,k}$ is denoted by $t_{n,k}$. Thus, there exists a graph of order n and size $t_{n,k}$ containing no $(k+1)$-clique. Turán [237] showed that for positive integers n and k with $n \ge k$, the graph $T_{n,k}$ is the unique graph of order n and maximum size having no $(k+1)$-clique.

Theorem 19.2 (**Turán's Theorem**) *Let n and k be positive integers with $n \ge 3$ and $n \ge k$. The Turán graph $T_{n,k}$ is the unique graph of order n and maximum size having no $(k+1)$-clique.*

Proof. We proceed by induction on k. Since the only graph of order n containing no 2-clique is $\overline{K}_n = T_{n,1}$, the result holds for $k = 1$. Assume, for an integer $k \ge 2$ and an integer n' with $n' \ge k-1$, that $T_{n',k-1}$ is the unique graph of order n' and maximum size containing no k-clique. Next, let $n \ge k$ and let G be a graph of order n and maximum size containing no $(k+1)$-clique. We show that $G = T_{n,k}$.

Let v be a vertex of G with $\deg v = \Delta(G) = \Delta$ and let $H = G[N_G(v)]$. Thus, H is a graph of order Δ containing no k-clique. We show that $\Delta \ge k-1$. Assume, to the contrary, that $\Delta \le k-2$. Since $n \ge k$, there exists $v \in V(G) - \{u\}$ that is not adjacent to u. Then the graph $G + uv$ contains a $(k+1)$-clique. On the other hand, each vertex $x \in V(G) - \{u, v\}$ has degree at most $\Delta \le k-2$ in $G+uv$ and $k+1 \ge 3$, which is impossible. Hence, $\Delta \ge k-1$. By the induction hypothesis, $H' = T_{\Delta,k-1}$ is the unique graph of order Δ and maximum size containing no k-clique. Thus, $|E(H)| \le |E(H')|$ and

$$|E(H)| = |E(H')| \text{ only if } H = H' = T_{\Delta,k-1}.$$

Define $G' = \overline{K}_{n-\Delta} \vee H'$. Hence, G' has order n. Also, since H' is a $(k-1)$-partite graph, G' is a k-partite graph and so contains no $(k+1)$-clique. Furthermore,

$$|E(G')| = |E(H')| + \Delta(n - \Delta).$$

For each $u \in V(G) - V(H)$, at most Δ edges of G do not belong to H. Hence,

$$|E(G) \leq |E(H)| + \Delta(n - \Delta).$$

Therefore,

$$|E(G)| \leq |E(H)| + \Delta(n - \Delta) \leq |E(H')| + \Delta(n - \Delta) = |E(G')|. \qquad (19.1)$$

Since G is a graph of order n and maximum size containing no $(k+1)$-clique, it follows that $|E(G)| \geq |E(G')|$ and so $|E(G)| = |E(G')|$. Thus, we have equality throughout (19.1). Because $|E(G)| = |E(H)| + \Delta(n - \Delta)$, it follows that $G = \overline{K}_{n-\Delta} \vee H$. Since $|E(H)| = |E(H')|$ and $H' = T_{\Delta,k-1}$ is the unique graph of order Δ and maximum size containing no k-clique, it follows that $H = H' = T_{\Delta,k-1}$. Thus, $G = \overline{K}_{n-\Delta} \vee T_{\Delta,k-1}$ and so G is a complete k-partite graph, which implies that $G = G'$.

Suppose that the partite sets of G are $V_1, V_2, \ldots, V_k$ where $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$. If $G \neq T_{n,k}$, then V_k contains at least two more vertices than V_1. Let $x \in V_k$. Replacing V_1 by $V_1 \cup \{x\}$ and V_k by $V_k - \{x\}$ produces a complete k-partite graph whose size exceeds that of G by $|V_k| - |V_1| - 1 \geq 1$, which is a contradiction. Thus, $T_{n,k}$ is the unique graph of order n and maximum size containing no $(k+1)$-clique. ■

As a consequence of Turán's theorem, we have the following:

Corollary 19.3 *Let k and n be integers with $2 \leq k+1 \leq n$. The minimum size of a graph G of order n which guarantees that G contains a $(k+1)$-clique is $t_{n,k} + 1$.*

19.2 Extremal Subgraphs

In Corollary 19.3, we saw that the minimum size of a graph of order n that contains a $(k+1)$-clique is $t_{n,k} + 1$. We now look at the minimum size of a graph G of a fixed order which guarantees that G contains particular subgraphs.

Graphs with Two Disjoint Cycles

We have already seen that the minimum size of a graph G of order $n \geq 3$ which guarantees that G contains a cycle is n. Although barely a teenager at the time, Lajos Pósa (see [80]) determined the minimum size of a graph G of order $n \geq 6$ which guarantees that G contains two disjoint cycles.

Theorem 19.4 *Every graph of order $n \geq 6$ and size at least $3n - 5$ contains two disjoint cycles.*

Proof. It suffices to show that every graph of order n and size $3n-5$ contains two disjoint cycles for $n \geq 6$. We employ induction on n. There are only two graphs of order 6 and size 13, one obtained by removing two nonadjacent edges from K_6 and the other obtained by removing two adjacent edges from K_6. In either case, the resulting graph has two disjoint triangles. Thus, the result is true for $n = 6$.

Assume, for a given integer $n > 6$, that for all integers k with $6 \leq k < n$, every graph of order k and size $3k - 5$ contains two disjoint cycles. Let G be a graph of order n and size $3n - 5$. Since

$$\sum_{v \in V(G)} \deg v = 6n - 10,$$

there exists a vertex v_0 of G such that $\deg v_0 \leq 5$. Suppose first that $\deg v_0 = 5$, and $N(v_0) = \{v_1, v_2, \ldots, v_5\}$. If the subgraph $G[N[v_0]]$ induced by the closed neighborhood $N[v_0]$ contains 13 or more edges, then we have already noted that $G[N[v_0]]$ has two disjoint cycles, implying that G has two disjoint cycles. If, on the other hand, $G[N[v_0]]$ contains 12 or fewer edges, then, since $\deg v_0 = 5$, some neighbor of v_0, say v_1, is not adjacent to two other neighbors of v_0, say v_2 and v_3. Add to G the edges v_1v_2 and v_1v_3 and delete the vertex v_0, obtaining the graph G'; that is, $G' = G + v_1v_2 + v_1v_3 - v_0$. The graph G' is a graph of order $n - 1$ and size $3n - 8$ and, by the inductive hypothesis, contains two disjoint cycles C_1 and C_2. At least one of these cycles, say C_1, does not contain the vertex v_1 and therefore contains neither the edge v_1v_2 nor the edge v_1v_3. Hence, C_1 is a cycle of G. If C_2 contains neither v_1v_2 nor v_1v_3, then C_1 and C_2 are disjoint cycles of G. If C_2 contains v_1v_2 but not v_1v_3, then by removing v_1v_2 and adding v_0, v_0v_1 and v_0v_2, we produce a cycle of G that is disjoint from C_1. The procedure is similar if C_2 contains v_1v_3 but not v_1v_2. If C_2 contains both v_1v_2 and v_1v_3, then, by removing v_1 from C_2 and adding v_0, v_0v_2 and v_0v_3, a cycle of G disjoint from C_1 is produced.

Suppose next that $\deg v_0 = 4$, where $N(v_0) = \{v_1, v_2, v_3, v_4\}$. If $G[N[v_0]]$ is not complete, then there are two vertices of $N(v_0)$ that are not adjacent, say v_1 and v_2. By adding v_1v_2 to G and deleting v_0, we obtain a graph G' of order $n - 1$ and size $3n - 8$, which, by the inductive hypothesis, contains two disjoint cycles. We may now proceed, as before, to show that G has two disjoint cycles. Assume then that $G[N[v_0]]$ is a complete graph of order 5. If some vertex of $V(G) - N[v_0]$ is adjacent to two or more neighbors of v_0, then G contains two disjoint cycles. Hence, we may assume that no vertex of $V(G) - N[v_0]$ is adjacent to more than one vertex of $N(v_0)$. Remove the vertices v_0, v_1 and v_2 from G, and note that the resulting graph G'' has order $n - 3$ and contains at least $(3n - 5) - (n - 5) - 9 = 2n - 9$ edges. However, $n \geq 6$ implies that $2n - 9 \geq n - 3$; so G'' contains at least one cycle C. The cycle C and the cycle (v_0, v_1, v_2, v_0) are disjoint and belong to G.

Finally, we assume that $\deg v_0 \leq 3$. The graph $G - v_0$ is a graph of order $n - 1$ and size m, where $m \geq 3n - 8$. Hence, by the inductive hypothesis, $G - v_0$ (and therefore G) contains two disjoint cycles. ■

To see that the bound $3n-5$ presented in Theorem 19.4 is sharp, observe that the complete 4-partite graph $K_{1,1,1,n-3}$ has order n and size $3n-6$. Every cycle in $K_{1,1,1,n-3}$ contains at least two of the three vertices having degree $n-1$. Thus, no two cycles of $K_{1,1,1,n-3}$ are disjoint.

Subgraphs with a Given Minimum Degree

Another example of a theorem in extremal graph theory gives the minimum number of edges that a graph G of order n must have to guarantee that G contains a subgraph with a specified minimum degree.

Theorem 19.5 *Let k and n be integers with $1 \le k < n$. Every graph of order n and size at least*

$$(k-1)n - \binom{k}{2} + 1$$

contains a subgraph with minimum degree k.

Proof. We proceed by induction on $n \ge k+1$. First, assume that $n = k+1$. Let G be a graph of order n and size at least

$$(k-1)n - \binom{k}{2} + 1 = (n-2)n - \binom{n-1}{2} + 1 = \binom{n}{2}.$$

Then $G = K_n = K_{k+1}$ and so G itself is a graph with minimum degree k.

Assume that every graph of order $n-1 \ge k+1$ and size at least

$$(k-1)(n-1) - \binom{k}{2} + 1$$

contains a subgraph with minimum degree k. Let G be a graph of order n and size m, where

$$m \ge (k-1)n - \binom{k}{2} + 1.$$

We show that G contains a subgraph with minimum degree k. If G itself is not such a graph, then G contains a vertex v with $\deg v \le k-1$. Then the order of $G - v$ is $n-1$ and its size is $m - \deg v$. Since

$$m - \deg v \ge (k-1)n - \binom{k}{2} + 1 - (k-1) = (k-1)(n-1) - \binom{k}{2} + 1,$$

it follows by the induction hypothesis that $G - v$, and therefore G as well, contains a subgraph with minimum degree k. ■

The bound given in Theorem 19.5 cannot be improved. If $k = 1$, then the graph $\overline{K}_n$ contains no subgraph with minimum degree 1. More generally, for $2 \le k < n$, the graph $\overline{K}_{n-k+1} \vee K_{k-1}$ has order n and size $(k-1)n - \binom{k}{2}$

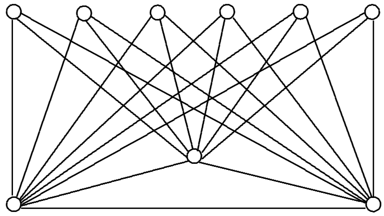


Figure 19.1: A graph of order 9 and size 21
containing no subgraph with minimum degree 4

but contains no subgraph with minimum degree k. For $n = 9$ and $k = 4$, the graph $\overline{K}_{n-k+1} \vee K_{k-1} = \overline{K}_6 \vee K_3$ (shown in Figure 19.1) has no subgraph with minimum degree 4.

By Theorem 3.20, if G is a graph such that $\delta(G) \geq k$ for some positive integer k, then G contains every tree of size k as a subgraph. Combining this result with Theorem 19.5 gives us the following corollary.

Corollary 19.6 *Let k and n be integers with $1 \leq k < n$. If G is a graph of order n and size at least*

$$(k-1)n - \binom{k}{2} + 1,$$

then G contains every tree of size k as a subgraph.

Corollary 19.6 is not best possible, however (see Exercise 4). A 1963 conjecture due to Paul Erdös and Vera Sós states that if k and n are integers with $1 \leq k < n$ and G is a graph of order n and size greater than $(k-1)n/2$, then G contains every tree of size k as a subgraph.

We saw in Chapter 13 that if G is a graceful graph of size m, then there exists a partition of $V(G)$ into two subsets V_e and V_0 such that the number of edges joining V_e and V_0 is $\lceil \frac{m}{2} \rceil$ (see Theorem 13.19). This implies, of course, that every graceful graph of size m has a bipartite subgraph with at least $\lceil \frac{m}{2} \rceil$ edges. This second statement is true, in fact, for every graph G of size m. This too can be considered a result in extremal graph theory. The proof applies an approach used in the proof of Turán's theorem (Theorem 19.2).

Theorem 19.7 *Every nontrivial graph of size m contains a bipartite subgraph with at least $\lceil \frac{m}{2} \rceil$ edges.*

Proof. Let H be a bipartite subgraph of maximum size in a graph G. We may assume, without loss of generality, that H is a spanning subgraph of G with partite sets U and W. If $\deg_H v \geq \frac{1}{2} \deg_G v$ for every vertex v of G, then

$$2|E(H)| = \sum_{v \in V(H)} \deg_H v \geq \frac{1}{2} \sum_{v \in V(G)} \deg_G v = m$$

and so H contains at least $\lceil \frac{m}{2} \rceil$ edges. Otherwise, there exists some vertex $v \in V(H) = V(G)$ for which $\deg_H v < \frac{1}{2} \deg_G v$. Suppose that $v \in U$. Thus, v is adjacent in G to more vertices in U than to vertices in W. Consider the spanning bipartite graph H' of G with partite sets U' and W', where $U' = U - \{v\}$ and $W' = W \cup \{v\}$, where $E(H')$ consists of all edges of G joining U' and W'. Then, since $\deg_H v < \deg_{H'} v$, it follows that $|E(H')| > |E(H)|$, contradicting the choice of H. ■

19.3 Cages

We have seen the interest in and the importance of cubic graphs on many occasions. For example, in Chapter 6, we saw that the graph of the dodecahedron played a key role in the origin of Hamiltonian graphs. This graph is a cubic graph of order 20 whose smallest cycle has length 5. We saw in Chapter 10 that the cubic graph $K_{3,3}$ is one of the two forbidden graphs in Kuratowski's characterization of planar graphs involving subdivisions of graphs and one of the forbidden minors in Wagner's characterization of planar graphs. The cubic graph K_4 is also a forbidden graph in the characterization of outerplanar graphs in terms of subdivisions of graphs. Of course, the smallest cycle in $K_{3,3}$ has length 4 and K_4 has triangles.

In Chapter 12 we noted that while it is easy to determine whether an r-regular graph contains a 1-factor when $r < 3$, the problem is considerably more challenging for cubic graphs. In Theorem 12.8, we saw in Petersen's theorem that every bridgeless cubic graph contains a 1-factor. In Chapter 13, we mentioned that Tait evidently believed that every bridgeless cubic graph is actually 1-factorable. However, Petersen showed that this was not the case by giving an example of a bridgeless cubic graph that is not 1-factorable, namely the Petersen graph, whose smallest cycle has length 5.

In Chapter 16, the famous Four Color Problem was discussed. We saw that this problem could be solved if it is possible to show that the regions of every bridgeless cubic planar graph could be colored with four or fewer colors in such a way that adjacent regions are colored differently. Furthermore, Kempe showed that the cubic maps of greatest interest were those whose regions had boundaries consisting of cycles of length 5 or more. This was where an error occurred in Kempe's attempted solution of the Four Color Problem. Heawood not only discovered this mistake, he went on to investigate colorings of maps embedded on surfaces of higher genus. Heawood displayed a cubic map embedded on the torus having seven regions (each of whose boundaries is a 6-cycle) and 14 vertices that requires seven colors to color its regions. This map appears in Figure 16.23 and is repeated in Figure 19.2(a). Furthermore, this map (graph) is redrawn in Figure 19.2(b).

In Chapter 17, we saw that Tait attempted to solve the Four Color Problem by proving that the regions of every bridgeless cubic plane graph could be colored with four or fewer colors if and only if the chromatic index of every such graph is 3. This gave rise to the topic of Tait colorings.

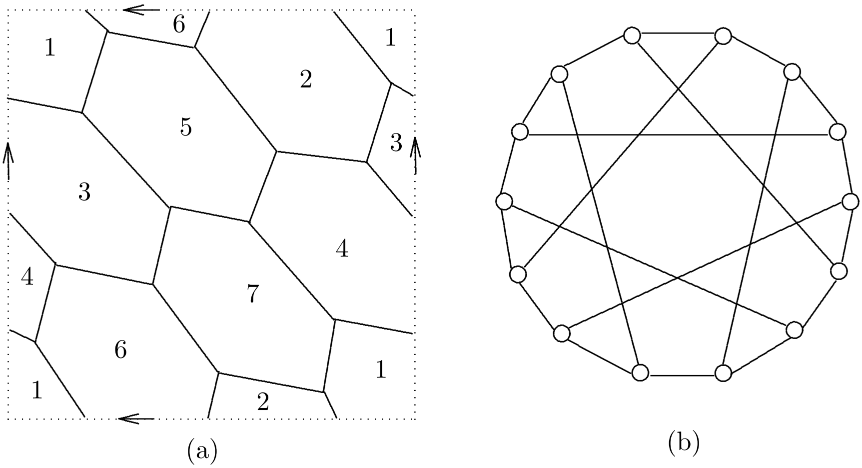


Figure 19.2: A map (graph) embedded on the torus

Consequently, it is not surprising that interest grew not only in the study of cubic graphs but in cubic graphs not containing cycles of small length. More generally, for integers $r \geq 2$ and $g \geq 3$, this led to an investigation of r-regular graphs having girth g – in particular to those of smallest order, the so-called *cages*. The study of this class of graphs was initiated by William Tutte [240] in 1947. The related problem of determining the minimum order of an r-regular Hamiltonian graph of girth g for given integers r and g was described by Francesco Kárteszi [140] in 1960.

For integers $r \geq 2$ and $g \geq 3$, an (r, g)**-graph** is an r-regular graph having girth g. For $r, g \geq 3$, the integer $M(r, g)$ is defined as

$$M(r,g) = \begin{cases} 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{(g-3)/2} & \text{if } g \text{ is odd} \\ 2\left[1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{(g-2)/2}\right] & \text{if } g \text{ is even} \end{cases}$$

$$= \begin{cases} 1 + \frac{r\left[(r-1)^{(g-1)/2} - 1\right]}{r-2} & \text{if } g \text{ is odd} \\ \frac{2r\left[(r-1)^{g/2} - 1\right]}{r-2} & \text{if } g \text{ is even.} \end{cases}$$

This number counts the number of vertices at distance $0, 1, 2, \ldots, (g-1)/2$ from a given vertex in an (r, g)-graph if g is odd and the number of vertices at distance $0, 1, 2, \ldots, (g-2)/2$ from one of two adjacent vertices in an (r, g)-graph if g is even. The number $M(r, g)$ is therefore a lower bound for the order of an (r, g)-graph. The number $M(r, g)$ is called the **Moore bound**, named for Edward F. Moore, whom we will encounter later in this section.

Theorem 19.8 *If G is an (r, g)-graph of order n, then $n \geq M(r, g)$.*

Proof. First, suppose that g is odd. Then $g = 2k+1$ for some positive integer k. Let $v \in V(G)$. For $1 \leq i \leq k$, the number of vertices at distance i from v is $r(r-1)^{i-1}$. Since the girth of G is $g = 2k+1$, no two vertices of G at distance i from v can be adjacent to a vertex of distance i or $i-1$ from v if $i \leq (g-5)/2$. From this, it follows that

$$\begin{aligned} n &\geq 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{k-1} \\ &= 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{(g-3)/2}. \end{aligned}$$

Next, suppose that g is even. Then $g = 2k$ where $k \geq 2$. Let $e = uv \in E(G)$. For $1 \leq i \leq k-1$, the number of vertices at distance i from u or v is $2(r-1)^i$. Here,

$$\begin{aligned} n &\geq 2 + 2(r-1) + 2(r-1)^2 + \cdots + 2(r-1)^{k-1} \\ &= 2\left[1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{(g-2)/2}\right]. \end{aligned}$$

Thus, $n \geq M(r, g)$. ■

For integers $r \geq 2$ and $g \geq 3$, the smallest order of an (r, g)-graph is denoted by $n(r, g)$. The r-regular graphs of order $n(r, g)$ having girth g are called **(r, g)-cages**. Thus, among the (r, g)-graphs, an (r, g)-cage is one of minimum order. The $(3, g)$-cages are the most studied of these graphs and are commonly referred to simply as **g-cages**. Therefore, a g-cage is a cubic graph of minimum order having girth g.

Existence of Cages

The most fundamental question now is the following: For which pairs r, g of integers with $r \geq 2$ and $g \geq 3$, does there exist an (r, g)-graph? Of course, if there exists even one (r, g)-graph, then there is an (r, g)-cage, perhaps more than one. If there exists an (r, g)-graph for given integers $r \geq 2$ and $g \geq 3$, then $n(r, g) \geq \max\{r+1, g\}$. Since C_g is a 2-regular graph of girth g, it follows that $n(2, g)$ exists and $n(2, g) = g$. Likewise, $n(r, 3) = r+1$ since K_{r+1} is an r-regular graph having girth 3. In fact, the complete graph K_4 is the unique 3-cage. The following result of Paul Erdös and Horst Sachs [83] not only establishes the existence of a cage for each pair r, g but provides an upper bound on their orders.

Theorem 19.9 *For every pair r, g of integers with $r \geq 2$ and $g \geq 3$, the number $n(r, g)$ exists and*

$$n(r, g) \leq 3(r-1) + 2\left[(r-1)^2 + (r-1)^3 + \cdots + (r-1)^{g-2}\right] + (r-1)^{g-1}.$$

Proof. Since $n(2, g) = g$ for $g \geq 3$ and $n(r, 3) = r+1$ for $r \geq 2$, we may assume that $r \geq 3$ and $g \geq 4$. Let

$$n = 3(r-1) + 2\left[(r-1)^2 + (r-1)^3 + \cdots + (r-1)^{g-2}\right] + (r-1)^{g-1}$$

and let S be the set of all graphs H of order n such that $g(H) = g$ and $\Delta(H) \leq r$. Thus, $n \geq g$. Since $C_g + (n-g)K_1 \in S$, it follows that $S \neq \emptyset$. For each graph H in S, let

$$V_r(H) = \{v \in V(H) : \deg_H v < r\}.$$

Should it occur that $V_r(H) = \emptyset$ for some $H \in S$, then this graph H has the desired properties. We may therefore assume that $V_r(H) \neq \emptyset$ for all $H \in S$. For a graph H in S, define $d(H)$ to be the maximum distance between two vertices of $V_r(H)$ if every two vertices are connected in H and define $d(H) = +\infty$ otherwise. Observe that if $|V_r(H)| = 1$, then $d(H) = 0$.

Among the graphs belonging to S, let S_1 be the set of those graphs having maximum size. Also, among the graphs belonging to S_1, let S_2 be the set of all those graphs H for which $|V_r(H)|$ is maximum. Finally, among those graphs belonging to S_2, let G be one for which $d(G)$ is maximum.

Let $u, v \in V_r(G)$ such that $d_G(u,v) = d(G)$. Suppose first that $d(G) \geq g-1 \geq 2$. Then the graph $G' = G + uv$ has order n, girth g and $\Delta(G') \leq r$, which implies that $G' \in S$. However, since the size of G' exceeds the size of G and $G \in S$, this produces a contradiction. Therefore, $d(G) \leq g-2$ and so $d_G(u,v) \leq g-2$.

Denote by W the set of all vertices w in G such that either $d_G(u,w) \leq g-2$ or $d_G(v,w) \leq g-1$. Thus, $u, v \in W$. The number of vertices distinct from u and whose distance from u is at most $g-2$ cannot exceed

$$(r-1) + (r-1)^2 + \cdots + (r-1)^{g-2}.$$

Moreover, the number of vertices distinct from v and whose distance from v is at most $g-1$ cannot exceed

$$(r-1) + (r-1)^2 + \cdots + (r-1)^{g-1}.$$

Therefore,

$$\begin{aligned}
|W| &\leq 2\left[(r-1) + (r-1)^2 + \cdots + (r-1)^{g-2}\right] + (r-1)^{g-1} \\
&= 3(r-1) + 2\left[(r-1)^2 + (r-1)^3 + \cdots + (r-1)^{g-2}\right] + \\
&\quad\ (r-1)^{g-1} - (r-1) \\
&= n - r + 1 < n.
\end{aligned}$$

Since $|W| < n$, there is a vertex $w_1 \in V(G) - W$. Hence, $d_G(u, w_1) \geq g-1$ and $d_G(v, w_1) \geq g$. Because $d_G(u, w_1) > d(G)$ and $u \in V_r(G)$, it follows that $w_1 \notin V_r(G)$ and so $\deg_G w_1 = r \geq 3$. Therefore, there exists an edge $e = w_1w_2$ such that $g(G-e) = g$. Since $d_G(v, w_1) \geq g$, it follows that $d(v, w_2) \geq g-1$. Since $d(G) \leq g-2$ and $v \in V_r(G)$, it follows that $w_2 \notin V_r(G)$ and $\deg_G w_2 = r$.

Let $G_1 = G - w_1w_2 + uw_1$. Observe that G_1 also belongs to S and, in fact, belongs to S_1. The set $V_r(G_1)$ contains every vertex in $V_r(G)$ except possibly u and, in addition, contains w_2. Because of the way G was chosen, it follows that $|V_r(G_1)| \leq |V_r(G)|$. This implies that

(i) $u \notin V_r(G_1)$, (ii) $\deg_G u = r - 1$ and (iii) $|V_r(G_1)| = |V_r(G)|$.

Hence, G_1 also belongs to S_2.

We claim that u is not the only vertex in $V_r(G)$, for suppose that it is. Since G contains $n-1$ vertices of degree r and one vertex of degree $r-1$, it follows that r and n are both odd. However, from the way n is defined, n is even when r is odd. Thus, as claimed, u is not the only vertex in $V_r(G)$ and so u and v are distinct vertices of $V_r(G)$.

Next, we claim that G_1 contains a $v - w_2$ path, for suppose that no such path exists. Then $d(G_1) = +\infty$. However then,

$$+\infty = d(G_1) \leq d(G) \leq g - 2,$$

which is impossible. So, as claimed, G_1 contains a $v - w_2$ path. Let P be a $v - w_2$ geodesic in G_1. If P is also a $v - w_2$ path in G, then the length of P is at least $d_G(v, w_2)$. However,

$$d_G(v, w_2) \geq g - 1 > d(G),$$

which is impossible since $v, w_2 \in V_r(G)$. Therefore, P is not a $v - w_2$ path in G, which implies that the edge uw_1 lies on P. Thus, in G, the path P contains a $u - v$ subpath of length $d_G(u, v)$. Since the length of P exceeds $d_G(u, v) = d(G)$, a contradiction is produced.

Hence, there is, in fact, some graph $H \in S$ with $V_r(H) = \emptyset$ and so H is an r-regular graph of order n having girth g. ■

Examples of Cages

Now that we know that (r, g)-cages exist for all pairs r, g of integers with $r \geq 2$ and $g \geq 3$, we determine some specific cages, beginning with $(r, 4)$-cages.

Theorem 19.10 *For each integer $r \geq 2$, $n(r, 4) = 2r$. Furthermore, the graph $K_{r,r}$ is the only $(r, 4)$-cage.*

Proof. Since the Moore bound $M(r, 4) = 2r$, it follows by Theorem 19.8 that $n(r, 4) \geq 2r$. Obviously, the graph $K_{r,r}$ is r-regular, has girth 4 and has order $2r$. Consequently, $n(r, 4) = 2r$. Furthermore, $K_{r,r}$ is an $(r, 4)$-cage. It remains to show that $K_{r,r}$ is the *only* $(r, 4)$-cage. Let G be an $(r, 4)$-cage of order $2r$ and let $u_1 \in V(G)$. Denote by $v_1, v_2, \ldots, v_r$ the vertices of G adjacent to u_1. Since $g(G) = 4$ and G is r-regular, it follows that $\{v_1, v_2, \ldots, v_r\}$ is an independent set and so v_1 is adjacent to none of the vertices v_i $(2 \leq i \leq r)$. Hence, G contains $r-1$ additional vertices $u_2, u_3, \ldots, u_r$. Similarly, $\{u_1, u_2, \ldots, u_r\}$ is an independent set and each vertex u_i $(1 \leq i \leq r)$ is adjacent to every vertex v_j $(1 \leq j \leq r)$. Therefore, $G = K_{r,r}$. ■

The most-studied (r, g)-cages are the $(3, g)$-cages, that is, the cubic cages. We now know then that K_4 is the unique 3-cage and that $K_{3,3}$ is the unique

4-cage. Since the Petersen graph is a cubic graph of girth 5, it is a $(3,5)$-graph. Because the order of the Petersen graph is 10, it follows that $n(3,5) \leq 10$. On the other hand, $M(3,5) = 10$ and so $n(3,5) \geq 10$. Hence, $n(3,5) = 10$ and so the Petersen graph is a 5-cage. In fact, it is the only 5-cage.

Theorem 19.11 *The Petersen graph is the unique* 5*-cage.*

Proof. Let G be a $(3,5)$-graph of order 10. We show that G is isomorphic to the Petersen graph. Let $v_1 \in V(G)$ and suppose that v_1 is adjacent to v_2, v_3 and v_4. Since $g(G) = 5$, each vertex v_i, $i = 2, 3, 4$, is adjacent to two new vertices of G. Suppose that v_2 is adjacent to v_5 and v_6, v_3 is adjacent to v_7 and v_8, and v_4 is adjacent to v_9 and v_{10}. Hence,

$$V(G) = \{v_i : \ i = 1, 2, \ldots, 10\}.$$

The fact that the girth of G is 5 and that every vertex of G has degree 3 implies that v_5 is adjacent to one of v_7 and v_8 and to one of v_9 and v_{10}. Without loss of generality, we may assume that v_5 is adjacent to v_8 and v_9. Necessarily, v_6 is adjacent to v_7 and v_{10}. Therefore, the edges v_7v_9 and v_8v_{10} are also present in G but no other edges are present. Thus, G is isomorphic to the Petersen graph (see Figure 19.3). ∎

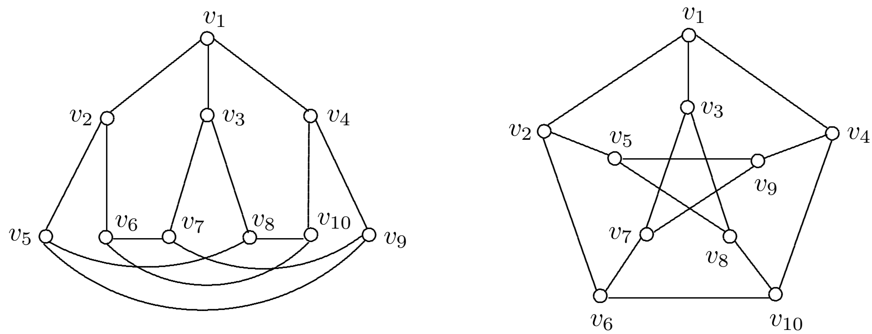


Figure 19.3: The Petersen graph: the unique 5-cage

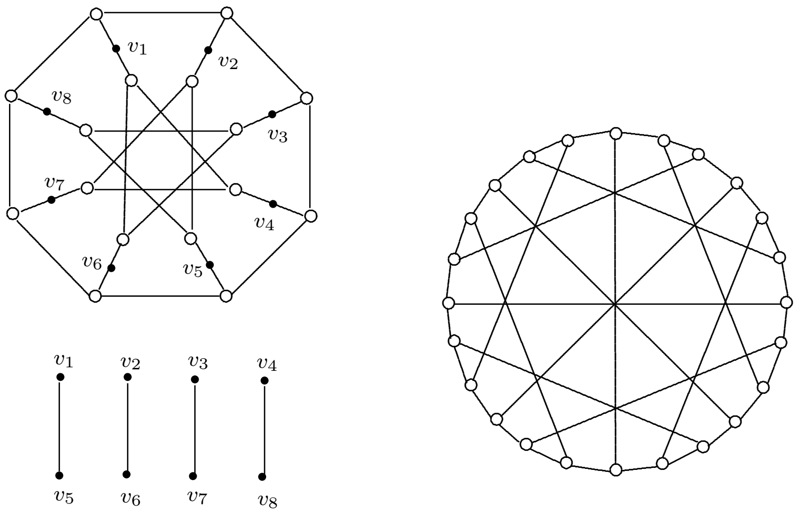
Only a few g-cages are known for which $g \geq 6$. For example, there is only one 6-cage, namely the **Heawood graph**, and this is shown in Figure 19.2(b). The Heawood graph is also the toroidal dual of the embedding of K_7 on the torus (see Figure 19.2). That is, this graph is constructed by inserting a vertex in each of the 14 regions and joining two vertices by an edge if the corresponding regions have a common boundary edge.

There is also only one 7-cage, known as the **McGee graph** and named after William F. McGee [165]. Although initially discovered by Horst Sachs, it was first published by McGee in 1960. There is also only one 8-cage, often known as the **Tutte-Coxeter graph** and first discovered by Tutte [240]. It is named for

Tutte and the famous geometer H. S. M. Coxeter, who studied combinatorial properties of this graph.

The most common way of drawing the Petersen graph P can be described by beginning with a 5-cycle $C = (u_1, u_2, u_3, u_4, u_5, u_1)$ and five additional vertices w_1, w_2, w_3, w_4, w_5 along with the five edges u_iw_i $(1 \le i \le 5)$. Then each vertex w_i is joined to $w_{i\pm 2}$ (subscripts expressed as $1, 2, 3, 4, 5$ modulo 5). This then results in the 5-cycle $C' =(w_1, w_3, w_5, w_2, w_4, w_1)$.

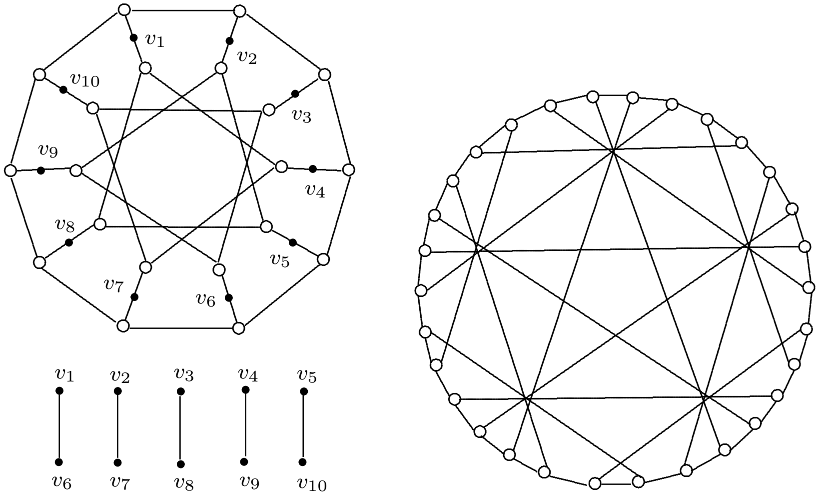
The McGee graph (the unique 7-cage) can be constructed in a somewhat similar manner. In this case, we begin with an 8-cycle $C = (u_1, u_2, \ldots, u_8, u_1)$, eight additional vertices $w_1, w_2, \ldots, w_8$ and the eight edges u_iw_i $(1 \le i \le 8)$. Then each vertex w_i is joined to $w_{i\pm 3}$ (subscripts expressed as $1, 2, \ldots, 8$ modulo 8). This produces the 8-cycle $C' =(w_1, w_4, w_7, w_2, w_5, w_8, w_3, w_6, w_1)$. See Figure 19.4(a). Each edge u_iw_i $(1 \le i \le 8)$ is then subdivided once, introducing the new vertices v_i. The edges v_iv_{i+4}, $1 \le i \le 4$, are then added to the graph, completing the construction of this 7-cage of order 24 shown in Figure 19.4(b). Consequently, $n(3, 7) = 24$.



(a) (b)

Figure 19.4: The McGee graph: the unique 7-cage

The Tutte-Coxeter graph (the unique 8-cage) can also be constructed in a manner similar to that of the 7-cage. In this case, we begin with a 10-cycle $C = (u_1, u_2, \ldots, u_{10}, u_1)$, ten additional vertices $w_1, w_2, \ldots, w_{10}$ and the ten edges u_iw_i $(1 \le i \le 10)$. Then each vertex w_i is joined to $w_{i\pm 3}$ (subscripts expressed as $1, 2, \ldots, 10$ modulo 10). This produces the 10-cycle $C' =(w_1, w_4, w_7, w_{10}, w_3, w_6, w_9, w_2, w_5, w_8, w_1)$. See Figure 19.5(a). Each edge u_iw_i $(1 \le i \le 10)$ is then subdivided once, introducing the new vertices v_i and the five edges v_iv_{i+5}, $1 \le i \le 5$, are then added to this graph, completing the construction of this 8-cage of order 30 shown in Figure 19.5(b). Thus, $n(3, 8) = 30$.



(a) (b)

Figure 19.5: The Tutte-Coxeter graph: the unique 8-cage

There is no unique 9-cage, however. In fact, there are 18 different 9-cages, each of order 58. There are three different 10-cages, each of order 70. The first 11-cage of order 112 was constructed by Alexandru Balaban [11] in 1973. Twenty-five years later, the **Balaban cage** was shown to be the only 11-cage by Brendan McKay, Wendy Myrvold and Jacqueline Nadon [167]. There is a unique 12-cage of order 126, commonly called the **Benson graph**, named for Clark T. Benson [20], who first constructed it in 1966.

Moore Graphs

There is a class of cages known as the Moore graphs that has received considerable attention. First, observe that for each vertex v in an r-regular graph G of diameter d, the number of vertices at distance i $(1 \leq i \leq d)$ from v is at most $r(r-1)^{i-1}$. Thus, the order of G is at most

$$1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{d-1}. \tag{19.2}$$

An r-regular graph of diameter d whose order is the number expressed in (19.2) is called a **Moore graph**. These graphs were named by Alan J. Hoffman and Robert R. Singleton [129] for Edward F. Moore, who posed the problem of describing and classifying these graphs.

Let G be an r-regular Moore graph of diameter d. For each vertex v in G, let $A_i(v)$ be the set of vertices whose distance from v is i $(0 \leq i \leq d)$. For

$1 \leq i \leq d-1$, $A_i(v)$ is an independent set of $r(r-1)^{i-1}$ vertices in G and

$$V(G) = \bigcup_{i=0}^{d} A_i(v).$$

The graph G necessarily has girth $g = 2d+1$ since there are not enough vertices to have a higher girth and a cycle of length less than $2d+1$ would result in too few vertices among the first d distance levels from a given vertex v. Hence,

$$M(r,g) = M(r, 2d+1) = 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{d-1}.$$

Thus, $n(r,g) = M(r,g)$. Consequently, any Moore graph must be a cage. That is, if G is an r-regular Moore graph of diameter d, then G is an (r,g)-cage, where $g = 2d+1$.

Every odd cycle C_{2k+1} $(k \geq 1)$ is a Moore graph. Indeed, C_{2k+1} is a 2-regular Moore graph of diameter k and girth $2k+1$. In addition, K_{r+1} $(r \geq 2)$ is an r-regular Moore graph of diameter 1 and girth 3. There are only two other known Moore graphs, namely the Petersen graph, which is a 3-regular Moore graph of diameter 2 and girth 5, and a 7-regular graph of diameter 2, girth 5 and order 50, known as the **Hoffman-Singleton graph**, named for Alan J. Hoffman and Robert R. Singleton. They constructed this graph while attempting to classify all Moore graphs.

This graph consists of ten pairwise disjoint 5-cycles denoted by $F_1, F_2, \ldots, F_5$ and $H_1, H_2, \ldots, H_5$ whose vertices are labeled as indicated in Figure 19.6. The vertex i $(0 \leq i \leq 4)$ in F_j $(1 \leq j \leq 5)$ is adjacent to the vertices $i + jk \pmod 5$ in H_k $(1 \leq k \leq 5)$. In addition to the ten 5-cycles, only the five additional edges incident with the vertex labeled 1 in F_3 are shown in Figure 19.6.

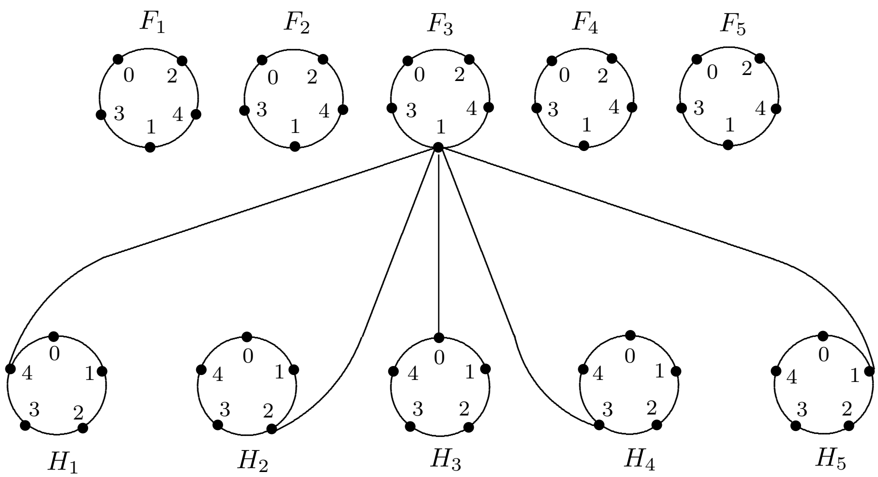


Figure 19.6: Constructing the Hoffman-Singleton graph

A theorem by Hoffman and Singleton [129] states everything that is known about r-regular Moore graphs of girth 5.

Theorem 19.12 *If G is an r-regular Moore graph ($r \geq 3$) of girth 5, then $r = 3, r = 7$ or possibly $r = 57$.*

As we already noted, the 2-regular Moore graph of girth 5 is C_5, the 3-regular Moore graph of girth 5 is the Petersen graph and the 7-regular Moore graph of girth 5 is the Hoffman-Singleton graph. It is not known whether there is a 57-regular Moore graph of girth 5 but if such a graph does exist, its order must be 3250.

The definition of Moore graph has been extended to permit a Moore graph to have even girth. For every two adjacent vertices u and v in an r-regular graph G of diameter d, the number of vertices w in G such that

$$\min\{d(u,w), d(v,w)\} = i \text{ for } 0 \leq i \leq d-1$$

is at most $2(r-1)^i$. Thus, the order of G is at most

$$2 + 2(r-1) + 2(r-1)^2 + \cdots + 2(r-1)^{d-1}. \qquad (19.3)$$

An r-regular graph G of diameter d whose order is the expression in (19.3) is also called a **Moore graph**. Such a Moore graph G necessarily has girth $g = 2d$. Hence,

$$M(r,g) = M(r,2d) = 2\left[1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{d-1}\right].$$

Thus, $n(r,g) = M(r,g)$. Therefore, in this case as well, a Moore graph is a cage. That is, if G is an r-regular Moore graph of diameter d, then the girth of G is $g = 2d$ and G is an (r,g)-cage.

In the more general definition of Moore graphs that allows even girth, the graph $K_{r,r}$ is an r-regular Moore graph of girth 4, the Heawood graph is a 3-regular Moore graph of girth 6 and the Tutte-Coxeter graph is a 3-regular Moore graph of girth 8. More generally, Robert Mark Damerell [64] and Eiichi Bannai and Tatsuro Ito [12] verified that other than the Moore graphs already mentioned, a Moore graph must have girth 5, 6, 8 or 12. The following theorem summarizes the information concerning Moore graphs.

Theorem 19.13 *There exists an r-regular Moore graph of odd girth g when*

- *$g = 3$ and $r \geq 3$, in which case K_{r+1} is the unique Moore graph;*
- *$g = 5$ and $r = 3$, in which case the Petersen graph is the unique Moore graph;*
- *$g = 5$ and $r = 7$, in which case the Hoffman-Singleton graph is the unique Moore graph; and*
- *possibly when $g = 5$ and $r = 57$.*

There exists an r-regular Moore graph of even girth g when

- $g = 4$ *and* $r \geq 4$, *in which case* $K_{r,r}$ *is the unique Moore graph;*
- $g = 6$ *and for all* r *for which there exists a projective plane of order* $r - 1$;
- $g = 8$ *and for all* r *for which there exists a certain projective geometry; and*
- $g = 12$ *and for all* r *for which there exists a certain projective geometry.*

As an additional example of a Moore graph referred to above, there is a unique 4-regular Moore graph of girth 6 and order $M(4, 6) = 26$. This graph is shown in Figure 19.7.

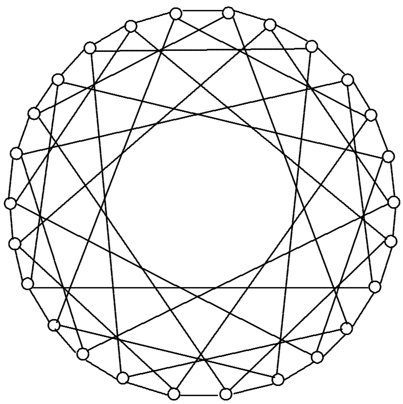


Figure 19.7: The 4-regular Moore graph of girth 6

Exercises for Chapter 19

Section 19.1. Turán's Theorem

1. For each ordered pair $(n, k) \in \{(7, 2), (7, 3), (7, 4), (8, 2), (8, 3), (8, 4)\}$, determine the Turán graph $T_{n,k}$ and its size $t_{n,k}$.

2. (a) Determine the minimum size m of a graph G of order $n = 8k+2 \geq 10$ and independence number $\alpha(G) = 4k+2$ for which G has a triangle.

   (b) Determine the minimum size m of a graph G of order $n = 8k+2 \geq 10$ and independence number $\alpha(G) = 4k+1+a$ where $1 \leq a \leq 4k$ for which G has a triangle.

Section 19.2. Extremal Subgraphs

3. Determine the minimum size m of a graph G of order 6 such that G contains a subdivision of K_5 as a subgraph.

4. By Corollary 19.6, if G is a graph of order n and size at least $(k-1)n - \binom{k}{2} + 1$, where $1 \leq k < n$, then G contains every tree of size k as a subgraph. Show that this bound on the size of G is not best possible by finding two integers k and n with $1 \leq k < n$ such that every graph of order n and size m contains every tree of size k as a subgraph but $m \leq (k-1)n - \binom{k}{2}$.

5. Show that every graph of order $n \geq 3$ and size $\lfloor n/2 \rfloor + 1$ contains P_3 as a subgraph. For each integer $n \geq 3$, describe all graphs of order n and size $\lfloor n/2 \rfloor$ not containing P_3 as a subgraph.

6. Show that every graph of order $n \geq 4$ and size n contains $2K_2$ as a subgraph. For each integer $n \geq 4$, describe all graphs of order n and size $n-1$ not containing $2K_2$ as a subgraph.

7. For each integer $n \geq 4$, determine the smallest integer m such that every graph of order n and size m contains $K_{1,3}$ as a subgraph. For each integer $n \geq 4$, describe all graphs of order n and size $m-1$ not containing $K_{1,3}$ as a subgraph.

8. Prove that every graph of order $n \geq 4$ and size at least $2n-2$ contains a subdivision of K_4 as a subgraph.

9. (a) Let G be a graph of order $n \geq 4$. Prove that if $\deg v \geq \frac{2n+1}{3}$ for every vertex v of G, then every edge of G belongs to a complete subgraph of order 4.

   (b) Show that the result in (a) is best possible in general by showing that $\frac{2n+1}{3}$ cannot be replaced by $\frac{2n}{3}$.

10. Prove that the minimum size of a graph of order $n \geq 3$ such that every vertex lies on a triangle is $\binom{n-1}{2} + 2$.

11. For each integer $n \geq 5$, determine the minimum positive integer m such that every graph of order n and size m contains two edge-disjoint triangles.

12. Prove that every graph of order at least 3 and size m contains a 3-partite subgraph of size at least $\lceil \frac{2m}{3} \rceil$.

Section 19.3. Cages

13. Let G be a connected graph with cycles. Show that $g(G) \leq 2\,\text{diam}(G)+1$.

14. (a) Prove that $n(3,6) = 14$.

    (b) Prove that the Heawood graph is the only 6-cage.

15. Let G be an (r,g)-graph ($r \geq 2$, $g \geq 3$) of order $n(r,g)$; that is, let G be an (r,g)-cage. Prove that if $H = G \,\square\, K_2$ is an (s,g)-graph, then H cannot be an (s,g)-cage.

16. Construct a cubic graph of order 12 with the maximum possible girth.

17. For integers $r \geq 2$ and $g \geq 3$, the number $n(r,g)$ can be defined as the smallest order of an r-regular graph containing an induced connected 2-regular subgraph of order g but of no smaller order. For integers r, k and g with $r \geq k \geq 2$ and $g \geq 3$, let $n(r,k,g)$ be the smallest order of an r-regular graph containing an induced connected k-regular graph of order g but of no smaller order. Certainly $n(r,k,g) \geq r+1$ and $n(r,2,g) = n(r,g)$.

    (a) Compute $n(4,3,4)$. (b) Compute $n(4,3,6)$.

18. For a set S of positive integers and an integer $g \geq 3$, an (S,g)-cage is a graph G of minimum order whose girth is g such that the elements of S occur an equal number of times as the degrees of the vertices of G. For $S = \{2,3\}$, give an example of each of the following (with justification).

    (a) an $(S,3)$-cage. (b) an $(S,4)$-cage.

    (c) an $(S,5)$-cage. (d) an $(S,6)$-cage.

Chapter 20

Ramsey Theory

Frank Plumpton Ramsey (1903–1930) was a British mathematician who had many interests, including English literature, philosophy, economics and politics. Ramsey [194] wrote a collection of 18 papers called *The Foundations of Mathematics, and Other Logical Essays*, in which he intended to improve upon *Principia Mathematica* by Bertrand Russell and Alfred North Whitehead. These papers ranged in dates from 1923 to 1929 and were edited after his death. The result for which he is best known occurred in a paper, presented to the London Mathematical Society and published in 1930 (the year he died). This paper was titled "On a problem of formal logic" [193] and his famous result occurred only as a minor lemma that he used to solve a problem in logic but this result led to an area in Extremal Graph Theory that became known as Ramsey Theory.

20.1 Classical Ramsey Numbers

One version of Ramsey's famous theorem is stated next.

Ramsey's Theorem

Theorem 20.1 (**Ramsey's Theorem**) *For any $k+1 \geq 3$ positive integers t, $n_1, n_2, \ldots, n_k$, there exists a positive integer n such that if each of the t-element subsets of the set $\{1, 2, \ldots, n\}$ is colored with one of the k colors $1, 2, \ldots, k$, then for some integer i with $1 \leq i \leq k$, there is a subset S of $\{1, 2, \ldots, n\}$ containing n_i elements such that every t-element subset of S is colored i.*

In order to see the connection that Ramsey's theorem has with graph theory, suppose that $\{1, 2, \ldots, n\}$ is the vertex set of the complete graph K_n. In the case of Theorem 20.1 for $t = 2$, each 2-element subset of the set $\{1, 2, \ldots, n\}$ represents an edge of K_n. Each edge of K_n is then assigned one of the colors

$1, 2, \ldots, k$, thereby producing a k-edge coloring of K_n (that is unlikely to be a *proper* k-edge coloring). It is this special case of Ramsey's theorem in which we have a special interest. We then restate Theorem 20.1 as a theorem in graph theory.

Theorem 20.2 (**Ramsey's Theorem**) *For any $k \geq 2$ positive integers n_1, n_2, …, n_k, there exists a positive integer n such that for every k-edge coloring of K_n (that is not necessarily a proper edge coloring), there is a complete subgraph K_{n_i} of K_n for some i $(1 \leq i \leq k)$ such that every edge of K_{n_i} is colored i.*

Our primary interest in Ramsey's theorem is in the case where $k = 2$, that is, when every edge of K_n is assigned one of two colors. It is common to use red and blue for these two colors. By a **red-blue edge coloring** (or simply a **red-blue coloring**) of a graph G, every edge of G is colored red or blue. Necessarily, adjacent edges may be colored the same. In fact, it is possible that all edges of G are colored red or that all edges of G are colored blue. For two positive integers s and t, the **Ramsey number** $R(s, t)$ is the minimum order n of a complete graph such that for every red-blue coloring of K_n, there is either a subgraph isomorphic to K_s, all of whose edges are colored red (a **red** K_s), or a subgraph isomorphic to K_t, all of whose edges are colored blue (a **blue** K_t). The integers s and t are therefore n_1 and n_2, respectively, in the statement of Theorem 20.2. The Ramsey number $R(s, s)$ is thus the minimum order n of a complete graph such that if every edge of K_n is colored with one of two given colors, then there is a subgraph isomorphic to K_s, all of whose edges are colored the same (a **monochromatic** K_s).

The Ramsey Number $R(3, 3)$

Perhaps the best known Ramsey number is $R(3, 3)$.

Theorem 20.3 $R(3, 3) = 6$.

Proof. Let $V(K_5) = \{v_1, v_2, \ldots v_5\}$ and define a red-blue coloring of K_5 by coloring each edge of the 5-cycle $(v_1, v_2, \ldots, v_5, v_1)$ red and the remaining edges blue (see Figure 20.1), where solid edges indicate red edges and dashed edges denote blue edges. Since this red-blue coloring produces neither a red K_3 nor a blue K_3, it follows that $R(3, 3) \geq 6$.

To show that $R(3, 3) \leq 6$, let there be given a red-blue coloring of K_6. Consider some vertex v_1 of K_6. Since v_1 is incident with five edges, it follows that at least three of these five edges are colored the same, say red. Suppose that v_1v_2, v_1v_3 and v_1v_4 are red edges. If any of the edges v_2v_3, v_2v_4 and v_3v_4 is colored red, then we have a red K_3; otherwise, all three of these edges are colored blue, producing a blue K_3. Hence, $R(3, 3) \leq 6$. Therefore, $R(3, 3) = 6$. ■

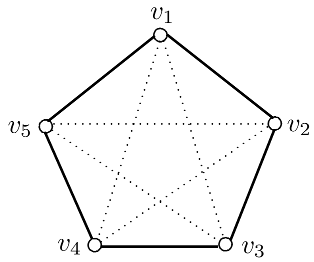


Figure 20.1: A red-blue coloring of K_5

Although the proof given of Theorem 20.3 is certainly the best known of this result, it is useful to describe another possible approach. To show that $R(3,3) = 6$, the major step is to verify the inequality $R(3,3) \leq 6$. To establish this inequality, it is required to show that every red-blue coloring of K_6 results in a monochromatic K_3. So, let there be given a red-blue coloring of K_6 and let the red and blue subgraphs of K_6 be denoted by G_R and G_B, respectively. Since the size of K_6 is $\binom{6}{2} = 15$, one of G_R and G_B has size 8 or more, say G_R does. Necessarily, G_R has cycles but suppose that it contains no triangles. Then G_R contains a cycle of length 6, 5 or 4. We consider these three cases.

Case 1. *G_R contains a 6-cycle $C = (u_1, u_2, \ldots, u_6, u_1)$.* Since u_1u_3, u_3u_5, $u_5u_1 \notin E(G_R)$, these edges belong to G_B and so K_6 has a blue triangle.

Case 2. *G_R contains a 5-cycle $C' = (v_1, v_2, v_3, v_4, v_5, v_1)$ but no 6-cycle.* Necessarily, G_R contains no chords of C'. Let v_6 be the remaining vertex of G_R. Then v_6 is joined to at least three vertices of C' in G_R, at least two of which are consecutive on C'. However then, G_R has a triangle, producing a contradiction.

Case 3. *G_R contains a 4-cycle $C'' = (w_1, w_2, w_3, w_4, w_1)$ but no cycle of length 5 or 6.* Let w_5 and w_6 be the two remaining vertices of K_6. At least one of w_5 and w_6 is joined to two vertices of C'' in G_R – necessarily, two nonconsecutive vertices of C''. We may assume that w_5 is joined to w_1 and w_3. However then, (w_2, w_4, w_5, w_2) is a triangle in G_B.

Therefore, $R(3,3) \leq 6$.

Theorem 20.3 provides a solution to a problem that appeared on an exam that has become well known over the years. Except for a 3-year period during World War II, the William Lowell Putnam mathematical competition for undergraduates has taken place every year since 1938. This exam, administered by the Mathematical Association of America, consists of (since 1962) twelve challenging mathematical problems. This competition was designed to stimulate a healthy rivalry among colleges and universities throughout the United States and Canada. The 1953 exam contained the following problem:

Problem A2 *The complete graph with* 6 *points (vertices) and* 15 *edges has each edge colored red or blue. Show that we can find* 3 *points such that the* 3 *edges joining them are the same color.*

Theorem 20.3 also provides the answer to a question regarding acquaintances at a party. What is the smallest number n of individuals in a gathering of people, every two of whom are either acquaintances or strangers, for which there must be either three mutual acquaintances or three mutual strangers? A gathering of n people can be modeled by the graph K_n and a red-blue coloring of K_n, where a red edge, say, indicates that two people are acquaintances and a blue edge indicates that two people are strangers. By Theorem 20.3, the smallest such integer n is 6.

The Ramsey number $R(s,t)$ of two positive integers s and t can be defined without regard to edge colorings. The Ramsey number $R(s,t)$ is also the smallest positive integer n such that for every graph G of order n, either G contains a subgraph isomorphic to K_s or its complement $\overline{G}$ contains a subgraph isomorphic to K_t. Assigning the color red to each edge of G and the color blue to each edge of $\overline{G}$ returns us to our initial definition of $R(s,t)$.

Bounds for Classical Ramsey Numbers

We will see in the next section that there are Ramsey numbers that are more general than $R(s,t)$. Historically, it is the Ramsey numbers $R(s,t)$ that were the first to be studied. The numbers $R(s,t)$ are referred to as the **classical Ramsey numbers**. By Ramsey's theorem, $R(s,t)$ exists for every two positive integers s and t. The Ramsey number $R(s,t)$ is thus the smallest positive integer n such that for *every* graph G of order n, either G contains an s-clique or $\overline{G}$ contains a t-clique. Equivalently, $R(s,t)$ is the smallest positive integer n such that every graph of order n contains an s-clique or an independent set of t vertices.

Observe that

$$R(s,t) = R(t,s)$$

for every two positive integers s and t. Moreover,

$$R(1,t) = 1 \text{ and } R(2,t) = t \tag{20.1}$$

for every positive integer t (see Exercise 2) and by Theorem 20.3, $R(3,3) = 6$.

The existence of the Ramsey numbers $R(s,t)$ was established in 1935 by Paul Erdős and George Szekeres [84], where upper bounds for $R(s,t)$ were obtained as well.

Theorem 20.4 *For every two integers $s, t \geq 2$, the Ramsey number $R(s,t)$ exists and*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Furthermore, if $R(s-1,t)$ and $R(s,t-1)$ are both even, then

$$R(s,t) \leq R(s-1,t) + R(s,t-1) - 1.$$

Proof. To establish the existence of $R(s,t)$, we apply induction on $s+t \geq 4$. We have already seen that $R(s,t)$ exists if s or t is 1 or 2, so $R(2,2)$ exists. Assume that $R(s', t')$ exists if $s', t' \geq 2$ and $s' + t' = k - 1$ for some integer $k \geq 5$ and let $s, t \geq 2$ such that $s+t = k$. In particular, $R(s-1,t)$ and $R(s,t-1)$ exist.

Let $n = R(s-1,t) + R(s,t-1)$ and let there be given a red-blue coloring of $G = K_n$. It suffices to show that G contains either a red K_s or a blue K_t. Let v be a vertex of G. Thus, the degree of v in G is $n-1 = R(s-1,t)+R(s,t-1)-1$. Let F be the spanning red subgraph of G and H the spanning blue subgraph of G. We now consider two cases, depending on the degree of v in F.

Case 1. $\deg_F v \geq R(s-1,t)$. Since the order of $G[N_F(v)]$ is at least $R(s-1,t)$, it follows that $G[N_F(v)]$ contains either a red K_{s-1} or a blue K_t. If $G[N_F(v)]$ contains a blue K_t, then so does G. If $G[N_F(v)]$ contains a red K_{s-1}, then F contains K_s and so G contains a red K_s.

Case 2. $\deg_F v \leq R(s-1,t)-1$. Thus $\deg_H v \geq R(s,t-1)$. Since the order of $G[N_H(v)]$ is at least $R(s,t-1)$, it follows that $G[N_H(v)]$ contains either a red K_s or a blue K_{t-1}. If $G[N_H(v)]$ contains a red K_s, then so does G. If $G[N_H(v)]$ contains a blue K_{t-1}, then H contains K_t and so G contains a blue K_t.

Therefore, $R(s,t)$ exists and

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Next, suppose that $R(s-1,t)$ and $R(s,t-1)$ are both even and once again let $n = R(s-1,t) + R(s,t-1)$. Let there be given a red-blue coloring of $G' = K_{n-1}$. Let F' be the spanning red subgraph of G' and H' the spanning blue subgraph of G'. Since F' has odd order, some vertex v of F' has even degree. If $\deg_{F'} v \geq R(s-1,t)$, then, proceeding as above, G' has either a red K_s or a blue K_t. On the other hand, if $\deg_{F'} v < R(s-1,t)$, then $\deg_{F'} v \leq R(s-1,t) - 2$ and so $\deg_{H'} v \geq R(s,t-1)$. Once again, proceeding as above, G' has either a red K_s or a blue K_t. ■

An upper bound for $R(s,t)$ can also be given as a binomial coefficient. For positive integers k and n with $k \leq n$, a well-known combinatorial identity is

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}. \tag{20.2}$$

Corollary 20.5 *For every two positive integers s and t,*

$$R(s,t) \le \binom{s+t-2}{s-1}.$$

Proof. We proceed by induction on $n = s+t$. Since $R(1,t) = 1$ and $R(2,t) = t$ for every positive integer t, it follows that $R(s,t) \le \binom{s+t-2}{s-1}$ when $n = s+t \le 5$. Thus, we may assume that $s \ge 3$ and $t \ge 3$ and so $n \ge 6$. Suppose that

$$R(s',t') \le \binom{s'+t'-2}{s'-1}$$

for $s'+t' = k-1$ where $k \ge 6$. We show for positive integers s and t with $s,t \ge 3$ and $k = s+t$ that

$$R(s,t) \le \binom{s+t-2}{s-1}.$$

By Theorem 20.4,

$$\begin{aligned} R(s,t) &\le R(s-1,t) + R(s,t-1) \\ &\le \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}, \end{aligned}$$

completing the proof. ■

As we have already noted, the bound given in Corollary 20.5 for $R(s,t)$ is exact if one of s and t is 1 or 2. The bound is also exact for $s = t = 3$. By Corollary 20.5,

$$R(3,t) \le \frac{t^2+t}{2}.$$

An improved bound for $R(3,t)$ is now presented.

Theorem 20.6 *For every integer $t \ge 3$,*

$$R(3,t) \le \frac{t^2+3}{2}. \tag{20.3}$$

Proof. We proceed by induction on t. For $t = 3$, $R(3,t) = 6 = (t^2+3)/2$, so that (20.3) holds if $t = 3$. Assume that

$$R(3,t-1) \le \frac{(t-1)^2+3}{2}$$

for some integer $t \ge 4$ and consider $R(3,t)$. By Theorem 20.4,

$$R(3,t) \le t + R(3,t-1). \tag{20.4}$$

Moreover, strict inequality holds if t and $R(3, t-1)$ are both even. Combining (20.4) and the induction hypothesis, we have

$$R(3,t) \leq t + \frac{(t-1)^2+3}{2} = \frac{t^2+4}{2}. \tag{20.5}$$

To complete the proof, it suffices to show that the inequality given in (20.5) is strict.

If t is odd, then $R(3,t) < (t^2+4)/2$ since t^2+4 is odd. Thus, we may assume that t is even. If $R(3,t-1) < ((t-1)^2+3)/2$, then clearly the inequality in (20.5) is strict. If, on the other hand,

$$R(3,t-1) = \frac{(t-1)^2+3}{2} = \frac{t^2}{2} - t + 2,$$

then $R(3,t-1)$ is even since t is even. Therefore, the inequality in (20.5) is strict, which implies the desired result. ∎

According to Theorem 20.6, $R(3,4) \leq 9$ and $R(3,5) \leq 14$. Actually, equality holds in both of these cases. The equality $R(3,4) = 9$ follows since there exists a graph of order 8 containing neither a triangle nor an independent set of four vertices, while the equality $R(3,5) = 14$ follows since there exists a graph of order 13 containing neither a triangle nor an independent set of five vertices. These graphs are shown in Figure 20.2. On the other hand, $R(3,6) \leq 19$ by Theorem 20.6, but the value of $R(3,6)$ is 18.

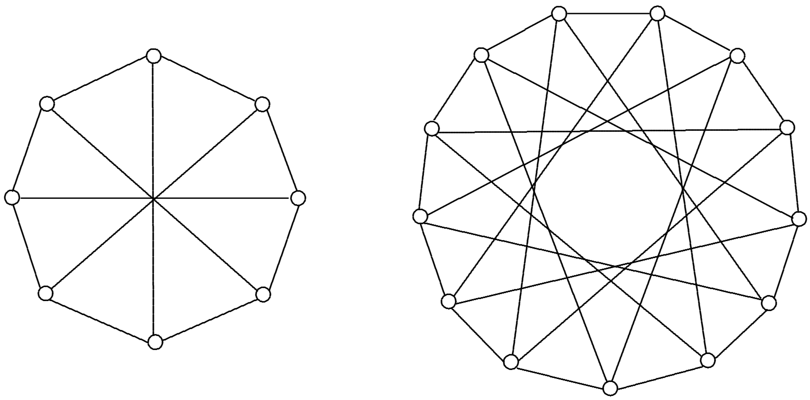


Figure 20.2: Graphs showing that $R(3,4) \geq 9$ and $R(3,5) \geq 14$

Corollary 20.5 gives an upper bound for the Ramsey numbers $R(t,t)$, namely $R(t,t) \leq \binom{2t-2}{t-1}$. There are three ways in which lower bounds for $R(t,t)$ have been obtained: the constructive method, a counting method and a probabilistic method. The next chapter will describe the probabilistic method of proof. In the constructive method, a lower bound for $R(t,t)$ is established by explicitly constructing a graph G of an appropriate order such that neither G nor $\overline{G}$ contains K_t as a subgraph. Better lower bounds, however, have been obtained using a counting method, which we briefly describe here. Suppose that we wish to establish the existence of a graph G of order n having some given property P. If we can estimate the number of graphs of order n that do not have property P and we can show that this number is strictly less than the total number of graphs of order n, then there must exist a graph G of order n having property P. Of course, this procedure offers no method for constructing G. In 1947 Erdős [78] provided one of the first applications of this counting technique to establish a lower bound for the Ramsey number $R(t,t)$. We prove a result weaker than that of Erdős but using the same counting method. In Chapter 21 we will revisit the result in Theorem 20.7 below from a "probabilistic" point of view and discuss Erdős' stronger 1947 result.

Theorem 20.7 *For every integer $t \geq 3$,*

$$R(t,t) > \lfloor 2^{t/2} \rfloor.$$

Proof. Let $n = \lfloor 2^{t/2} \rfloor$. We demonstrate the existence of a graph G of order n such that neither G nor $\overline{G}$ contains K_t as a subgraph. Consider the set $\mathcal{G}$ of all graphs of order n having a fixed vertex set V, where then $|V| = n$. Two graphs G_1 and G_2 in $\mathcal{G}$ are then different if $E(G_1) \neq E(G_2)$. Since the set E of all edges with vertex set V consists of $\binom{n}{2}$ elements, $|\mathcal{G}|$ is the number of all subsets of E, which then is $2^{\binom{n}{2}}$. For each subset S of V with $|S| = t$, consider those graphs in $\mathcal{G}$ for which S induces a complete graph. Since the $\binom{t}{2}$ edges joining two vertices in S belong to each such graph H, the remaining edges of H must come from a set of $\binom{n}{2} - \binom{t}{2}$ edges. So, the total number of such graphs is therefore $2^{\binom{n}{2}-\binom{t}{2}}$. Thus, if N denotes the number of graphs with vertex set V that contain a subgraph isomorphic to K_t, then since there are $\binom{n}{t}$ choices for the set S, it follows that

$$N \leq \binom{n}{t} 2^{\binom{n}{2}-\binom{t}{2}} = \left[\frac{n(n-1)\cdots(n-t+1)}{t!}\right] 2^{\binom{n}{2}-\binom{t}{2}} < \frac{n^t}{t!} 2^{\binom{n}{2}-\binom{t}{2}}. \quad (20.6)$$

Since $n = \lfloor 2^{t/2} \rfloor \leq 2^{t/2}$, it follows that

$$n^t \leq 2^{t^2/2}. \quad (20.7)$$

Next, we verify by induction on $t \geq 3$ that

$$2^{t^2/2} < \left(\frac{1}{2}\right) t! 2^{\binom{t}{2}}. \quad (20.8)$$

Since

$$2^{9/2} = \sqrt{2} \cdot 2^4 < \left(\frac{3}{2}\right) \cdot 16 = 3 \cdot 2^3 = \frac{1}{2}(3!) 2^{\binom{3}{2}},$$

the inequality holds for $t = 3$. Suppose that

$$2^{k^2/2} < \frac{1}{2} \cdot k! \cdot 2^{\binom{k}{2}}$$

for some integer $k \geq 3$. We show that

$$2^{(k+1)^2/2} < \frac{1}{2}(k+1)!2^{\binom{k+1}{2}}.$$

Observe that

$$\begin{aligned} 2^{(k+1)^2/2} &= 2^{k^2/2} \cdot 2^{(2k+1)/2} = \sqrt{2} \cdot 2^{k^2/2} \cdot 2^k \\ &< \sqrt{2}\left[\frac{1}{2} \cdot k! \cdot 2^{\binom{k}{2}}\right] \cdot 2^k = \sqrt{2} \cdot \frac{1}{2} \cdot k! \cdot 2^{\binom{k}{2}+k} \\ &< \frac{1}{2}(k+1) \cdot k! \cdot 2^{\binom{k+1}{2}} = \frac{1}{2}(k+1)!2^{\binom{k+1}{2}}. \end{aligned}$$

Thus, the inequality in (20.8) holds for every integer $t \geq 3$ and so, by (20.7),

$$n^t < \left(\frac{1}{2}\right) t!2^{\binom{t}{2}}. \tag{20.9}$$

Combining (20.6) and (20.9), we conclude that

$$N < \left(\frac{1}{2}\right) 2^{\binom{n}{2}}.$$

If we list the N graphs with vertex set V that contain a subgraph isomorphic to K_t, together with their complements, then there are at most $2N < 2^{\binom{n}{2}}$ different graphs in the list. Since there are $2^{\binom{n}{2}}$ graphs with vertex set V, we conclude that there is a graph G with vertex set V such that neither G nor $\overline{G}$ appears in the aforementioned list; that is, neither G nor $\overline{G}$ contains a subgraph isomorphic to K_t. ■

Relatively few classical Ramsey numbers $R(s,t)$ are known for $s,t \geq 3$. The table in Figure 20.3 gives the known values of $R(s,t)$ for integers s and t with $s,t \geq 3$. Although it is known that $43 \leq R(5,5) \leq 49$, the value of $R(5,5)$ remains unknown.

s	3	3	3	3	3	3	3	4	4
t	3	4	5	6	7	8	9	4	5
$R(s,t)$	6	9	14	18	23	28	36	18	25

Figure 20.3: Some classical Ramsey numbers

Multi-Colored Classical Ramsey Numbers

Ramsey's theorem states that the Ramsey number $R(s,t)$ for two positive integers s and t can be extended to more than two integers. For $k \geq 3$ positive integers $n_1, n_2, \ldots, n_k$, the **Ramsey number** $R(n_1, n_2, \ldots, n_k)$ is defined as the smallest positive integer n such that if every edge of K_n is colored with one of k given colors $1, 2, \ldots, k$, then for some i ($1 \leq i \leq k$) K_n contains a subgraph K_{n_i} all of whose edges are colored i. It is a consequence of Ramsey's theorem that $R(n_1, n_2, \ldots, n_k)$ exists for every $k \geq 2$ positive integers $n_1, n_2, \ldots, n_k$.

Theorem 20.8 *For every $k \geq 2$ positive integers $n_1, n_2, \ldots, n_k$, the Ramsey number $R(n_1, n_2, \ldots, n_k)$ exists.*

For the Ramsey numbers $R(n_1, n_2, \ldots, n_k)$ with $k \geq 3$ and $n_i \geq 3$ for $1 \leq i \leq k$, only the Ramsey number $R(3,3,3)$ is known. The value of this Ramsey number was established by Robert E. Greenwood and Andrew M. Gleason [108].

Theorem 20.9 $R(3,3,3) = 17$.

Proof. Let there be given an edge coloring of K_{17} with the three colors red, blue and green and let v be a vertex of K_{17}. Since v has degree 16, it follows that v is incident with six edges that are colored the same, say vv_i ($1 \leq i \leq 6$) are colored green. Let $H = K_6$ be the subgraph induced by $\{v_1, v_2, \ldots, v_6\}$. If any edge of H is colored green, then K_{17} has a green triangle. Thus, we may assume that no edge of H is colored green. Hence, every edge of H is colored red or blue. Since $H = K_6$ and $R(3,3) = 6$ (by Theorem 20.3), it follows that H, and K_{17} as well, contains either a red triangle or a blue triangle. Therefore, K_{17} contains a monochromatic triangle and so $R(3,3,3) \leq 17$.

To verify that $R(3,3,3) = 17$, it remains to show that there is a 3-edge coloring of K_{16} for which there is no monochromatic triangle. In fact, there is an isomorphic factorization of K_{16} into a triangle-free graph that is commonly called the **Clebsch graph** (named for Alfred Clebsch [59] who discovered it in 1868). This graph is also referred to as the **Greenwood-Gleason graph**. This graph can be constructed by beginning with the Petersen graph with vertices u_i and v_i ($1 \leq i \leq 5$), as illustrated in Figure 20.4 by solid vertices and solid edges. We then add six new vertices, namely x and w_i ($1 \leq i \leq 5$). The Clebsch graph CG (a 5-regular graph of order 16) is constructed as shown in Figure 20.4. This graph has the property that for every vertex v of CG, the subgraph $CG - N[v]$ is isomorphic to the Petersen graph. ■

20.2 More General Ramsey Numbers

While the Ramsey number $R(s,t)$ for positive integers s and t concerns red-blue colorings of complete graphs that result in a red K_s or a blue K_t, there are related concepts that don't require the red and blue subgraphs to be complete.

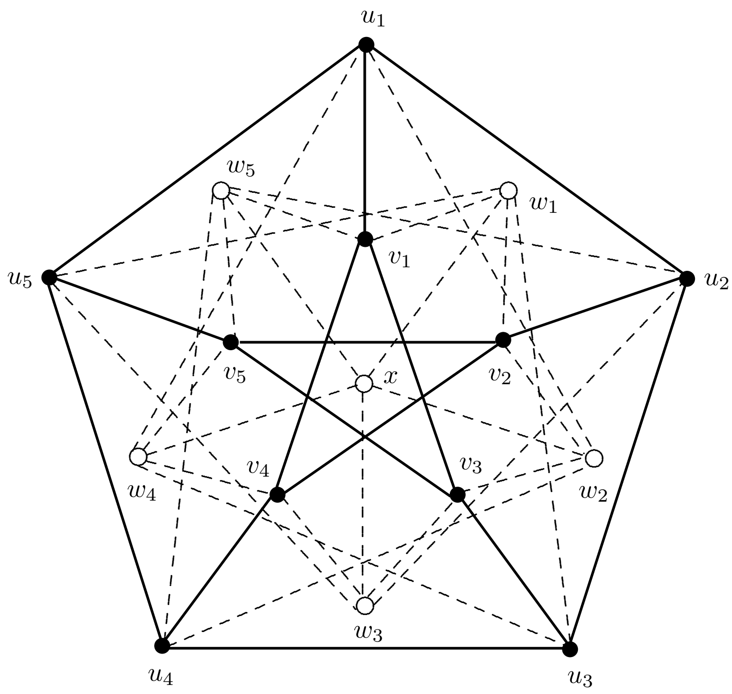


Figure 20.4: The Clebsch graph

For two graphs F and H, the **Ramsey number** $R(F, H)$ is the minimum order n of a complete graph such that for every red-blue coloring of K_n, there is either a subgraph isomorphic to F, all of whose edges are colored red (a *red* F), or a subgraph isomorphic to H, all of whose edges are colored blue (a *blue* H). The Ramsey number $R(F, H)$ exists for every two graphs F and H. In fact, if F has order s and H has order t, then

$$R(F, H) \leq R(s, t).$$

To illustrate a Ramsey number of this type, we determine the Ramsey number $R(P_4, K_4)$.

Example 20.10 $R(P_4, K_4) = 10$.

Proof. To show that $R(P_4, K_4) \geq 10$, consider the red-blue coloring of K_9 in which the red subgraph is $3K_3$. Since the order of every component of $3K_3$ is 3, there is no red P_4 in this red-blue coloring of K_9. The blue subgraph in this case is $K_{3,3,3}$, which does not contain a 4-clique. Hence, there is also no blue K_4 in this red-blue coloring of K_9. Therefore, $R(P_4, K_4) \geq 10$.

We now show that $R(P_4, K_4) \leq 10$. Let there be given a red-blue coloring of $G = K_{10}$. We show that there is either a red P_4 or a blue K_4. Suppose first that some vertex v of G is incident with four or more red edges, say vv_1, vv_2, vv_3 and vv_4 are red. If any edge joining two vertices of $S = \{v_1, v_2, v_3, v_4\}$ is red, then a red P_4 results; otherwise, $G[S] = K_4$ is blue.

Hence, we may assume that every vertex of G is incident with six or more blue edges. Let $u \in V(G)$ and suppose that $uu_1, uu_2, \ldots, uu_6$ are blue edges of G. Let $A = \{u_1, u_2, \ldots, u_6\}$ and let $H = G[A] = K_6$. Since $R(3,3) = 6$ by Theorem 20.3, it follows that H contains either a red K_3 or a blue K_3. Since u is joined to every vertex of H by a blue edge, if H contains a blue K_3, then G contains a blue K_4. Hence, we may assume that H contains a red K_3, which we denote by F_1. If some vertex of F_1 is joined to a vertex of $V(G) - V(F_1)$ by a red edge, then a red P_4 results. Thus, we may assume that G contains a blue $K_{3,7}$ where $V(F_1)$ is the partite set of order 3. Since each vertex of F_1 is joined to six vertices of $V(G) - V(F_1)$ by blue edges, we may use the same reasoning as above to conclude that G contains a red K_3, which we denote by F_2, where $V(F_2) \subseteq V(G) - V(F_1)$ and so F_1 and F_2 are disjoint. We may assume then that every vertex of $V(G) - V(F_2)$ is joined to the vertices of F_2 by blue edges. Hence, G contains a blue $K_{3,3,4}$. Thus, every vertex of $V(F_1) \cup V(F_2)$ in the blue subgraph G_B of G has degree 7 and the remaining four vertices of G_B have degree at least 6. Therefore, the size of G_B is at least $(6 \cdot 7 + 4 \cdot 6)/2 = 33$. By Turán's theorem (Theorem 19.2), either G_B contains a 4-clique (and so G contains a blue K_4) or G_B has size 33 and so is the Turán graph $T_{10,3} = K_{3,3,4}$. In the latter case, the red subgraph is $2K_3 + K_4$, which contains a red K_4 and a red P_4 as well, and so G contains a red P_4. ■

Tree-Complete Graph Ramsey Numbers

We saw in Example 20.10 that $R(P_4, K_4) = 10$. Vašek Chvátal [57] extended this considerably by finding the exact value of $R(F, H)$ whenever F is any tree and H is any complete graph.

Theorem 20.11 *Let T be a tree of order $p \geq 2$. For every integer $n \geq 2$,*

$$R(T, K_n) = (p-1)(n-1) + 1.$$

Proof. First, we show that $R(T, K_n) \geq (p-1)(n-1)+1$. Consider a red-blue coloring of the complete graph $K_{(p-1)(n-1)}$ such that the resulting red subgraph is $(n-1)K_{p-1}$; that is, the red subgraph consists of $n-1$ copies of K_{p-1}. Since each component of the red subgraph has order $p-1$, it contains no connected subgraph of order greater than $p-1$. In particular, there is no red tree of order p. The blue subgraph is then the complete $(n-1)$-partite graph $K_{p-1,p-1,\ldots,p-1}$, where every partite set contains exactly $p-1$ vertices. Hence, there is no blue K_n either. Since this red-blue coloring of $K_{(p-1)(n-1)}$ produces neither a red tree T nor a blue K_n, it follows that $R(T, K_n) \geq (p-1)(n-1)+1$.

We now show that $R(T, K_n) \leq (p-1)(n-1)+1$ for an arbitrary but fixed tree T of order $p \geq 2$ and an integer $n \geq 2$. We verify this inequality by induction on n. For $n = 2$, we show that $R(T, K_2) \leq (p-1)(2-1)+1 = p$. Since $R(K_p, K_2) = R(p, 2) = p$, as noted in (20.1), every red-blue coloring of K_p produces a red K_p or a blue K_2. Thus, $R(T, K_2) \leq p$. Therefore, the inequality $R(T, K_n) \leq (p-1)(n-1)+1$ holds when $n = 2$. Assume for an integer $k \geq 2$ that

$$R(T, K_k) \leq (p-1)(k-1)+1.$$

Consequently, every red-blue coloring of $K_{(p-1)(k-1)+1}$ contains either a red T or a blue K_k. We now show that

$$R(T, K_{k+1}) \leq (p-1)k+1.$$

Let there be given a red-blue coloring of $K_{(p-1)k+1}$. We show that there is either a red tree T or a blue K_{k+1}. We consider two cases.

Case 1. *There exists a vertex v in $K_{(p-1)k+1}$ that is incident with at least $(p-1)(k-1)+1$ blue edges.* Suppose that vv_i is a blue edge for $1 \leq i \leq (p-1)(k-1)+1$. Consider the subgraph H induced by the set $\{v_i : 1 \leq i \leq (p-1)(k-1)+1\}$. Thus, $H = K_{(p-1)(k-1)+1}$. By the induction hypothesis, H contains either a red T or a blue K_k. If H contains a red T, so does $K_{(p-1)k+1}$. On the other hand, if H contains a blue K_k, then, since v is joined to every vertex of H by a blue edge, there is a blue K_{k+1} in $K_{(p-1)k+1}$.

Case 2. *Every vertex of $K_{(p-1)k+1}$ is incident with at most $(p-1)(k-1)$ blue edges.* So every vertex of $K_{(p-1)k+1}$ is incident with at least $p-1$ red edges. Thus, the red subgraph of $K_{(p-1)k+1}$ has minimum degree at least $p-1$. By Theorem 3.20, this red subgraph contains a red T. Therefore, $K_{(p-1)k+1}$ contains a red T as well. ■

General Ramsey numbers involving three or more graphs exist as well. For $k \geq 3$ graphs $G_1, G_2, \ldots, G_k$, the **Ramsey number** $R(G_1, G_2, \ldots, G_k)$ is the smallest positive integer n such that if every edge of K_n is colored with one of k given colors $k = 1, 2, \ldots, k$, then for some i $(1 \leq i \leq k)$, the graph K_n contains a subgraph (isomorphic to) G_i all of whose edges are colored i. The next result gives the Ramsey number of the two stars $K_{1,3}$ and $K_{1,5}$ and the 5-cycle.

Theorem 20.12 $R(K_{1,3}, K_{1,5}, C_5) = 15$.

Proof. First, consider the edge coloring of K_{14} in which every edge of $K_{7,7}$ is colored green. At this point every edge of $2K_7$ is uncolored. Since K_7 can be factored into three Hamiltonian cycles by Theorem 13.5, we can color the edges red in one Hamiltonian cycle in each copy of K_7 and the remaining edges blue. Then the red subgraph is 2-regular and the blue graph is 4-regular. With this edge coloring, there is no red $K_{1,3}$, no blue $K_{1,5}$ and no green C_5. Thus, $R(K_{1,3}, K_{1,5}, C_5) \geq 15$.

Next, let there be given an edge coloring of $G = K_{15}$ with the colors red, blue and green and suppose that there is no red $K_{1,3}$ and no blue $K_{1,5}$. Then every vertex of G is incident with at least 8 green edges. Thus, in the green subgraph H of G, $\deg_H v \geq 8$ for every vertex v of H which results in $\deg_H u + \deg_H v \geq 16$ for every pair u, v of nonadjacent vertices of H. Since H has order 15, it follows by Theorem 6.21 that H is pancyclic and so G contains a green C_5. Therefore, $R(K_{1,3}, K_{1,5}, C_5) = 15$. ■

We now determine the Ramsey number of $k \geq 2$ graphs, all of which are stars.

Theorem 20.13 *Let $s_1, s_2, \ldots, s_k$ be $k \geq 2$ positive integers, t of which are even, and let $s = \sum_{i=1}^{k}(s_i - 1)$. Then*

$$R(K_{1,s_1}, K_{1,s_2}, \ldots, K_{1,s_k}) = \begin{cases} s+1 & \text{if } t \text{ is positive and even} \\ s+2 & \text{otherwise.} \end{cases}$$

Proof. Let $n = R(K_{1,s_1}, K_{1,s_2}, \ldots, K_{1,s_k})$. We consider two cases.

Case 1. *$t = 0$ or t is odd.* We show here that $n = s+2$. Let there be given a k-edge coloring of K_{s+2} with the colors $1, 2, \ldots, k$ and let v be a vertex of K_{s+2}. Since $\deg v = s + 1 = \sum_{i=1}^{k}(s_i - 1) + 1$, it follows that v is incident with at least s_i edges colored i for some i $(1 \leq i \leq k)$. Thus, K_{s+2} contains K_{1,s_i} as a subgraph all of whose edges are colored i for some i $(1 \leq i \leq k)$. Therefore, $n \leq s+2$.

Next, we show that $n \geq s+2$. First, suppose that $t = 0$. Then $s+1$ is odd and, by Theorem 13.5, the complete graph K_{s+1} can be factored into $s/2$ Hamiltonian cycles. For $i = 1, 2, \ldots, k$, let F_i be the $(s_i - 1)$-factor obtained from $(s_i - 1)/2$ Hamiltonian cycles. Assigning the color i to each edge in F_i produces a k-edge coloring of K_{s+1} having no subgraph K_{1,s_i}, all of whose edges are colored i. Thus, $n \geq s+2$ and so $n = s+2$.

Next, suppose that t is odd. Hence, $s+1$ is even and by Theorem 13.2, the complete graph K_{s+1} can be factored into $s = \sum_{i=1}^{k}(s_i - 1)$ 1-factors. For $i = 1, 2, \ldots, k$, let F_i be the $(s_i - 1)$-factor obtained from $(s_i - 1)$ 1-factors. Assigning the color i to the edges in F_i produces a k-edge coloring of K_{s+1} having no subgraph K_{1,s_i}, all of whose edges are colored i. Thus, $n \geq s+2$ and so $n = s+2$.

Case 2. *t is positive and even.* We show here that $n = s+1$. In this case, $n = s+1$ is odd. First, we show that $n \leq s+1$. Assume, to the contrary, that there is a k-edge coloring of K_{s+1} containing no subgraph K_{1,s_i}, all of whose edges are colored i, for every i with $1 \leq i \leq k$. Since every vertex of K_{s+1} has degree $s = \sum_{i=1}^{k}(s_i - 1)$, it follows that K_{s+1} contains a spanning $(s_i - 1)$-regular subgraph for every i $(1 \leq i \leq k)$. Because some integer $s_i - 1$ is odd, at least one of these subgraphs has an odd number of odd vertices, which is impossible. Thus, $n \leq s+1$.

Next, we show that $n \geq s+1$. Since t is positive and even, s is even and K_s has a 1-factorization into $s-1$ 1-factors. We may assume that $s_k \geq 2$ is even. Let G_i be the $(s_i - 1)$-regular spanning subgraph obtained from $s_i - 1$ of these

1-factors for $1 \le i \le k-1$ and let G_k be the (s_k-2)-regular spanning subgraph obtained from s_k-2 of these 1-factors. Since $\sum_{i=1}^{k-1}(s_i-1)+(s_k-2) = s-1$, it follows that $\{G_1, G_2, \ldots, G_k\}$ is a factorization of K_s. Assigning the color i to the edges in G_i for each i $(1 \le i \le k)$ produces a k-edge coloring of K_s having no subgraph K_{1,s_i}, all of whose edges are colored i for all i $(1 \le i \le k)$. Thus, $n \ge s+1$ and so $n = s+1$. ■

For example,

$$\begin{aligned} R(K_{1,3}, K_{1,5}, K_{1,7}) &= (3-1)+(5-1)+(7-1)+2 = 14, \text{where } t = 0 \\ R(K_{1,4}, K_{1,5}, K_{1,7}) &= (4-1)+(5-1)+(7-1)+2 = 15, \text{where } t = 1 \\ R(K_{1,4}, K_{1,6}, K_{1,7}) &= (4-1)+(5-1)+(7-1)+1 = 14, \text{where } t = 2. \end{aligned}$$

Corollary 20.14 *For integers $s, t \ge 2$,*

$$R(K_{1,s}, K_{1,t}) = \begin{cases} s+t-1 & \textit{if } s \textit{ and } t \textit{ are both even} \\ s+t & \textit{otherwise.} \end{cases}$$

Exercises for Chapter 20

Section 20.1. Classical Ramsey Numbers

1. Show that if G is a graph of order $R(s,t)-1$, then
   (a) $K_{s-1} \subseteq G$ or $K_t \subseteq \overline{G}$,
   (b) $K_s \subseteq G$ or $K_{t-1} \subseteq \overline{G}$.

2. Show that $R(1,t) = 1$ and $R(2,t) = t$ for every positive integer t.

3. If $2 \le s' \le s$ and $2 \le t' \le t$, then prove that $R(s',t') \le R(s,t)$. Furthermore, prove that equality holds if and only if $s' = s$ and $t' = t$.

4. Show that $R(3,4) = 9$.

5. The graph shown in Figure 20.5 has order 17 and contains neither four mutually adjacent vertices nor an independent set of four vertices. Thus, $R(4,4) > 17$. Show that $R(4,4) = 18$.

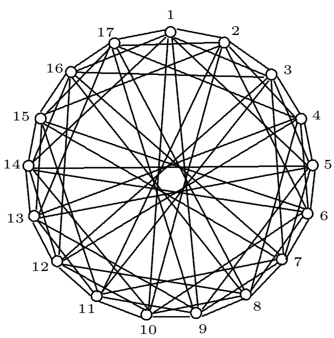


Figure 20.5: The graph in Exercise 5

6. The value of the Ramsey number $R(5,5)$ is unknown. Establish a lower bound A and an upper bound B (with explanations) for $R(5,5)$ such that $B - A < 25$.

7. For a red-blue coloring c of K_6, let t_c denote the number of monochromatic triangles produced. By Theorem 20.3, $t_c \ge 1$ for every red-blue coloring c of K_6. Determine $\min\{t_c\}$ over all red-blue colorings c of K_6.

8. We saw in Exercise 4 that $R(3,4) = 9$. What is $R(2,3,4)$?

9. Since $R(3,3) = 6$, every red-blue coloring of K_6 results in a monochromatic K_3. Does every red-blue coloring of $K_6 - e$ result in a monochromatic K_3?

10. In one part of the proof of Theorem 20.3 verifying that $R(3,3) = 6$, it was shown that every red-blue coloring of K_6 produces a monochromatic K_3. This was accomplished by selecting a vertex v of K_6 and considering the colors of the five edges incident with v. In particular, the situation where three edges of K_6 incident with v were colored the same was considered.

    (a) Show that if four edges incident with v are colored the same, then K_6 contains at least two monochromatic triangles.

    (b) Show that if the red subgraph G_R or blue subgraph G_B is 3-regular, then K_6 contains at least two monochromatic triangles. [Recall that there are only two 3-regular graphs of order 6, namely $K_{3,3}$ or $K_3 \square K_2$.]

    (c) Show that if neither G_R nor G_B contains a vertex of degree 4 or more or is 3-regular, then either G_R or G_B contains four vertices of degree 3 and two nonadjacent vertices of degree 2. Show in this case that K_6 contains two monochromatic triangles.

Section 20.2. More General Ramsey Numbers

11. For the graphs F and H shown in Figure 20.6, determine $R(F, H)$.

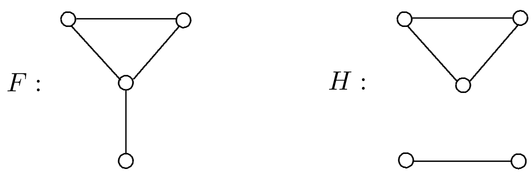


Figure 20.6: The graphs F and H in Exercise 11

12. Determine $R(C_3, C_4)$.

13. Use the fact that the classical Ramsey number $R(3, 4) = 9$ to determine the value of $R(K_3, C_5)$.

14. Let F and H be two nontrivial graphs, where $x \in V(F)$ and $y \in V(H)$. Let $F' = F - x$ and $H' = H - y$. Prove that $R(F, H) \leq R(F', H) + R(F, H')$.

15. Let $n \geq 3$ be an integer. Prove that

$$R(K_{1,n}, K_{1,n}) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ 2n - 1 & \text{if } n \text{ is even} \end{cases}$$

    without using Theorem 20.13 and Corollary 20.14.

16. Let T_s be a tree of order s and let t be a positive integer. Prove that $R(T_s, K_{1,t}) \leq s + t - 1$.

17. For integers s and t with $1 \le s \le t$, define the Ramsey chromatic number $R_\chi(s,t)$ to be the smallest positive integer n such that for every red-blue coloring of K_n, either the red subgraph has chromatic number at least s or the blue subgraph has chromatic number at least t. Trivially, $R_\chi(1,t) = 1$ and $R_\chi(2,t) = t$.

    (a) Does $R_\chi(s,t)$ exist for all integers s and t with $1 \le s \le t$?

    (b) Prove that $R_\chi(s,t) \le R_\chi(s-1,t) + R_\chi(s,t-1)$.

    (c) Determine $R_\chi(3,3)$.

    (d) For $3 \le s \le t$, determine a formula for $R_\chi(s,t)$ in terms of s and t.

18. Show for graphs $G_1, G_2, \ldots, G_k$ $(k \ge 2)$ that

$$R(G_1, G_2, \ldots, G_k, K_2) = R(G_1, G_2, \ldots, G_k).$$

19. Show for positive integers $n_1, n_2, \ldots, n_k$ $(k \ge 2)$ that

$$R(K_{n_1}, K_{n_2}, \ldots, K_{n_k}, T_s) = 1 + (r-1)(s-1)$$

    where T_s is any tree of order $s \ge 1$ and $r = R(n_1, n_2, \ldots, n_k)$.

20. The **monochromatic Ramsey number** $MR(F,H)$ of two graphs F and H is the minimum order n of a complete graph such that for every red-blue coloring of K_n, there is either a monochromatic F or a monochromatic H.

    (a) What is the relationship between $MR(F,H)$ and $R(F,H)$?

    (b) Show that if either F or H is isomorphic to a subgraph of the other, then $MR(F,H)$ equals a Ramsey number.

    [For Exercises 21 –23, see Exercise 20 for the definition of monochromatic Ramsey number.]

21. (a) Determine the Ramsey number $R(K_{1,3}, K_3)$.

    (b) Determine monochromatic Ramsey number $MR(K_{1,3}, K_3)$.

22. (a) Determine the Ramsey number $R(P_4, K_3)$.

    (b) Determine monochromatic Ramsey number $MR(P_4, K_3)$.

23. For the graphs F and H in Figure 20.7, determine $MR(F,H)$.

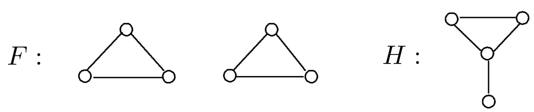


Figure 20.7: The graphs F and H in Problem 23

24. Let F and H be two graphs of size m_1 and m_2, respectively, and let $k = \max\{m_1, m_2\}$. The **rainbow Ramsey number** $RR(F, H)$ of F and H is the minimum order n of a complete graph such that every k-edge coloring of K_n results in a subgraph isomorphic to F whose edges are colored differently (a *rainbow* F) or a monochromatic H. Determine the rainbow Ramsey number $RR(K_{1,k}, K_{1,k})$ for each integer $k \geq 3$.

25. Determine $RR(K_{1,3}, P_4)$. [See Exercise 24 for the definition of rainbow Ramsey number.]

26. The **bipartite Ramsey number** $BR(F, H)$ of two bipartite graphs F and H is the smallest positive integer r such that every red-blue coloring of $K_{r,r}$ results in a red F or a blue H.

    (a) Determine $R(K_{1,3}, C_4)$.

    (b) Determine $BR(K_{1,3}, C_4)$.

27. Determine the bipartite Ramsey number $BR(P_4, P_4)$. [See Exercise 26 for the definition of bipartite Ramsey number.]

28. For each integer $k \geq 2$, determine the minimum positive integer n such that every k-edge coloring of K_n results in a monochromatic subgraph G of K_n with $\chi(G) \geq 3$.

Chapter 21

The Probabilistic Method

In this final chapter, we describe a powerful nonconstructive proof technique known as the probabilistic method. This method is generally credited to Paul Erdős. The topic of random graphs is also discussed.

21.1 The Probabilistic Method

Since the probabilistic method involves defining an appropriate probability space and using techniques and results from probability theory, it is useful to describe some concepts from this area. This material will also be instrumental in the proofs in Section 21.2.

Probability Spaces

A **probability space** consists of a finite set S of objects, together with a **probability function** $P : S \to [0, 1]$. Each subset of S is referred to as an **event**. The **probability of an event** A, denoted by $P[A]$, is defined by

$$P[A] = \sum_{s \in A} P(s),$$

where the probabilities of the sets S and $\emptyset$ are given by $P[S] = 1$ and $P[\emptyset] = 0$. In particular, we frequently use the fact that if $P[A] > 0$ for some event A, then $A \neq \emptyset$.

For an event A, the **complementary event** $\overline{A} = S - A$ and so, by the definition of a probability space, $P[\overline{A}] = 1 - P[A]$. For events A and B, $P[A \cup B]$ means, of course, the probability that A or B (or both) occurs and $P[A \cap B]$ is the probability that both A and B occur. Two events A and B are said to be **independent** if $P[A \cap B] = P[A] \cdot P[B]$. More generally, events $A_i, i \in I$, are

mutually independent if for every subset J of I,

$$P[\cap_{i\in J} A_i] = \prod_{i\in J} P[A_i].$$

Many of the proofs of this chapter depend on results from basic set theory and calculus. We first review the necessary set theory. For events A and B, $A \cup B$ is the event that A or B (or both) occurs. It follows that

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \leq P[A] + P[B].$$

More generally, for events $A_1, A_2, \ldots, A_k$, $k \geq 2$, we have

$$P[\cup_{i=1}^{k} A_i] \leq \sum_{i=1}^{k} P[A_i].$$

For events A and B, De Morgan's laws for sets give us that

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \text{ and } \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

More generally, for events $A_1, A_2, \ldots, A_k$, $k \geq 2$,

$$\overline{\cup_{i=1}^{k} A_i} = \cap_{i=1}^{k} \overline{A}_i \quad \text{and} \quad \overline{\cap_{i=1}^{k} A_i} = \cup_{i=1}^{k} \overline{A}_i.$$

Consequently

$$P[\overline{\cup_{i=1}^{k} A_i}] = P[\cap_{i=1}^{k} \overline{A}_i] \quad \text{and} \quad P[\overline{\cap_{i=1}^{k} A_i}] = P[\cup_{i=1}^{k} \overline{A}_i].$$

When using the probabilistic method (described in this section) or defining a model for a random graph (described in Section 21.2), we generally need to bound the probability $P[A]$ of an event A. Most often, we wish to show that $P[A] > 0$ or $P[A] < 1$. There are some results from calculus that are used repeatedly in this chapter to establish such bounds, including:

$$1 - p < e^{-p} \text{ for a positive real number } p \text{ with } p < 1 \tag{21.1}$$

$$\lim_{n\to\infty} n^{\alpha}\beta^{n} = 0 \text{ for positive real numbers } \alpha \text{ and } \beta < 1 \tag{21.2}$$

$$\lim_{n\to\infty} \frac{\ln n}{n^{\alpha}} = 0 \text{ for a positive real number } \alpha. \tag{21.3}$$

We now illustrate the probabilistic method in graph theory by giving several examples from a number of different areas of graph theory.

Ramsey Numbers

In Chapter 20, it was shown for every integer $t \geq 3$ that $\lfloor 2^{t/2} \rfloor$ is a strict lower bound for the Ramsey number $R(t,t) = R(K_t, K_t)$. This was done by showing the existence of a graph G of order $n = \lfloor 2^{t/2} \rfloor$ such that neither G nor $\overline{G}$ contains K_t as a subgraph. This fact was established by counting the number of different labeled graphs of order n that contain a subgraph isomorphic to K_t, together with the complements of these graphs, and showing that there are fewer than $2^{\binom{n}{2}}$ of these graphs. Since the number of different labeled graphs of order n is $2^{\binom{n}{2}}$, it follows that there *exists* a graph G of order n such that neither G nor $\overline{G}$ contains K_t as a subgraph, even though no such a graph G was ever exhibited. This, however, shows that $R(t,t) > \lfloor 2^{t/2} \rfloor$.

Recall that a red-blue coloring of a graph G is an edge coloring of G where every edge of G is colored red or blue (and where two adjacent edges may be assigned the same color). The next proof establishes, for every integer $t \geq 3$ and $n = \lfloor 2^{t/2} \rfloor$, the existence of a red-blue coloring of K_n having no monochromatic K_t. This implies that $R(t,t) > \lfloor 2^{t/2} \rfloor$.

In the proof of the following result, it will be convenient to employ the inequality

$$2^{t^2/2} < \left(\frac{1}{2}\right) t!\ 2^{\binom{t}{2}}, \tag{21.4}$$

which holds for every integer $t \geq 3$ and which can be verified by induction.

Theorem 21.1 *For every integer $t \geq 3$,*

$$R(t,t) > \lfloor 2^{t/2} \rfloor.$$

Proof. Let $n = \lfloor 2^{t/2} \rfloor$. We show that there is a red-blue coloring of K_n that contains no monochromatic K_t, that is, there is neither a red K_t nor a blue K_t.

Here, the probability space S is the set of red-blue colorings of K_n, where $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. The probabilities are defined by setting

$$P[v_iv_j \text{ is red}] = P[v_iv_j \text{ is blue}] = \frac{1}{2}$$

for each pair v_i, v_j of distinct vertices of K_n and letting these events be mutually independent. Therefore, each of the $2^{\binom{n}{2}}$ red-blue colorings of K_n is equally likely with probability $\left(\frac{1}{2}\right)^{\binom{n}{2}}$.

For a fixed t-element subset T of $V(K_n)$, let A_T denote the event that the subgraph of K_n induced by T is either a red K_t or a blue K_t. The probability that the subgraph of K_n induced by T is a red K_t is $\left(\frac{1}{2}\right)^{\binom{t}{2}}$ since each of the $\binom{t}{2}$ edges joining two vertices of T must be red. Similarly, the probability that the subgraph of K_n induced by T is a blue K_t is $\left(\frac{1}{2}\right)^{\binom{t}{2}}$. Consequently, the probability that the subgraph of K_n induced by T is either a red K_t or a blue K_t is

$$P[A_T] = \left(\frac{1}{2}\right)^{\binom{t}{2}} + \left(\frac{1}{2}\right)^{\binom{t}{2}} = 2^{1-\binom{t}{2}}.$$

Next, let $\mathcal{T}$ be the set of all $\binom{n}{t}$ t-element subsets of $V(K_n)$ and consider the event $\cup_{T\in\mathcal{T}} A_T$. Then

$$\begin{aligned} P\left[\bigcup_{T\in\mathcal{T}} A_T\right] &\leq \sum_{T\in\mathcal{T}} P[A_T] = \binom{n}{t} 2^{1-\binom{n}{t}} \\ &= \frac{n!}{t!(n-t)!}\left(2^{1-\binom{n}{t}}\right) < \frac{n^t}{t!}\left(2^{1-\binom{n}{t}}\right). \end{aligned}$$

Since $n = \lfloor 2^{t/2} \rfloor \leq 2^{t/2}$, it follows that $n^t \leq 2^{t^2/2}$. Moreover, since $t \geq 3$, it follows by (21.4) that

$$2^{t^2/2} < \left(\frac{1}{2}\right) t!\ 2^{\binom{t}{2}}.$$

Therefore,

$$\begin{aligned} P\left[\bigcup_{T\in\mathcal{T}} A_T\right] &< \frac{n^t}{t!}\left(2^{1-\binom{t}{2}}\right) \leq \frac{2^{t^2/2}}{t!}\left(2^{1-\binom{t}{2}}\right) \\ &< \left(\frac{1}{2}\right) 2^{\binom{t}{2}}\left(2^{1-\binom{t}{2}}\right) = 1. \end{aligned}$$

Since $P\left[\bigcup_{T\in\mathcal{T}} A_T\right] < 1$, it follows that $P\left[\overline{\bigcup_{T\in\mathcal{T}} A_T}\right] > 0$ and so

$$P\left[\bigcap_{T\in\mathcal{T}} \overline{A}_T\right] > 0.$$

Hence, $\bigcap_{T\in\mathcal{T}} \overline{A}_T \neq \emptyset$ and therefore some element of the probability space S belongs to $\bigcap_{T\in\mathcal{T}} \overline{A}_T$, implying that there is a red-blue coloring of K_n with no monochromatic K_t. ■

The proof of Theorem 21.1 illustrates a basic technique employed in the probabilistic method. An appropriate probability function is defined on a set of objects. The set of objects in this proof was the set of red-blue colorings of K_n. An event A is then defined representing the desired structure. In the proof of Theorem 21.1, we wanted to show the existence of a red-blue coloring of K_n that contains *no* monochromatic K_t. Thus, the desired event here is $A = \bigcap_{T\in\mathcal{T}} \overline{A}_T$. By showing that $P[A] > 0$, we conclude that $A \neq \emptyset$, that is, there exists a red-blue coloring of K_n with no monochromatic K_t. In general, our goal is to show that $P[A] > 0$ so that we can conclude then that an object with the desired characteristics exists.

As mentioned earlier, this proof technique is generally credited to Erdős. Using a a proof similar to that given in Theorem 21.1 but with more careful approximations, he actually showed that

$$R(t,t) \geq \left(\frac{t}{e}\right) 2^{\frac{t-1}{2}}.$$

In 1975, Joel Spencer [223] proved that this bound can be improved by a factor of 2 with the aid of the Lovász local lemma [224, p. 57]. The bound obtained by Spencer remains the best lower bound for $R(t,t)$ currently known.

List Colorings

We saw in Theorem 15.18 that if r and k are positive integers such that $r \geq \binom{2k-1}{k}$, then $\chi_\ell(K_{r,r}) \geq k+1$. Here, we use the probabilistic method to verify that $\lceil 2\log_2 r \rceil$ is an upper bound for $\chi_\ell(K_{r,r})$, a result initially stated without proof in Theorem 15.19.

Theorem 21.2 *For every integer $r \geq 3$,*

$$\chi_\ell(K_{r,r}) \leq \lceil 2\log_2 r \rceil.$$

Proof. Let U_0 and U_1 denote the partite sets of $K_{r,r}$, where then $|U_0| = |U_1| = r$. For each vertex v of $K_{r,r}$, let $L(v)$ denote a list of $\lceil 2\log_2 r \rceil$ colors that are allowable for v. Let $L = \bigcup L(v)$, where the union is taken over all vertices v of $K_{r,r}$. Thus, L is the set of all colors under consideration. Suppose that $|L| = n$. We now show that there is a proper coloring of $K_{r,r}$ where each vertex v is assigned a color from its list $L(v)$.

For each of the 2^n subsets L_0 of L, let $L_1 = L - L_0$. Suppose that there is a subset L_0 of L such that

$$L_0 \cap L(u) \neq \emptyset \text{ for each } u \in U_0 \tag{21.5}$$

and

$$L_1 \cap L(w) \neq \emptyset \text{ for each } w \in U_1. \tag{21.6}$$

In such a case, each vertex $u_0 \in U_0$ is assigned a color in $L_0 \cap L(u_0)$ and each vertex $u_1 \in U_1$ is assigned a color in $L_1 \cap L(u_1)$. Because $L_0 \cap L_1 = \emptyset$, this results in a proper coloring of $K_{r,r}$ and so $\chi_\ell(K_{r,r}) \leq \lceil 2\log_2 r \rceil$.

We now use the probabilistic method to show that there is, in fact, a subset L_0 of L satisfying (21.5) and (21.6).

Let S be the probability space consisting of all pairs L_0, L_1 of such subsets of L. The probability of a color $c \in L$ belonging to L_0 and L_1 is defined by setting

$$P[c \in L_0] = P[c \in L_1] = \frac{1}{2},$$

and letting these events be mutually independent. Then each of the 2^n pairs L_0, L_1 is equally likely with probability $\left(\frac{1}{2}\right)^n$.

For each $v \in U_i$ where $i = 0, 1$,

$$\begin{aligned} P[L_i \cap L(v) = \emptyset] &= \left(\frac{1}{2}\right)^{|L(v)|} = \left(\frac{1}{2}\right)^{\lceil 2\log_2 r\rceil} \\ &\le \left(\frac{1}{2}\right)^{2\log_2 r} = \frac{1}{r^2} < \frac{1}{2r}, \end{aligned}$$

where the last inequality follows since $r \ge 3$. For each vertex v of $K_{r,r}$, let A_v denote the event that $L_i \cap L(v) = \emptyset$ for $i = 0, 1$. Then $P[A_v] < \frac{1}{2r}$ for every vertex v of $K_{r,r}$. Now, consider the event $\bigcup A_v$, where the union is taken over all vertices v of $K_{r,r}$. Then

$$P\left[\bigcup A_v\right] \le \sum P[A_v] < 2r\left(\frac{1}{2r}\right) = 1.$$

Since $P\left[\bigcup A_v\right] < 1$, it follows that $P\left[\overline{\bigcup A_v}\right] > 0$, that is, $P\left[\bigcap \overline{A}_v\right] > 0$. Hence, $\bigcap \overline{A}_v \ne \emptyset$ and so some element of the probability space S belongs to $\bigcap \overline{A}_v$. Such an element is therefore a pair L_0, L_1 of subsets of L that satisfies (21.5) and (21.6). ■

Theorem 21.2 can be proved without using probabilistic techniques (see Exercise 2), as could Theorem 21.1 (see Theorem 20.7). Both can be established using counting. However, in most important applications of the probabilistic methods, tools of probability are essential and proofs without them are unknown or very difficult. The last three results in this section involve some aspects of probability beyond counting.

Random Variables

By a **random variable** X on a probability space S, we mean a real-valued function on S. The **expected value** $E[X]$ of X is the weighted average

$$E[X] = \sum_{s \in S} X(s) \cdot P[\{s\}].$$

We write "$X = k$" to denote the event consisting of all elements of S where the random variable X has the value k. Consequently, it follows that

$$E[X] = \sum kP[X = k], \tag{21.7}$$

where the sum is taken over all possible values k of X. It is the second expression for $E[X]$ described in (21.7) that will be most convenient to use.

If $X_1, X_2 \ldots, X_k$ are random variables on a probability space S and $X = X_1 + X_2 + \cdots + X_k$, then it follows from the definition of expectation that $E[X] = \sum_{i=1}^{k} E[X_i]$. This property is often referred to as *linearity of expectation.*

Finally, if $E[X] = t$ for a random variable X on a probability space S, then there are elements $s_1, s_2 \in S$ for which $X(s_1) \geq t$ and $X(s_2) \leq t$. This observation is useful in many proofs involving the expected value of a random variable.

Hamiltonian Paths in Tournaments

In Theorem 7.23, we saw that every tournament contains a Hamiltonian path. In what may be considered the first use of the probabilistic method, Tibor Szele [231] showed, for every positive integer n, the existence of a tournament of order n with many Hamiltonian paths. This result was stated without proof as Theorem 7.25 and is now proved as Theorem 21.3. The linearity of expectation is important in this proof, as is the idea of indicator random variables.

Theorem 21.3 *For every positive integer n, there exists a tournament of order n containing at least $n!/2^{n-1}$ Hamiltonian paths.*

Proof. Let S be the probability space consisting of the $2^{\binom{n}{2}}$ different labeled tournaments with vertex set $\{v_1, v_2, \ldots, v_n\}$. The probabilities are defined by setting

$$P[(v_i, v_j) \in E(T)] = P[(v_j, v_i) \in E(T)] = \frac{1}{2}$$

for each pair v_i, v_j of distinct vertices of T and letting these events be mutually independent. Each of the $2^{\binom{n}{2}}$ tournaments, then, is equally likely with probability $\left(\frac{1}{2}\right)^{\binom{n}{2}}$.

Let X be the random variable on the probability space S that gives the number of Hamiltonian paths in a tournament T. In addition, for each of the $n!$ permutations σ of $V(T)$, let X_σ be the **indicator random variable** that depends on whether σ gives a Hamiltonian path in T, that is, $X_\sigma = 1$ if σ describes a Hamiltonian path in T and $X_\sigma = 0$, otherwise. Then $P[X_\sigma = 1] = \left(\frac{1}{2}\right)^{n-1}$ since each of the $n-1$ arcs in the potential Hamiltonian path must be directed consistently. Thus,

$$E[X_\sigma] = 0 \cdot P[X_\sigma = 0] + 1 \cdot P[X_\sigma = 1] = \left(\frac{1}{2}\right)^{n-1}.$$

Furthermore, $X = \sum X_\sigma$ where the sum is taken over all permutations σ of $V(T)$. Therefore,

$$E[X] = E\left[\sum X_\sigma\right] = \sum E[X_\sigma] = \frac{n!}{2^{n-1}}.$$

Since the expected value of the number of Hamiltonian paths is $n!/2^{n-1}$, at least one tournament of order n has at least $n!/2^{n-1}$ Hamiltonian paths. ■

In the proof of Theorem 21.3, we define $n!$ indicator random variables X_σ for the particular purpose of calculating the expected value $E[X]$ of $X = \sum X_\sigma$, where the sum is taken over all $n!$ permutations σ of $V(T)$. These random variables are called indicator random variables because they indicate whether a permutation σ describes a Hamiltonian path in T. Furthermore, $X = \sum X_\sigma$ gives the number of Hamiltonian paths in T.

In general, if X is a random variable on a probability space S that counts substructures (such as Hamiltonian paths in a random tournament), then X can often be written as a sum of indicator random variables. These indicator random variables take on values in $\{0, 1\}$, where a value of 1 indicates that one of the possible substructures occurs. The value of using an indicator random variable X_ℓ is that $E[X_\ell] = 0 \cdot P[X_\ell = 0] + 1 \cdot P[X_\ell = 1] = P[X_\ell = 1]$. The linearity of expectation therefore simplifies calculating $E[X]$.

Domination

In Theorem 12.18, an upper bound for the domination number of a graph G was given in terms of the order and minimum degree of G. This result is due to Vladimir Amautov [7] and, independently, to Charles Payan [183]. We now present a proof of this theorem using the probabilistic method.

The proofs of Theorems 21.4 and 21.5 include many approximating inequalities to determine bounds on probabilities under consideration. The proofs of the next two theorems will make use of the inequality $1 - p < e^{-p}$ for each real number p with $p \in (0, 1)$, stated in (21.1).

Theorem 21.4 *If G is a graph of order n and $\delta(G) \geq 2$, then*

$$\gamma(G) \leq \frac{n(1 + \ln(\delta(G) + 1))}{\delta(G) + 1}.$$

Proof. Let $\delta = \delta(G)$ and $p = \dfrac{\ln(\delta + 1)}{\delta + 1}$. Thus, $0 < p < 1$. Consider the probability space S whose elements are the 2^n subsets of $V(G) = \{v_1, v_2, \ldots, v_n\}$. For a set U of vertices of G, probabilities are assigned by defining $P[v_i \in U] = p$ and then letting these events be mutually independent. For a set $U \subseteq V(G)$, let Y_U be the set of vertices of G not in U that have no neighbors in U, that is, let $Y_U = \{w \in V(G) - U : \ N(w) \cap U = \emptyset\}$. Then $U \cup Y_U$ is a dominating set of G. We next show that

$$E[|U| + |Y_U|] \leq \frac{n(1 + \ln(\delta + 1))}{\delta + 1}.$$

Certainly, the expected value $E[|U|]$ of $|U|$ is np. The random variable $|Y_U|$ is the sum $\sum_{v \in V(G)} X_v$ of indicator variables X_v, where $X_v = 1$ if and only if

v and all its neighbors are *not* in U. Since $\deg v$ is at least $\delta(G)$ for each vertex v, it follows that

$$P[X_v = 1] \le (1-p)^{\deg v + 1} \le (1-p)^{\delta+1}.$$

By the linearity of expectation,

$$E[|Y_U|] = \sum_{v \in V(G)} E[X_v] \le n(1-p)^{\delta+1}.$$

Since $(1-p)^{\delta+1} \le e^{-p(\delta+1)}$, we have

$$E[|U| + |Y_U|] \le np + ne^{-p(\delta+1)} = \frac{n(1+\ln(\delta+1))}{\delta+1}$$

since $p = \frac{\ln(\delta+1)}{\delta+1}$. Thus, there is a set U, corresponding to Y_U, for which $U \cup Y_U$ is a dominating set of order at most $\frac{n(1+\ln(\delta+1))}{\delta+1}$ and so $\gamma(G) \le \frac{n(1+\ln(\delta+1))}{\delta+1}$. ■

One new idea was involved in the proof of Theorem 21.4. Consider the random variable $|U|$ on the 2^n subsets U of $V(G)$. As we noted, $E[|U|]$ equals np, that is, $E[|U|] = \frac{n\ln(\delta+1)}{\delta+1}$. Consequently, there is a set $U \subseteq V(G)$ for which $|U| \le \frac{n\ln(\delta+1)}{\delta+1}$. However, U need not be a dominating set. The set U was then altered (in this case, by adding Y_U) to obtain the desired dominating set. The proof of Theorem 21.5 below employs the same technique of alteration. This proof also uses an inequality due to the Russian mathematician Andrey Markov.

Markov's inequality states that for a random variable X on a probability space S and positive number t,

$$P[X \ge t] \le \frac{E[X]}{t}. \tag{21.8}$$

As expected, "$X \ge t$" is used to denote the event consisting of all elements of S where the random variable X has value at least t. Markov's inequality will be employed in the next section for the case when $t = 1$, giving $P[X \ge 1] \le E[X]$.

Chromatic Number and Girth

It was proved in Theorem 15.1 that there exists a triangle-free k-chromatic graph for every positive integer k. Furthermore, our next theorem, due to Erdős [79] and Lovász [158], which generalizes this result, was stated as Theorem 15.2 without proof.

Recall that $\alpha(G)$ denotes the (vertex) independence number of a graph G and $g(G)$ denotes the girth of G.

Theorem 21.5 *For every two integers $k \ge 2$ and $\ell \ge 3$, there exists a graph G with $\chi(G) = k$ and $g(G) > \ell$.*

Proof. For $k = 2$, any even cycle of length greater than ℓ has the desired property. Thus, we may assume that $k \geq 3$. Let Θ be a real number for which $0 < \Theta < \frac{1}{\ell}$ and for a fixed integer $n \geq 3$, let $p = n^{\Theta - 1}$. Consider the probability space S whose elements are the $2^{\binom{n}{2}}$ different labeled graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$. The probabilities of graphs G in S are determined by setting $P[v_iv_j \in E(G)] = p$ and then letting these events be mutually independent. Thus, each of the $2^{\binom{n}{2}}$ different labeled graphs of size m occurs with probability $p^m(1-p)^{\binom{n}{2}-m}$.

Let X be the random variable on S that gives the number of cycles of length at most ℓ in a graph $G \in S$. For a fixed integer i, $3 \leq i \leq \ell$, there are $\binom{n}{i}$ i-element subsets of $\{v_1, v_2, \ldots, v_n\}$. For each such subset U, there are $\frac{(i-1)!}{2}$ different cyclic orderings of the vertices in U corresponding to possible i-cycles in G. We obtain this bound as follows. For each of the i vertices u of U, there are $(i-1)!$ permutations of the vertices of U with initial vertex u that can result in an i-cycle in G. Since the reverse ordering of these vertices produces the same cycle, there are $\frac{(i-1)!}{2}$ possible i-cycles consisting of the vertices of U. Thus, there are

$$\binom{n}{i}\frac{(i-1)!}{2} = \frac{n(n-1)\cdots(n-i+1)}{2i}$$

potential i-cycles and so X can be expressed as the sum of

$$\sum_{i=3}^{\ell}\frac{n(n-1)\cdots(n-i+1)}{2i}$$

indicator variables. Furthermore, since an i-cycle occurs with probability p^i, the linearity of expectation gives us

$$E[X] = \sum_{i=3}^{\ell}\frac{n(n-1)\cdots(n-i+1)}{2i}p^i. \tag{21.9}$$

Since $p = n^{\Theta-1}$, it follows by (21.9) that

$$\begin{aligned} E[X] &= \sum_{i=3}^{\ell}\frac{n(n-1)\cdots(n-i+1)}{2i}\left(n^{\Theta-1}\right)^i \\ &= \sum_{i=3}^{\ell}\frac{n(n-1)\cdots(n-i+1)n^{\Theta i}}{2i\ n^i} < \sum_{i=3}^{\ell}\frac{n^{\Theta i}}{2i}. \end{aligned}$$

By the choice of Θ, it follows that $\Theta\ell = 1 - \epsilon$ for some real number ϵ with $0 < \epsilon < 1$. Thus,

$$\frac{E[X]}{n/2} \leq \sum_{i=3}^{\ell}\frac{n^{\Theta i}}{ni} \leq \sum_{i=3}^{\ell}\frac{n^{\Theta\ell}}{ni} = \frac{K}{n^{\epsilon}}$$

where $K = \sum_{i=3}^{\ell} \frac{1}{i}$ and so $\lim_{n\to\infty} \frac{E[X]}{n/2} \le \lim_{n\to\infty} \frac{K}{n^{\epsilon}} = 0$.

By Markov's inequality in (21.8),

$$P[X \ge n/2] \le \frac{E[X]}{n/2}.$$

Thus, for n sufficiently large, $P[X \ge n/2] < 1/2$.

Let $t = \lceil 3(\ln n)/p \rceil$. The probability that a given t-element subset of $V(G) = \{v_1, v_2, \ldots, v_n\}$ is independent is $(1-p)^{\binom{t}{2}}$. Since there are $\binom{n}{t}$ such sets, it follows that

$$P[\alpha(G) \ge t] \le \binom{n}{t}(1-p)^{\binom{t}{2}}.$$

However, $1 - p < e^{-p}$ (by 21.1) and so

$$P[\alpha(G) \ge t] < \binom{n}{t} e^{-p\binom{t}{2}} < \left(ne^{-p(t-1)/2}\right)^t.$$

Since

$$t = \left\lceil \frac{3\ln n}{p} \right\rceil = \left\lceil \frac{3\ln n}{n^{\Theta-1}} \right\rceil = \lceil 3(\ln n)n^{1-\Theta} \rceil$$

and $1 - \Theta > 0$, it follows that $\lim_{n\to\infty} t = \infty$. Therefore, $\lim_{n\to\infty} \left(\frac{1}{e}\right)^{p(t-1)/2} = 0$. Consequently, by a generalization of (21.2), we have $\lim_{n\to\infty} ne^{-p(t-1)/2} = 0$ and it follows that we can choose n such that $P[\alpha(G) \ge t] < \frac{1}{2}$ and $P[X \ge n/2] < \frac{1}{2}$. Thus, there is a graph $G \in S$ with fewer than $n/2$ cycles of length at most ℓ and $\alpha(G) < t$. Since

$$t = \lceil 3(\ln n)/p \rceil = \lceil 3(\ln n)n^{1-\Theta} \rceil$$

and $\lim_{n\to\infty} \frac{\ln n}{n^{\alpha}} = 0$ for a positive real number α (see (21.3)), we may assume that n is sufficiently large to ensure that $t < \frac{n}{2k}$.

Remove one vertex from each cycle of G of length at most ℓ and denote the resulting graph by G_1. Then G_1 has girth greater than ℓ, that is, $g(G_1) > \ell$. Furthermore,

$$\alpha(G_1) \le \alpha(G) < \frac{n}{2k}.$$

Moreover, by Theorem 14.9,

$$\chi(G_1) \ge \frac{|V(G_1)|}{\alpha(G_1)} \ge \frac{n/2}{n/(2k)} = k.$$

Finally, we remove vertices from G_1, if necessary, to produce a graph G_2 with $g(G_2) > \ell$ and $\chi(G_2) = k$. ■

The proofs we have presented in this section show that theorems from many areas of graph theory can be established using the probabilistic method of proof. Furthermore, we used only very simple probabilistic concepts and results. Many more examples can be obtained using more powerful tools such as the second moment method, martingales or the Lovász local lemma.

21.2 Random Graphs

In Section 21.1, we found it useful to define appropriate probability spaces in order to establish the *existence* of graphs with certain desired properties. In this section, we provide a formal model for a random graph and answer questions concerning the probability that a random graph has a certain property of interest, such as being nonplanar, or being k-connected for some positive integer k.

Introduction to Random Graphs

For a positive integer n and a real number p with $0 < p < 1$, the **random graph** $G(n,p)$ denotes the probability space whose elements are the $2^{\binom{n}{2}}$ different labeled graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$. The probabilities of graphs $G \in G(n,p)$ are determined by setting $P[v_iv_j \in E(G)] = p$, with these events being mutually independent. The probability of any such graph of size m is therefore $p^m(1-p)^{\binom{n}{2}-m}$. It is important to keep in mind that when we speak of "a random graph", we are actually referring to the probability space $G(n,p)$.

For example, if $p = \frac{1}{2}$, then each of the $2^{\binom{n}{2}}$ different labeled graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$ and size m is assigned the probability $\left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{\binom{n}{2}-m} = \left(\frac{1}{2}\right)^{\binom{n}{2}}$. Thus, the probability of each of these $2^{\binom{n}{2}}$ graphs is $1/2^{\binom{n}{2}}$, as expected.

On the other hand, suppose that $0 < p < 1$ and $p \neq 1/2$, say $p = 1/4$. Then, in $G(n,p) = G(n, \frac{1}{4})$, the probability of each of the $2^{\binom{n}{2}}$ different graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$ of size m is $(\frac{1}{4})^m(\frac{3}{4})^{\binom{n}{2}-m}$. We illustrate below the probability space $G(n,p)$ for $n = 3$ and $p = 1/4$.

Here, $G(3, 1/4)$ denotes the probability space whose elements are the $2^3 = 8$ different labeled graphs $G_1, G_2, \ldots, G_8$ with vertex set $\{v_1, v_2, v_3\}$. Figure 21.1 shows the eight different labeled graphs.

We write $P[G]$ to mean $P[G \in G(n,p)]$, where then

$$P[G_1] = (\tfrac{3}{4})^3,\ P[G_2] = P[G_3] = P[G_4] = \tfrac{1}{4}(\tfrac{3}{4})^2,$$
$$P[G_5] = P[G_6] = P[G_7] = (\tfrac{1}{4})^2\ \tfrac{3}{4} \text{ and } P[G_8] = (\tfrac{1}{4})^3.$$

For the graphs G_i $(1 \leq i \leq 8)$ in Figure 21.1, note that $\sum_{i=1}^{8} P[G_i] = 1$. This will, of course, be true in general for $G(n,p)$, as we now show.

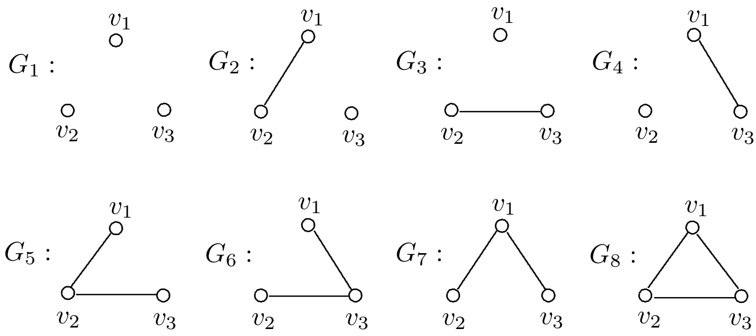


Figure 21.1: The graphs in $G(3, 1/4)$

For each possible value m for the number of edges in a graph $G(n,p)$, $0 \le m \le \binom{n}{2}$, there are $\binom{\binom{n}{2}}{m}$ different labeled graphs with m edges. The probability of each such graph, as we mentioned earlier, is $p^m(1-p)^{\binom{n}{2}-m}$. Therefore,

$$\begin{aligned}\sum_{G\in G(n,p)} P[G] &= (1-p)^{\binom{n}{2}} + \binom{n}{2}p(1-p)^{\binom{n}{2}-1} \\ &\quad + \binom{\binom{n}{2}}{2}p^2(1-p)^{\binom{n}{2}-2} + \cdots + p^{\binom{n}{2}} \\ &= \sum_{m=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}.\end{aligned}$$

It follows, by the Binomial Theorem, that

$$\sum_{G\in G(n,p)} P[G] = [p + (1-p)]^{\binom{n}{2}} = 1.$$

In this section, we discuss properties possessed by "almost all graphs". Specifically, for a given graph theoretic property Q, we say that **almost all graphs** (in $G(n,p)$) have property Q if

$$\lim_{n\to\infty} P[G \in G(n,p) \text{ has property } Q] = 1.$$

We shall see, for example, in Theorem 21.6 for any positive constant $p < 1$ that almost all graphs (in $G(n,p)$) are connected with diameter 2.

A useful device to verify that almost all graphs have some property Q is to define an appropriate integer-valued random variable X on $G(n,p)$ so that $G \in G(n,p)$ has property Q if $X = 0$. Thus, for $G \in G(n,p)$,

$$P[X = 0] \le \; P[G \in G(n,p) \text{ has property } Q].$$

Consequently, for $G \in G(n,p)$,

$$\lim_{n\to\infty} P[X=0] = 1 \text{ implies that } \lim_{n\to\infty} P[G \in G(n,p) \text{ has property } Q] = 1.$$

Since X is an integer-valued function on $G(n,p)$,

$$\lim_{n\to\infty} P[X=0] = 1 \text{ if and only if } \lim_{n\to\infty} P[X \geq 1] = 0.$$

By Markov's inequality, $P[X \geq 1] \leq E[X]$. From this, it follows that

$$\text{if } \lim_{n\to\infty} E[X] = 0, \text{ then } \lim_{n\to\infty} P[X \geq 1] = 0, \tag{21.10}$$

and so almost all graphs have property Q.

Connectedness

Our first result shows that for each constant p ($0 < p < 1$), almost all graphs are connected with diameter 2. This strengthens an earlier result of Edgar Gilbert and Frank Harary [105] that almost all graphs are connected.

Theorem 21.6 *For each positive constant* $p < 1$*, almost all graphs* (*in* $G(n,p)$) *are connected with diameter* 2.

Proof. For each $G \in G(n,p)$, let the random variable $X(G)$ be the number of (unordered) pairs of vertices in G with no common neighbor. Certainly, if $X(G) = 0$, then G is connected with diameter 2 (or G is the single exception K_n). Thus, by (21.10), it suffices to show that $\lim_{n\to\infty} E[X] = 0$.

List the $\binom{n}{2}$ pairs of vertices of G. Then X can be written as the sum of $\binom{n}{2}$ indicator variables X_i, $1 \leq i \leq \binom{n}{2}$, where $X_i = 1$ if the ith pair has no common neighbor and $X_i = 0$, otherwise. Then $X = X_1 + X_2 + \cdots + X_{\binom{n}{2}}$ and, by the linearity of expectation,

$$E[X] = \sum_{i=1}^{\binom{n}{2}} E[X_i].$$

If the ith pair is $\{u,v\}$, then $P[X_i = 1]$ is the probability that no other vertex is adjacent to both u and v. For a fixed vertex z ($z \neq u,v$), the probability that z is not adjacent to both u and v is $1-p^2$. This probability is independent of the probability that any other vertex is not adjacent to both u and v. Thus, the probability that none of the other $n-2$ vertices z ($z \neq u,v$) are adjacent to both u and v is $\left(1-p^2\right)^{n-2}$ and, since X_i is an indicator variable, $E[X_i] = P[X_i = 1] = \left(1-p^2\right)^{n-2}$. It then follows that

$$E[X] = \binom{n}{2}\left(1-p^2\right)^{n-2}.$$

As n tends to infinity, the first factor in the expression for $E[X]$ tends to ∞ (as a polynomial in n) and the second factor tends to 0 exponentially. As we saw earlier in (21.2), this implies that $\lim_{n\to\infty} E[X] = 0$. ■

The basic idea used to define the random variable X in the proof of Theorem 21.6 was generalized by Andreas Blass and Frank Harary [29] in order to study other properties of almost all graphs.

For fixed nonnegative integers k and ℓ, let $Q_{k,\ell}$ denote the property that if S and T are disjoint sets of vertices of a graph with $|S| \le k$ and $|T| \le \ell$, then there is a vertex $z \notin S \cup T$ that is adjacent to every vertex of S and to no vertex of T. As is normally done, the statement of Theorem 21.7 below does not include the phrase "in $G(n, p)$". Since we are referring to "almost all graphs", this is implied from the definition.

Theorem 21.7 *For each pair k, ℓ of nonnegative integers and each positive constant $p < 1$, almost all graphs have property $Q_{k,\ell}$.*

Proof. Define a pair $\{S, T\}$ of disjoint subsets of vertices of a graph with $|S| \le k$ and $|T| \le \ell$ to be *bad* if no vertex $z \notin S \cup T$ is adjacent to every vertex of S and to no vertex of T. For each graph G in $G(n, p)$, let $X(G)$ be the number of bad pairs $\{S, T\}$ in G. We wish to show that almost all graphs have no bad pairs of sets and, as in the proof of Theorem 21.6, we need only show that $\lim_{n\to\infty} E[X] = 0$.

For each graph G in $G(n, p)$, list all pairs $\{S, T\}$ of disjoint subsets of vertices of G. The variable $X(G)$ can be written as the sum of indicator variables X_i, where $X_i = 1$ if the ith pair $\{S, T\}$ is bad and $X_i = 0$ otherwise. For a fixed pair $\{S, T\}$ and a vertex $z \notin S \cup T$, the probability that z is adjacent to every vertex of S and to no vertex of T is $p^{|S|}(1-p)^{|T|}$. Since

$$p^{|S|}(1-p)^{|T|} \ge p^k(1-p)^\ell,$$

the probability that the pair S, T is bad, that is, that no such z exists for the pair $\{S, T\}$, is

$$\left(1 - p^{|S|}(1-p)^{|T|}\right)^{n-|S|-|T|} \le \left(1 - p^k(1-p)^\ell\right)^{n-k-\ell}.$$

In other words,

$$P[X_i = 1] \le \left(1 - p^k(1-p)^\ell\right)^{n-k-\ell}.$$

Since the number N of pairs $\{S, T\}$ is no more than $n^{k+\ell}$, it follows that

$$\begin{aligned} E[X] &= \sum_{i=1}^{N} E[X_i] \le \sum_{i=1}^{N} \left(1 - p^k(1-p)^\ell\right)^{n-k-\ell} \\ &\le n^{k+\ell}\left(1 - p^k(1-p)^\ell\right)^{n-k-\ell}. \end{aligned}$$

Thus, by applying (21.2), we have $\lim_{n\to\infty} E[X] = 0$. ■

The following is a useful observation (see Exercise 6):

Theorem 21.8 *Let Q_1 and Q_2 be graphical properties. If almost all graphs have property Q_1 and almost all graphs have property Q_2, then almost all graphs have both properties Q_1 and Q_2.*

As an example of an application of Theorems 21.7 and 21.8, we prove that for each graph H and each positive constant p $(0 < p < 1)$, almost all graphs contain H as an induced subgraph.

Theorem 21.9 *For each graph H and each positive constant $p < 1$, almost all graphs contain H as an induced subgraph.*

Proof. We proceed by induction on the order t of H. For $t = 1$, all graphs contain $H = K_1$ as an induced subgraph. Assume that for every graph H' of order $t - 1 \geq 1$, almost all graphs contain H' as an induced subgraph and consider a graph H of order t. Select a vertex v of H and let $H' = H - v$. By the induction hypothesis, almost all graphs contain H' as an induced subgraph. Let $S = \{u \in V(H) : uv \in E(H)\}$ and let $T = V(H) - S - \{v\}$, where $|S| = k$ and $|T| = \ell$. Since almost all graphs have property $Q_{k,\ell}$ by Theorem 21.7, it follows that almost all graphs contain a vertex z that is adjacent to every vertex in the copy of S in H' and to no vertex in the copy of T in H'. Therefore, by Theorem 21.8, almost all graphs contain H as an induced subgraph. ■

If Q is a graphical property (for example, planarity) that implies that certain graphs (such as K_5 and $K_{3,3}$) do not exist as induced subgraphs, then Theorem 21.9 immediately implies that for each positive constant $p < 1$, almost no graphs have property Q. Here, of course, we mean that

$$\lim_{n\to\infty} P[G \in G(n,p) \text{ has property } Q] = 0.$$

Corollary 21.10 *For each positive constant $p < 1$, almost no graphs are planar.*

Corollary 21.11 *For each positive integer k and each positive constant $p < 1$, almost no graphs are k-colorable.*

Corollary 21.12 *For each positive integer k and each positive constant $p < 1$, almost no graphs have genus at most k.*

Other results can be obtained in a manner similar to that used in the proof of Theorem 21.7.

Theorem 21.13 *For each positive integer k and each positive constant $p < 1$, almost all graphs are k-connected.*

It should be noted that for each positive constant $p < 1$, there are properties possessed by almost all graphs that *cannot* be established with the aid of Theorem 21.7. For example, Blass and Harary [29] showed that almost all graphs are Hamiltonian but Theorem 21.7 cannot be used to verify this result.

Perhaps surprisingly, the results of this section have the property that they do not depend on the particular (constant) value of p. We finish this section with a brief discussion of a more "sensitive" model $G(n,p(n))$, that is, $G(n,p)$, where rather than p being a constant, $p = p(n)$ is a function of n. We begin with an example involving the existence of copies of K_4 in the random graph $G(n,p(n))$.

Let $p(n)$ be a function of n and, for each graph G in $G(n,p(n))$, let the random variable $X(G)$ denote the number of copies of K_4 in G. By (21.10),

$$\text{if } \lim_{n\to\infty} E[X] = 0, \text{ then } \lim_{n\to\infty} P[X \geq 1] = 0.$$

For a graph $G \in G(n,p(n))$ and each 4-element subset S of $V(G)$, let A_S be the event that $G[S]$ is complete. Since all six possible edges must be present in $G[S]$, we have $P[A_S] = (p(n))^6$. Let X_S be the indicator variable with $X_S = 1$ if $G[S]$ is complete and $X_S = 0$, otherwise. Then $X = \sum X_S$, where the sum is taken over all 4-element subsets S of $V(G)$. Furthermore, since X_S is an indicator variable, $E[X_S] = P[X_S = 1] = P[A_S] = (p(n))^6$. By the linearity of expectation,

$$E[X] = \sum E[X_S] = \binom{n}{4}(p(n))^6 < n^4(p(n))^6.$$

Consequently, if

$$\lim_{n\to\infty} n^4(p(n))^6 = 0, \tag{21.11}$$

then $\lim_{n\to\infty} E[X] = 0$ and so $\lim_{n\to\infty} P[X \geq 1] = 0$. Certainly, if $\lim_{n\to\infty} \frac{p(n)}{n^{-2/3}} = 0$, then (21.11) will hold. We conclude that if $\lim_{n\to\infty} \frac{p(n)}{n^{-2/3}} = 0$, then

$$\lim_{n\to\infty} P[G \in G(n,p(n)) \text{ contains a copy of } K_4] = 0.$$

The interesting fact is that, with the aid of the second moment method ([223, p. 19]), it can be shown that if $\lim_{n\to\infty} \frac{p(n)}{n^{-2/3}} = \infty$, then

$$\lim_{n\to\infty} P[G \in G(n,p(n)) \text{ contains a copy of } K_4] = 1.$$

We might say that if $p(n)$ is "significantly smaller than $n^{-2/3}$", then almost no graphs contain a copy of K_4, while if $p(n)$ is "significantly larger than $n^{-2/3}$", then almost all graphs contain a copy of K_4. This leads to the concept of a threshold function for a property.

Generally, let Q be a graph theoretic property that is preserved by the addition of edges to a graph. A function $r(n)$ is called a **threshold function** for Q if

(i) $\lim_{n\to\infty} \frac{p(n)}{r(n)} = 0$ implies that $\lim_{n\to\infty} P[G \in G(n,p(n)) \text{ has } Q] = 0$

and

(ii) $\lim_{n\to\infty} \frac{p(n)}{r(n)} = \infty$ implies that $\lim_{n\to\infty} P[G \in G(n, p(n))$ has $Q] = 1$.

In other words, a function $r(n)$ is called a threshold function for Q if

(i) $\lim_{n\to\infty} \frac{p(n)}{r(n)} = 0$ implies that almost no graphs have property Q

and

(ii) $\lim_{n\to\infty} \frac{p(n)}{r(n)} = \infty$ implies that almost all graphs have property Q.

Figure 21.2 indicates many well-known graph properties and corresponding threshold function ([224, p. 17] and [149, 189]).

Property	**Threshold Functions**
Contains a path of length k	$r(n) = n^{-(k+1)/k}$
Is not planar	$r(n) = 1/n$
Is Hamiltonian	$r(n) = (\ln n)/n$
Is connected	$r(n) = (\ln n)/n$
Contains a copy of K_k	$r(n) = n^{-2/(k-1)}$

Figure 21.2: Threshold functions

Exercises for Chapter 21

Section 21.1. The Probabilistic Method

1. Show that if $\binom{n}{t}2^{1-\binom{t}{2}} < 1$, then $R(t,t) > n$.

2. Prove Theorem 21.2 without using the probabilistic method.

3. We have seen that $\chi_\ell(K_{10,10}) \geq 4$. How does this compare to the upper bound given for $\chi_\ell(K_{10,10})$ by Theorem 21.2?

4. Show that if $\binom{n}{s}2^{\binom{s}{2}} + \binom{n}{t}2^{-\binom{t}{2}} < 1$, then $R(s,t) > n$.

5. A nontrivial tournament T of order n has *property* U_k $(1 \leq k \leq n-1)$ if for every set U of k vertices of T, there is a vertex $w \notin U$ that is adjacent to every vertex of U. Show that if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a tournament T of order n with property U_k.

Section 21.2. Random Graphs

6. (a) Show that $P[A \cap B] \geq 1 - (P[\overline{A}] + P[\overline{B}])$.

   (b) Prove Theorem 21.8:

   Let Q_1 and Q_2 be graphical properties. If almost all graphs have property Q_1 and almost all graphs have property Q_2, then almost all graphs have both properties Q_1 and Q_2.

   (c) Let $Q_1, Q_2, \ldots, Q_k$, $k \geq 2$, be graphical properties. Prove that if almost all graphs have property Q_i for all $i = 1, 2, \ldots, k$, then almost all graphs have all of these k properties.

7. Show that Theorem 21.6 is a corollary of Theorem 21.7.

8. Prove Corollary 21.11: *For each positive integer k and each positive constant $p < 1$, almost no graphs are k-colorable.*

9. Prove Corollary 21.12: *For each positive integer k and each positive constant $p < 1$, almost no graphs have genus at most k.*

10. Without using the fact that a threshold function for the property of containing a copy of K_k is $n^{-2/(k-1)}$, show for every integer $k \geq 5$ that if $\lim_{n\to\infty} \frac{p(n)}{n^{-2/(k-1)}} = 0$, then almost no graph in $G(n, p(n))$ contains a copy of K_k.

11. Prove or disprove: If $r(n)$ is a threshold function for some property Q, then $r(n)$ is unique.

12. Let $T(n, \frac{1}{2})$ denote the probability space of the $2^{\binom{n}{2}}$ labeled tournaments T of order n with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$, where the probabilities are defined by setting $P[(v_i, v_j) \in E(T)] = P[(v_j, v_i) \in E(T)] = \frac{1}{2}$. In Exercise 5, a nontrivial tournament T of order n was said to have property U_k $(1 \le k \le n-1)$ if for every set U of k vertices of T, there is a vertex $w \notin U$ that is adjacent to every vertex of U. Show for a fixed positive integer k, that almost all tournaments have property U_k, that is, $\lim_{n\to\infty} P[T \in T(n, \frac{1}{2}) \text{ has } U_k] = 1$.

13. Show for a fixed positive real number $p < 1$ that almost all graphs have eccentricity sequence $2, 2, \ldots, 2, 1, 1, \ldots, 1$, where i terms equal 2 and $n-i$ terms equal 1, for some integer i with $2 \le i \le n$.

Hints and Solutions to Odd-Numbered Exercises

Chapter 1 Section 1.1

1. Let the vertices of the graph represent the boxes where two vertices are adjacent if the corresponding boxes contain a wire segment of the same color.

Section 1.2

3. The degree of each remaining vertex is 3.

Section 1.3

5. **Proof.** Let G be a graph with r vertices of degree r, $r+1$ vertices of degree $r+1$ and $r+2$ vertices of degree $r+2$. Thus, the order of G is $3r+3$. First, we show that r is odd. Assume, to the contrary, that r is even. Then G contains an odd number $r+1$ of odd vertices, which is impossible. Thus, r is odd and G contains $2r+2$ vertices of odd degree. ■

7. (a) $G_1 \cong G_2$. (b) $H_1 \not\cong H_2$.

9. (a) Apply the definition of isomorphism to $G[S]$ and $H[T]$.
   (b) Consider $r = 3$.

Section 1.4

11. Consider the average degree $2m/n$ of G.

13. Let $V(G) = \{u, v, w, x\}$ and $E(G) = \{uv, vw, wx, xu, vx\}$. Let $e = vx$. Then $G - e = C_4$ and $G - u = C_3$.

15. If G is not r-regular, then let G' be another copy of G and join corresponding vertices whose degrees are less than r.

17. The Petersen graph.

Section 1.5

19. $n = 22$.

21. Let the partite sets of a 3-partite graph G of order $n = 3k$ and size m be A, B and C, where $|A| = a$, $|B| = b$ and $|C| = c$ and $a+b+c = 3k$. We may assume that $a \geq b \geq c$ and let $a = k+x$, $c = k-y$ and $b = 3k-a-c = k-x+y$. Then show that $m \leq ab + ac + bc \leq 3k^2 - x^2 + xy - y^2 \leq 3k^2$.

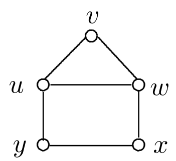
Section 1.6

23. Observe that if G has a partite set with three or more vertices, then $\overline{G}$ is not bipartite.
25. Observe that $\overline{G}$ in (a) is C_7 or $C_3 + C_4$, while $\overline{G}$ in (b) is one of $C_9, C_6 + C_3, C_5 + C_4$ or $3C_3$.
27. Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$, replace v_i by a copy H_i of C_5 and join every vertex of H_i to every vertex of H_j if $v_i v_j$ is an edge of C.
29. Use induction and the hint given in Exercise 15.
31. Since there is a self-complementary graph of order n for every integer n with $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ by Exercise 29, we may assume that $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$. Use induction on n.
33. (a) The degree of u_1 in $G_1 + G_2$ is $\deg_{G_1} u_1$.
(b) The degree of u_1 in $G_1 \vee G_2$ is $\deg_{G_1} u_1 + n_2$.
(c) The degree of (u_1, u_2) in $G_1 \,\square\, G_2$ is $\deg_{G_1} u_1 + \deg_{G_2} u_2$.

Section 1.7

35. $G - uw - xy + ux + wy$.

$G:$



37. $G = P_4$.
39. Assume that for each graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \le i \le n$, there is a vertex v_k $(1 \le k \le n)$ such that both (1) and (2) fail.
41. (b), (c) and (e) are graphical.
43. If a graph G of order 6 contains two vertices of degree 5, it has no vertex of degree 1.
45. Observe that $\sum_{i=1}^{3} d_i = 17$ and $3(3-1) + \sum_{i=1}^{7} \min\{3, d_i\} = 16$.
47. Let $S = \{a_1, a_2, \cdots, a_n\}$ and $k = \text{lcm}\ \{a_1 + 1, a_2 + 1, \cdots, a_n + 1\}$. For $S = \{2, 6, 7\}$, $k = 4$.
49. In one direction, assume that s_1 and s_2 are bigraphical sequences. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{w_1, w_2, \ldots, w_t\}$. Suppose that there is no bipartite graph H with partite sets V_1 and V_2 and a vertex u of H of degree a_1 in V_1 adjacent to vertices of degrees $b_1, b_2, \ldots, b_{a_1}$.
51. (a) $rK_1 + (s/2)K_2$. (b) The minimum size is $s/2$.
(c) The maximum size is $\binom{n}{2} - s/2$.

Section 1.8

53. Only the sequence in (c) is a degree sequence of an irregular multigraph.
55. Each of the sequences in (a), (c), (e), (g) and (h) is the degree sequence of a multigraph.
57. (a) $m = 20$. (b) $9 \le m \le 11$.

(c) Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ where $\deg u_1 \geq \deg u_2 \geq \cdots \geq \deg u_s$ and $\deg w_1 \geq \deg w_2 \geq \cdots \geq \deg w_t$. Then $m = \min\left\{\sum_{i=1}^{s} i \deg u_i, \quad \sum_{i=1}^{t} i \deg w_i\right\}$.

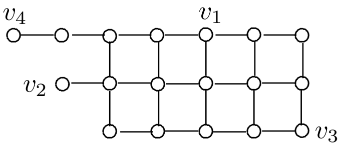
Chapter 2 Section 2.1

1. Let $G = K_{2,3}$ and let u and v be the two vertices of G with $\deg u = \deg v = 3$
3. (a) The statement is true. (b) The statement is false.
5. For $i \neq j$, the (i, j)-entry of A^2 is the number of different $v_i - v_j$ paths of length 2 and the (i, j)-entry of A^3 is the number of different $v_i - v_j$ walks of length 3. The vertices v_1 and v_4 belong to one triangle while the vertices v_2 and v_3 belong to two triangles.
7. $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_5\}$.
9. Let $P_{k+1} = (u_1, u_2, \ldots, u_{k+1})$ be a path of order $k+1$ and let G be the graph obtained from P_{k+1} by adding k new vertices $v_1, v_2, \ldots, v_k$ and joining v_i to u_i and u_{i+1} for $1 \leq i \leq k$.
11. Consider a longest path in G.
13. If G contains a $u - v$ path and a $v - w$ path, then G contains a $u - w$ path.
15. Suppose that the statement is false and consider a longest path in G.
17. Let $V(G) = \{u = v_1, v_2, \cdots, v_n = v\}$ and let P_i be a $v_i - v_{i+1}$ path for $i = 1, 2, \cdots, n-1$.
19. (a) Consider a path connecting two vertices of distinct degrees. (b) No.
21. Suppose that G has k or more components and consider a component of minimum order.
23. For every nonempty proper subset S of $V(G)$, there is a vertex in S adjacent to a vertex in $G - S$.

Section 2.2

25. Observe that $d(u, v) + d(u, w) + d(v, w) = [d(u, v) + d(v, w)] + d(u, w) \geq 2d(u, w)$.
27. $G = K_{k-1,k+1}$.
29. (a) $d_1(v) = \deg v$. (b) For $1 \leq k \leq n-1$, let m_k denote the number of pairs u, v of vertices of G such that $d(u, v) = k$. Then $\sum_{v \in V(G)} d_k(v) = 2m_k$. (c) $n^2 - n$.
31. Let G be obtained by identifying an end-vertex of P_{b-a+1} with a vertex of C_{2a}.
33. For $n \geq 4$, let e_1, e_2 and e_3 be the edges of a triangle in K_n. Let $F = K_n - e_1 - e_2$ and $G = K_n - e_1 - e_2 - e_3$.
35. Let $H = K_{2,k}$ where the partite sets of H are $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_k\}$ and let G be the graph obtained by H by adding two vertices v_1 and v_2 and the two edges u_1v_1 and u_2v_2. Then $\deg w_i = e(w_i) = 2$ for $i = 1, 2, \ldots, k$.
37. $n = 2 + (r-1)\sum_{i=1}^{s-1}(r-2)^i$.

39. See the graph below.



41. Let $2k+1 \geq 3$ and let G be obtained by identifying the center of $K_{1,2k+1}$ and an end-vertex of P_{2k+1}.
43. If F can be transformed into G by a 2-switches and G can be transformed into H by b 2-switches, then F can be transformed into H by $a+b$ 2-switches.

Chapter 3 Section 3.1

1. Apply Theorem 3.2.
3. Apply Theorems 3.2 and 3.3.
5. Let H be the graph obtained by adding two vertices u and w to G and joining u to the two vertices of U and joining w to the two vertices of W.
7. (a), (b) Apply Theorem 3.2.
9. Let $u, w \in V(\overline{G}) - \{v\}$. Show that there is a $u - w$ path in $\overline{G}$ not containing v.
11. In the case when G is a nonseparable graph and the two elements of G are the two edges $e = uv$ and $f = xy$, subdivide e and f, introducing two new vertices s and t and producing a new nonseparable graph H.
13. The statement is false.
15. The statement is false.
17. (a) Use Exercise 1. (b) The statement is false.

Section 3.2

19. Either $\deg u \geq 2$ or $\deg v \geq 2$, say the latter. Then v is adjacent to a vertex w distinct from u. Assume that v is not a cut-vertex and obtain a contradiction to Theorem 2.10.
21. (a) There is only one example. (b) $T = K_{1,3}$ or T is the tree in (a).
23. There are 20 forests of order 6.
25. A graph G is a forest if and only if every connected subgraph of G is an induced subgraph of G.
27. In one direction, assume that at least one of e_1 and e_2 is not a bridge and observe that $G - e_1 - e_2$ has at most two components.
29. Apply Theorem 3.9.
31. (a) Observe that $b(v_i) \leq \deg v_i$ and apply Theorem 1.4.
(b) Observe that $\sum_{i=1}^{n} b(v_i) = 2m$ if and only if every edge of G is a bridge.
33. Apply Theorem 3.8.
35. Show that $\deg u = \deg v = m - (n - 2)$.
37. Apply Theorem 3.20.
39. The 4-cycle C_4 is the only graph with this property.

41. If T is not a path, then T contains vertices that are not on P. For $w \in V(T) - V(P)$, let P_w denote the unique shortest path from w to P. For a vertex u of T, consider two cases, depending on whether u belongs to P, to show $e(u) = \max\{d(u, u_0), d(u, u_t)\}$.

Section 3.3

43. The Prüfer codes of the first 8 trees are (2), (1), (3), (1,1), (2, 2), (4,4), (3, 3), (3,4).
45. The labeled tree with this Prüfer code is a caterpillar with three vertices of degree 3.
47. Observe that every vertex v of a tree T appears $\deg v - 1$ times in its Prüfer code.
49. $G = P_n$.
51. If G is a unicyclic graph of order n, then G has size n and at least three vertices of G lie on a cycle.
53. Let v be a vertex of degree 2 in the unicyclic graph of order 5 containing a 4-cycle.
55. Let u and v be the two end-vertices in the unicyclic graph of order 5 containing a vertex of degree 4.
57. (a) There are six paths and three stars. (b) $(n-1)^{n-2}$.
59. In one direction, suppose that e is an edge that does not belong to every spanning tree of G.
61. $n/2$.
63. Show that for each positive integer $k \neq 2$, there is a connected graph with exactly k spanning trees.
65. For $2 \leq r < t$, let G be the graph obtained by attaching $r - 1$ pendant edges to a vertex of K_{t-r+2}.

Section 3.4

67. A minimum spanning tree has weight 20.
69. First, assume that T is the unique minimum spanning tree of G. Obtain a contradiction if the weight of some edge e that is not in T is at most the weight of some edge f on the cycle $T + e$. For the converse, assume that T is not unique. Consider another minimum spanning tree T' of G and obtain a contradiction.
71. By Kruskal's algorithm, there is only one choice at each step for the edge selected.

Chapter 4 **Section 4.1**

1. If G is a complete k-partite graph of order n whose largest partite set contains n_k vertices, then $\kappa(G) = \lambda(G) = \delta(G) = n - n_k$.
3. (a) Let $H = G \vee K_1$. Show that if S is a set of vertices of H with $|S| \leq k$, then $H - S$ is connected.
   (b) Let $H = G \vee K_1$. Show that if S is a set of edges of H with $|S| \leq k$, then $H - S$ is connected.

5. Observe that $k \le \kappa(G) \le \delta(G)$ and apply Theorem 3.20.
7. Assume, to the contrary, that G is not (ℓ, k)-connected. Consider a set S of $\kappa_k(G)$ vertices such that $G - S$ contains at least k components and produce a contradiction to the degree condition.
9. Such a graph G can be constructed by beginning with two copies of K_{c+1}.
11. Every k-connected graph of order n has minimum degree at least k and so has size at least $kn/2$. Show that there exists a k-connected graph G of (even) order n and size $kn/2$.
13. (a) Let W be a set of $k-2$ vertices of $G - e$. Show that $(G - e) - W$ is connected.

(b) Let X be a set of $k - 2$ edges of $G - e$. Since $|X \cup \{e\}| = k - 1$, the graph $G - e - X$ is connected.
15. $\overline{\kappa}(G) \ge \kappa(G)$, $\overline{\lambda}(G) \ge \lambda(G)$ and $\overline{\lambda}(G) \ge \overline{\kappa}(G)$.

Section 4.2

17. For a set W of $k-1$ vertices of H, show that $H - W$ is connected.
19. In one direction, let G be a graph of order $n \ge 2k$ that is not k-connected. Then G contains a vertex-cut S with $|S| = k-1$. Consider a component G_1 of $G - S$ of minimum order and consider appropriate sets V_1 and V_2.
21. Let $u \in S_1$ and $v \in S_2$. Then there exist k internally disjoint $u-v$ paths in G. Consider appropriate subpaths of these paths.
23. Assume first that G is k-edge-connected. Consider a $u-v$ separating set of edges and apply Theorem 4.17. For the converse, suppose that G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G. Then apply Theorem 4.17.
25. The statement is true.
27. Consider the step where each edge of S is subdivided and then the k new vertices are identified.
29. Let u and v be two adjacent vertices of a 3-connected graph G. Then consider three internally disjoint $u - v$ paths. The graph $G = C_n$, $n \ge 4$, shows that a 2-connected graph need not contain a chorded cycle.
31. (a) By Theorem 4.11, $\kappa(G) = k$. Let S be a vertex-cut of G with $|S| = k$ and let u and v be vertices belonging to different components of $G - S$. Let $w \in V(G) - (S \cup \{u, v\})$ and consider $S \cup \{w\}$.

(b) Let u and v be two vertices of G with $d(u, v) = \operatorname{diam} G = k$ and consider k internally disjoint $v - u$ paths.

Chapter 5 Section 5.1

1. Yes.
3. This is possible and one such route is: A, B, A, B, A, C, A, C, A, D, B, D, C D, A.

Section 5.2

5. Show that each vertex of G is even.

7. The graph $G \square H$ is Eulerian if and only if both G and H are Eulerian or every vertex of G and H is odd.
9. No.
11. For each vertex v in G, either $\deg v \equiv 0 \pmod 4$ or $\deg v \equiv 2 \pmod 4$. Consider the sum of the degrees of the vertices of G.
13. (a) and (b). Both statements are false.
15. (a) $m - n/2$. (b) The Petersen graph.
17. (a) Every edge in G belongs to four cycles, while every edge in H belongs to 15 cycles.
(b) The graph G is not Eulerian and H is Eulerian.

Chapter 6 Section 6.2

1. (a) In the dodecahedron in Figure 6.2, follow the consonants in alphabetical order.
(b) Consider the path (R, Q, P, C, B, G).
3. Apply Theorem 2.6.
5. For $k \geq 1$, consider the graph $G = K_1 \vee 2K_k$.
7. First, observe that $G - v$ is connected for every vertex v of G. Next, consider the Petersen graph.
9. Assume, to the contrary, that the result is false. Among all counterexamples of order $2k$, let G be one of maximum size and consider a Hamiltonian path in G. The bound is sharp.
11. Consider $G = K_{k,k+1}$.
13. Let $H = G \vee K_1$ and apply Theorem 6.10.

Section 6.3

15. (a) Apply Dirac's theorem (Corollary 6.2).
(b) Apply Theorem 6.11.
(c) The graph G is Hamiltonian if and only if $n_k \leq \sum_{i=1}^{k-1} n_i$.
17. (a) Let $G = P_5 = (v_1, v_2, v_3, v_4, v_5)$ and let $S = \{v_2, v_4\}$.
(b) **Theorem** If G is a graph with a Hamiltonian path, then $k(G-S) \leq |S| + 1$ for every nonempty proper subset S of $V(G)$.
19. (a) Since $\delta(G) > n/2$, it follows by Dirac's theorem (Corollary 6.2) that G has a a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{101}, v_1)$. Consider the vertex v_1 and the set $S_i = \{v_i, v_{i+25}, v_{i+50}, v_{i+75}\}$ for $2 \leq i \leq 26$.
(b) **Theorem** Let k be a positive integer. If G is a graph of order $4k+1$ such that $\delta(G) \geq 2k + 1$, then every vertex of G lies on a cycle of length $k + 2$.
21. Observe that every vertex-cut of G is also a vertex-cut of H.
23. Let $G = K_{r,s}$ with partite sets U and W such that $|U| = r$ and $|W| = s$ with $r \leq s$. Then $t(G) \leq \frac{|U|}{k(G-U)} = \frac{r}{s}$. Show that $t(G) \geq r/s$.
25. (a) Let $n = \sum_{i=1}^{k} n_i$. Then $t(K_{n_1,n_2,\ldots,n_k}) = \frac{n-n_k}{n_k}$.
(b) Let $r = a/b$, where a and b are positive integers. Consider the result in (a).

27. If G has a Hamiltonian cycle C, then at least one of the five pairs $\{s_i, t_i\}$ $(1 \le i \le 5)$ of vertices contains no vertex that is adjacent to either p or q on C (see Figure 6.8).

Section 6.4

29. Apply Theorem 6.15.

31. Suppose that the statement is false. Then there exist two vertices u and v for which there is no Hamiltonian $u-v$ path. Consider the Hamiltonian graph $G-v$.

33. (a) Consider two cases: (i) $(n-1)/n \le r < 1$ and (ii) $1/2 \le r < (n-1)/n$ and let $k = \lceil 2rn \rceil - n + 1$. (b) Let $r = (n+k-1)/2n$.

35. Let $H = K_3 \square K_2$, where $\{u, v, w\}$ are the vertices in one copy of K_3 and $\{u', v', w'\}$ are the vertices in the other copy of K_3, where $uu', vv', ww' \in E(G)$. Let G be the graph obtained by adding a new vertex x to H where x is joined to w and w'.

37. $n+1$.

Section 6.5

39. The subdivision graph $S(G)$ of a graph G is Eulerian if and only if G is Eulerian.

41. The exceptions are the trees.

43. Let $u, v \in V(G)$ such that $d_G(u, v) = \text{diam}(G) = \ell$. Consider a $u-v$ geodesic $P = (u = v_0, v_1, \ldots, v_\ell = v)$.

45. Apply Exercise 2.

47. First observe by Exercise 8(b) that every self-complementary graph is connected and $\text{diam}(G) = 2$ or $\text{diam}(G) = 3$. If $\text{diam}(G) = 2$, then G^2 is complete. Hence it may be assumed that $\text{diam}(G) = 3$. If G is 2-connected, then apply Theorem 6.23. Hence we can assume that G has cut-vertices and contains vertices having eccentricity 3.

49. No.

51. Observe that (a, b, c, a) is a triangle in $L(G)$ if and only if the subgraph induced by the three edges a, b, c is either K_3 or $K_{1,3}$.

53. (a) For a graph G of order n, $G \cong L(G)$ if and only if $G = C_n$.
 (b) Apply (a) to $L^2(G)$.

55. First observe that $L^2(G)$ is connected by Exercise 50. Show that if every vertex of $L^3(G)$ has even degree, then every vertex of $L^2(G)$ has even degree.

57. (a) Assume, to the contrary, that there is a k-edge-connected graph G, $k \ge 2$, such that $L(G)$ is not k-connected. Let S be a vertex-cut in $L(G)$ with $|S| = \ell \le k-1$.
 (b) Let $\{U, W\}$ be a partition of $V(L(G))$ and apply Theorem 4.2.

59. (a) Observe that $T(G^2) \cong (S(G^2))^2$ and apply Theorem 6.23.
 (b) Consider $H = (S(G))^2$ and observe that $T(G) \cong H$.

Chapter 7 Section 7.1

1. (a) Let D be the digraph with $V(D) = \{v_1, v_2, v_3, v_4, v_5\}$ where $(v_i, v_j) \in E(D)$ if and only if $1 \le i < j \le 5$. (b) Yes.

3. Let D be the digraph of order $2k$ such that $V(D)$ is partitioned into two sets U and W where $|U| = |W| = k$, where every vertex $u \in U$ is adjacent to every vertex of W.

5. The statement is false.

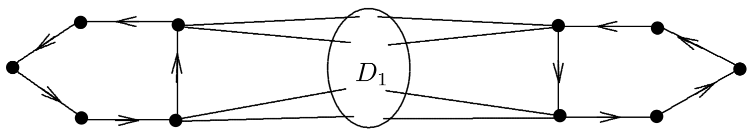
7. The outdegree and indegree of a vertex in a regular tournament of order n is $(n-1)/2$.

9. (a) The sum of the entries in row i is $\text{od}\, v_i$ and the sum of the entries in column i is $\text{id}\, v_i$.

(b) A is the adjacency matrix of a digraph if and only if A is a zero-one matrix where every entry on the main diagonal is 0.

11. Any orientation of an odd cycle produces a directed path of length 2.

13. Begin with the following.



Section 7.2

15. Follow the proof of Theorem 2.1.

17. (a) By Robbins' theorem (Theorem 7.5), G has a strong orientation D. Consider D and $\vec{D}$. (b) Consider $G = C_n$ where $n \geq 3$.

19. (a) Consider two cases: (1) G contains exactly one bridge and (2) G contains exactly two bridges. (b) Consider $G = K_{1,3}$.

Section 7.3

21. Follow the proof of Theorem 5.1.

23. Add a new vertex w to D together with the arcs (v, w) and (w, u). Then apply Theorem 7.6.

25. (a) Yes. (b) No.

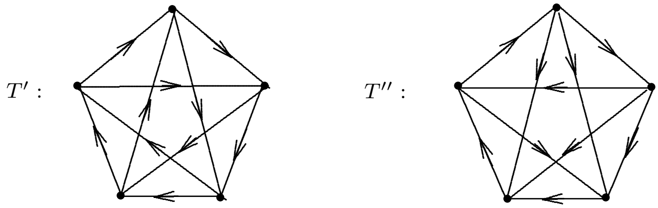
27. Assume that $\text{od}\, v_i \geq \text{id}\, v_i$ for $1 \leq i \leq k$ and that $\text{od}\, v_i < \text{id}\, v_i$ for $k+1 \leq i \leq n$. Construct a digraph D' by adding t new vertices x_i $(1 \leq i \leq t)$ so that D' is Eulerian.

29. Label the vertices of D as $v_1, v_2, \ldots, v_n$ so that $r(v_i) = i$ for $1 \leq i \leq n$. Show that D contains the Hamiltonian path $(v_n, v_{n-1}, \ldots, v_2, v_1)$.

31. Let D_1 and D_2 be copies of the digraph K_k^* obtained by replacing each edge uv of K_k by the symmetric pair (u, v) and (v, u) of arcs. Let D be obtained by identifying a vertex of D_1 and a vertex of D_2.

Section 7.4

33. See the tournaments below.



35. If D is an r-regular tournament of order n, then $n = 2r + 1$.

37. If $\widetilde{T}$ is not transitive, then $\widetilde{T}$ contains a triangle (S_1, S_2, S_3, S_1). Consider $v_i \in V(S_i)$ for $i = 1, 2, 3$.

39. (b) and (c) are score sequences of tournaments.

41. The tournament T is the transitive tournament of order n.

43. (a) One. (b) Suppose that there is a tournament that contains three vertices not having both positive outdegree and positive indegree. (c) None.

45. Consider a shortest $u - v$ path.

47. Consider the set of vertices to which v is adjacent.

49. Consider the sets of vertices to which u is adjacent and from which u is adjacent.

51. Assume that $(u, v) \in E(T)$ and then consider the set of vertices to which v is adjacent.

Section 7.5

53. Note that every vertex of maximum outdegree in a tournament is a king

55. Let T be the tournament with $V(T) = \{u, v, w, x\}$ and $E(T)$=$\{(v, u)$, (u, w), (u, x), (w, v), (x, v), $(w, x)\}$.

57. Assume that $n \geq 6$ is even. One possible construction of T: Let T_{n-3} be a regular tournament of order $n - 2$. Then the tournament T of order n has $V(T) = \{v_1, v_2, v_3\} \cup V(T_{n-3})$ and $E(T) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\} \cup E(T_{n-3}) \cup E_1 \cup E_2 \cup E_3$ where $E_1 = \{(v, v_1) : v \in E(T_{n-3})\}$, $E_2 = \{(v, v_2) : v \in E(T_{n-3})\}$ and $E_3 = \{(v_3, v) : v \in E(T_{n-3})\}$.

Section 7.6

59. If T is strong, then T is Hamiltonian. If T is not strong, then T contains at least two strong components, at least one of which is nontrivial.

61. The strong component S_2 has $k/15$ Hamiltonian paths.

63. (a) and (b). Both statements are false.

65. There must be a strong component of order k, another strong component of order $k - 1$ or k and k trivial strong components. So: $n \geq k + (k - 1) + k = 3k - 1$.

Chapter 8 Section 8.1

1. No.

3. (a) $a = 1$, $b = 2$ and $c = 1$. (b) $\text{val}(f) = 4$.

5. Show that for the flow f', the net flow out of each vertex in N' equals the net flow out of this vertex for the flow f.

7. (a) Let $\mathcal{U}$ be the set of all maximal paths P in D whose initial vertex is u and such that no arc of P belongs to A. Consider the set $X = \bigcup_{P\in\mathcal{U}} V(P)$.

(b) Let D be a $u - v$ path of length 2.

9. (a) Show that $\text{val}(f) = \text{cap}(K)$ and apply Corollary 8.3.

(b) Use the proof of Theorem 8.6 to show that $f(\overline{X}, X) = 0$ and $c(X, \overline{X}) = f(X, \overline{X})$.

Section 8.2

11. Apply Algorithm 8.7 three times to obtain a maximum flow f'' with $\text{val}(f'') = 6$ and a minimum cut $[X, \overline{X}]$ with $X = \{u, s\}$.

13. (a) The statement is false. (b) The statement is true.

Section 8.3

15. The maximum number of pairwise arc-disjoint $u - v$ paths in D is 1.

17. Suppose that $S = \{s_1, s_2, \ldots, s_r\}$ and $T = \{t_1, t_2, \ldots, t_q\}$. Add a new vertex s and the arcs (s, s_i), $1 \le i \le r$, and add a new vertex t and the arcs (t_i, t), $1 \le i \le q$. Let $C = \sum_{e\in E(D)} c(e)$

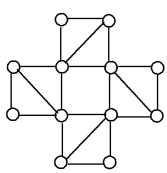
Chapter 9 Section 9.1

1. $\text{Aut}(G_1) = \{\epsilon, (t\ u), (y\ z), (t\ u)(y\ z)\}$.

$\text{Aut}(G_2) = \{\epsilon$, (1 2), (6 7), (1 2)(6 7), (3 5)(1 6)(2 7), (3 5)(2 6)(1 7), (3 5)(1 7 2 6), (3 5)(1 6 2 7)$\}$.

3. Yes. $K_4 - e$.

5. See the figure below.



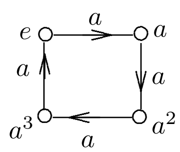
7. Such a pair (k, n) is realizable except when $k = n$ and $2 \le n \le 5$.

9. Let $n = 2k$ for some positive integer k and let G be the bipartite graph with partite sets $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$ where u_i $(1 \le i \le k)$ is adjacent to w_j if $i + j > k$.

11. The graph G_3 has two cut-vertices. In G_4, consider the subgraphs induced by the neighborhoods of its vertices.

Section 9.2

13. (a) Let $\Delta_1 = \{a\}$. The Cayley color graph $D_{\Delta_1}(\Gamma)$ is shown below.



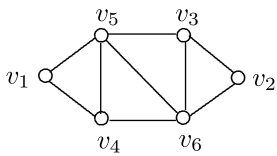
(b) Let $\Delta_2 = \{a, a^2, a^3\}$.

15. The group of color-preserving automorphisms of the Cayley color graph $D_\Delta(\Gamma)$ consists of ϵ, $\alpha = (g_1\ g_2\ g_3\ g_4)(g_5\ g_6\ g_7\ g_8)$, $(g_1\ g_3)(g_2\ g_4)(g_5\ g_7)(g_6\ g_8)$, $(g_1\ g_4\ g_3\ g_2)(g_5\ g_8\ g_7\ g_6)$, $\beta = (g_1\ g_5)(g_2\ g_6)(g_3\ g_7)(g_4\ g_8)$, $(g_1\ g_6\ g_3\ g_8)(g_2\ g_7\ g_4\ g_5)$, $(g_1\ g_7)(g_2\ g_8)(g_3\ g_5)(g_4\ g_6)$, $(g_1\ g_8\ g_3\ g_6)(g_2\ g_5\ g_4\ g_7)$.

17. Let $\Gamma \cong \mathbb{Z}_n = \{e, a, a^2, \cdots, a^{n-1}\}$ and $\Delta = \{a\}$.

Section 9.3

19. See the graph below.



21. The graph $G = K_{3,4}$.

23. (a) Observe that $\overline{\overline{G} - v} = G - v$.
(b) Apply Theorem 9.10 and (a).

25. Apply the proof of Theorem 9.10 to obtain $G = 2C_4 + P_3$.

27. No.

29. The only solution is the graph obtained from $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ by adding the two edges v_2v_5 and v_3v_5.

Chapter 10 Section 10.1

1. To prove that a graph G is planar if each block of G is planar, employ induction on the number of blocks of G. Consider an end-block.

3. Let G be the graph consisting of two paths $P = (u, v, w, x)$ and $Q = (y_1, y_2, z_2, z_1)$ where the end-vertices y_1 and z_1 are joined to every vertex of P.

5. (a) Apply the Euler Identity (Theorem 10.1).

7. Determine the size of G.

9. Apply the Euler Identity (Theorem 10.1).

Section 10.2

11. Consider $K_{2,2,2}$ and $P_4 \vee K_2$.

13. Suppose that there is a planar graph of order $n \geq 3$ and size $m = 3n - 6$ that is not maximal planar.

15. The statement is false.

17. The graph is nearly maximal planar.

19. Suppose that the result is false. Assign to each vertex v of G a charge of $6 - \deg v$. For each vertex v of G having degree 5, distribute its charge of $+1$ equally to four neighbors of v not having degree 5 or 6.

Section 10.3

21. (a) The statement is true. (b) $G[U] = \overline{K}_k$.

23. If G is nonplanar, then G contains a subgraph H that is a subdivision of K_5. Compute the size of H.

25. The only such graph is $G = K_5$.

27. (a) Give a proof by contradiction.

(b) Let G be the graph obtained from $K_{3,3}$ by subdividing one edge of $K_{3,3}$.

29. The graph C_n^2 is nonplanar if and only if $n \geq 5$ and n is odd.

31. The statement is false.

33. (a) No conclusion can be made.

(b) The statement is true.

(c) The graph H is nonplanar.

(d) Give a proof by contradiction.

35. The graph $G = P_n$.

Section 10.4

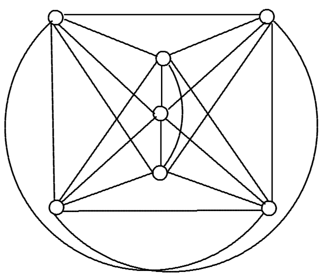
37. Consider a planar embedding of $K_{2,3}$ and suppose that G contains a Hamiltonian cycle C.

39. Suppose that the Grinberg graph G is Hamiltonian and observe that $3(r_5 - r_5') + 6(r_8 - r_8') + 7(r_9 - r_9') = 0$.

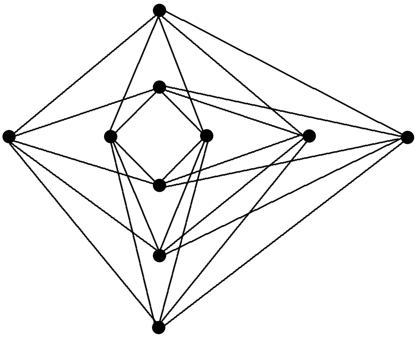
41. Suppose that this graph G has a Hamiltonian cycle C that contains both e and f. Then one of the regions having e on its boundary is in the interior of C and the other is in the exterior of C.

Chapter 11 Section 11.1

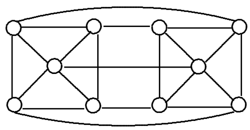
1. See the figure below.



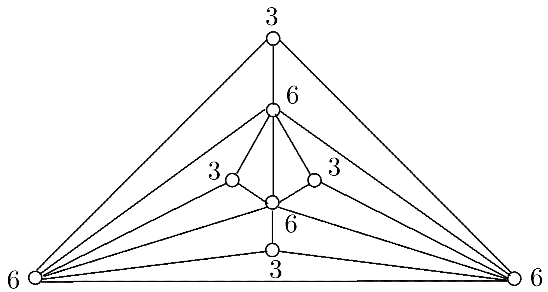
3. See the figure below.



5. $\mathrm{cr}(K_{1,2,3}) = 1$.
7. (a) See the figure below.



(b) The statement is false.
9. See the figure below.



Section 11.2
11. Since $K_{3,3} \subseteq K_{4,4}$, it follows that $\gamma(K_{4,4}) \geq 1$. Exhibit an embedding of $K_{4,4}$ on the torus to show that $\gamma(K_{4,4}) = 1$.
13. Since the boundary of every region contains at least k edges and every edge is on the boundary of at most two regions, $kr \leq 2m$. The result follows by applying Corollary 11.12.
15. Apply Corollary 11.15.
17. $\gamma(C_4 \,\Box\, C_6 \,\Box\, K_2) = 7$.
19. For $n \geq 6k + 6$, let G be a maximal planar graph of order n. Apply Theorem 11.13.

21. $\gamma(G) = 2$.
23. Yes.

Section 11.3
25. (a) Apply the definition of a minor-closed family of graphs. (b) $\{K_3\}$.
27. Yes; $\{K_4, K_{2,3}\}$.

Chapter 12 Section 12.1

1. Suppose that there exists a tree with two distinct perfect matchings. Apply Theorem 12.1.
3. Consider the graph H obtained from G by adding two new vertices u and w where u is joined to every vertex of U and w is joined to every vertex of W.
5. (a) Proceed by contradiction and apply Hall's theorem (Theorem 12.3).
   (b) Consider $G = K_{k,k-1} + K_1$.
7. After observing that $\alpha'(G) \leq |U|$, consider the cases where $\text{def}(U) \leq 0$ and $\text{def}(U) > 0$.
9. $M(C_3) = C_3$, $M(2P_3) = C_4$, $M(C_5) = C_5$, $M(K_{2,3}) = C_6$.
11. (a) The statement is true. (b) The statement is false.

Section 12.2
13. (a) The statement is false. (b) The converse of Errera's theorem is false.
15. (a) Let $S = \{u, v\}$.
   (b) Proceed by contradiction and apply Tutte's theorem (Theorem 12.7).
   (c) See the graph G in (a).
17. (a) Apply the First Theorem of Graph Theory (Theorem 1.4).
   (b) Proceed by contradiction and apply Tutte's theorem (Theorem 12.7).
19. (a) 3. (b) Yes.

Section 12.3
21. In one direction, let G be a graph that is not bipartite and let C be a smallest odd cycle of G.
23. By Exercise 22, $\frac{n}{1+\Delta(G)} \leq \alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Apply Gallai's theorem (Theorem 12.11).
25. $\alpha(G) + \beta(G) = n$ and $\alpha'(G) + \beta'(G) = n - k$.

Section 12.4
27. $\gamma(Q_3) = 2$ and $\gamma(Q_4) = 4$.
29. Let G be a graph of order n. Then $\gamma(G) = 1$ if and only if $\Delta(G) = n - 1$.
31. Let $d = \text{diam}(G) \geq 3$ and let v be a vertex of G such that $e(v) = d$. For a vertex w of G with $d(v, w) \geq 3$, consider $S = \{v, w\}$.
33. Use the fact that $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) = n$.
35. (a) The statement is true.
   (b) Consider the double star where every vertex has degree 1 or 3.

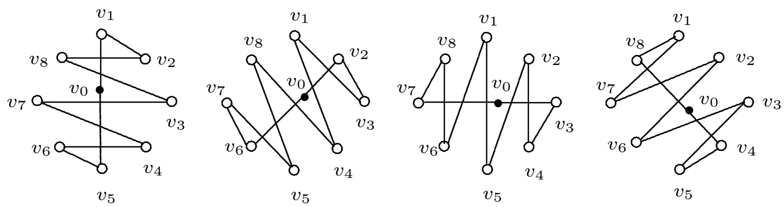
37. For $k \geq 4$, let G be the graph obtained from $P_3 = (u, v, w)$ by adding $3(k-2)$ new vertices, where $k-2$ of these vertices are adjacent to u, another $k-2$ vertices are adjacent to v and the final $k-2$ vertices are adjacent to w.
39. The statement is true.
41. (a) To show that $\gamma_t(G) \leq 2\gamma(G)$, let $S = \{v_1, v_2, \ldots, v_k\}$ be a minimum dominating set of G. For $u_i \in N(v_i)$ $(1 \leq i \leq k)$, consider $S \cup \{u_1, u_2, \ldots, u_k\}$.

(b) For each $n \geq 3$, let H_n be the corona of the cycle C_n and let F_n be the graph obtained by adding a pendant edge to each end-vertex of H_n.
43. No.

Chapter 13 Section 13.1

1. Suppose that there is a 1-factorization $\{F_1, F_2, F_3\}$ of the Petersen graph and consider the spanning subgraph with edge set $E(F_1) \cup E(F_2)$.
3. (a) Apply Theorem 13.1.

(b) Observe that if Q_n is k-factorable, then $k \mid n$. For the converse, apply (a).
5. Let $n = 4k$ for some integer $k \geq 2$. Let $H_1 = K_{2k-1}$ and $H_2' = K_{2k+1}$, let C be a Hamiltonian cycle in H_2' and let $H_2 = H_2' - E(C)$. Consider $G = H_1 + H_2$.
7. See the figure below.



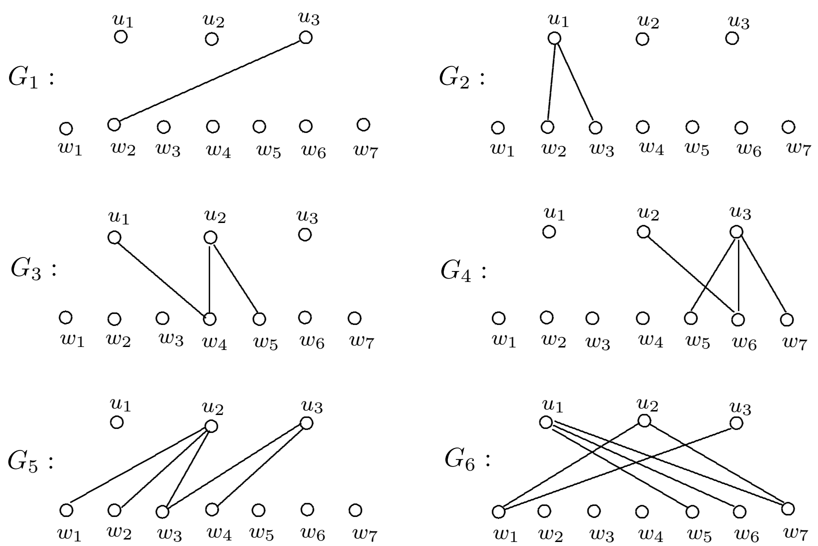
9. Observe first that there is a k-regular graph of order $n-2$.
11. Proceed by contradiction and consider the size of each factor.
13. Use Theorem 13.7.
15. Let $C' = (v_1, v_2, v_4, v_6, v_8, v_7, v_5, v_3, v_1)$ and $C'' = (v_1,\ v_7,\ v_5,\ v_6,\ v_4,\ v_3,\ v_2,\ v_8,\ v_1)$. Then C' and C'' form a Hamiltonian-factorization of C_8^2.
17. For $n \geq 3$, let G be the graph obtained by adding a new vertex v to K_{n-1} and joining v to two distinct vertices u and w of K_{n-1}.

Section 13.2

19. The graph of the octahedron in Figure 13.7 can be decomposed into three copies of P_5, namely $P = (u, x, z, w, v)$, $P' = (v, z, y, u, w)$ and $P''' = (w, y, x, v, u)$.
21. If there exists a Steiner triple system of order $n \geq 3$, then the complete graph K_n is K_3-decomposable and so $3 \mid \binom{n}{2}$.
23. (a) Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. For $1 \leq i \leq n-1$, let G_i be the star of size i with central vertex v_{i+1} containing the edges $v_j v_{i+1}$ for $1 \leq j \leq i$.

(b) By Theorem 13.5, there exists a Hamiltonian factorization of K_n.

25. See the figure below.



Section 13.3

27. $k = 3, 4, 9$.

29. $k = 0, 2, 4, 6, 8$.

31. (a) Consider $K_{1,2k}$. (b) Let G be the graph obtained by subdividing k edges of $K_{1,2k}$ once each.

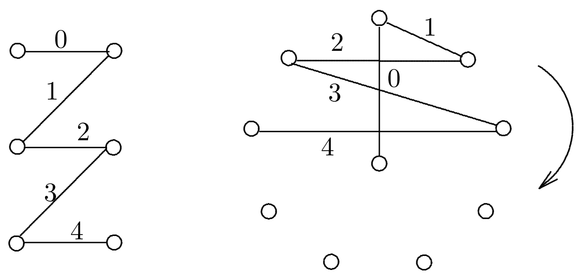
Section 13.4

33. Show that it is not possible for three edges to be labeled $1, 3, 5$.

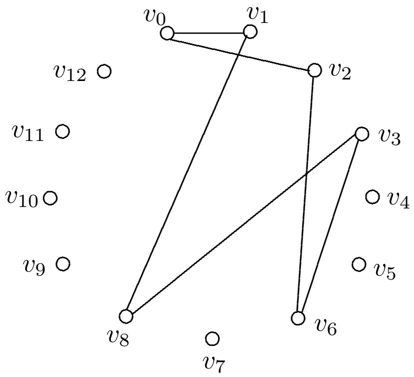
35. (a) Label the seven vertices of K_7 by $0, 1, 2, \ldots, 6$ and observe that the vertex labels $0, 1$ and 3 produce a graceful labeling of K_3.

(b) Proceed as in (a) with a regular 9-gon rather than a regular 7-gon.

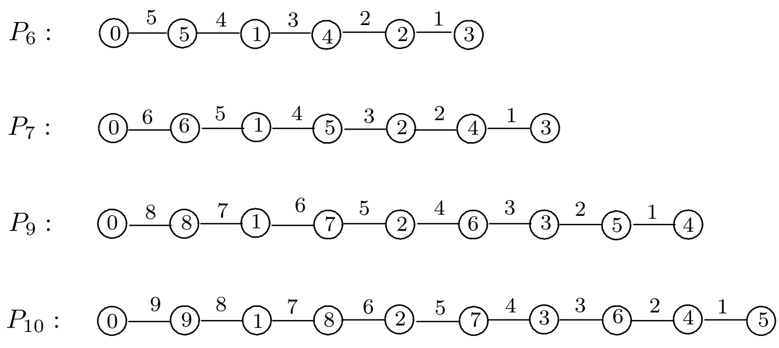
37. See the figure below.



39. Let $m = 2k$. Then $K_{3m+1} = K_{6k+1}$. Arrange the vertices $v_1, v_2, \cdots, v_{6k+1}$ cyclically about a regular $(6k+1)$-gon and join every two vertices by a straight line segment, producing the complete graph K_{6k+1}. Let $C = (v_1, v_2, \cdots, v_{6k+1}, v_1)$. Assign each edge xy of K_{6k+1} the value $d_C(x, y)$. Use the fact that G is graceful to place G in K_{6k+1} so that the edge labels coincide.
41. The Petersen graph is cyclically F-decomposable when F is any of the graphs $P_4, 3K_2, P_3 + K_2$.
43. See the figure below.



45. See the figure below.



47. (a) If F is a forest of order n with $k \geq 2$ components, then the size of F is $m = n - k \leq n - 2$.

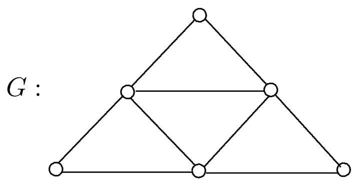
(b) The graph $P_3 + C_3$ is not graceful.

49. Apply the definition of a graceful labeling to f and $\overline{f}$.

Chapter 14 Section 14.1

1. Suppose that the vertices of G are colored 1, 2, 3. For each vertex v colored 3, say, consider the colors of the neighbors of v.
3. $\chi(H) = k$.
5. Consider a $v_0 - v_d$ geodesic $P = (v_0, v_1, v_2, \ldots, v_d)$ of length d in G.

7. If G is not r-regular, let G' be another copy of G and join corresponding vertices whose degrees are less than r. Permute the colors of the vertices of G'. Continue in this manner.
9. (a) Begin with a coloring of G (of H) and produce a coloring of H (of G). (b) (i) Let $G = P_2$. (ii) Let $G = C_3$. (iii) Let $G = C_4$.
11. (a) The statement is false. (b) The statement is true.
13. Assume that $\chi(G_1) = \max\{\chi(G_i) : 1 \le i \le k\}$ and for each i ($1 \le i \le k$), provide a proper coloring of G_i using the colors $1, 2, \ldots, \chi(G_1)$. Then $\chi(G) \le \chi(G_1)$.
15. Proceed by induction on $k \ge 2$.
17. The statement is false.
19. $n = 2k^2 + k$, $\alpha(G) = k$, $\omega(G) = 2k$ and $\chi(G) = 2k + 1$.
21. See the graph G below.



23. Assume, to the contrary, that $\chi(G) \le k-1$ and consider the color classes in a $\chi(G)$-coloring of G. This bound is sharp.
25. (a) Consider C_5.

(b) Suppose that $\overline{\chi}(G) = k$ and let $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ into k maximal independent sets. Assigning the color i to each vertex in V_i for $1 \le i \le k$, we obtain a proper k-coloring of G.

(c) For $n, s, t \ge 1$, $\overline{\chi}(K_n) = n$ and $\overline{\chi}(K_{s,t}) = 2$.

27. (a) Consider some k-coloring of G and the resulting color classes V_1, V_2, …, V_k. (b) For one direction, suppose that $\chi(G[V(G) - S]) = k - 1$ for some independent set S of vertices of G. Then color the vertices of $G[V(G) - S]$ with the $k-1$ colors $1, 2, \ldots, k-1$ and assign each vertex of S the color k.
29. If $\chi(G) = 2$, then G is a bipartite graph and is not Hamiltonian-connected.
31. A graph G is constructed with $V(G) = \{F_1, F_2, \ldots, F_{10}\}$, where F_i is adjacent to F_j ($i \ne j$) if F_i and F_j cannot be placed in the same tank. Then the minimum number of tanks required is $\chi(G)$.

Section 14.2

33. (a) Assume, to the contrary, that $\chi(G - v) \le k - 2$.

(b) Assume, to the contrary, that $\chi(G - e) \le k - 2$.

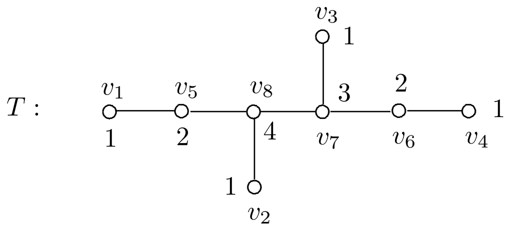
35. First, observe that any 3-critical graph of order $n \ge 3$ contains an odd cycle C.
37. The statement is true.
39. Both (a) and (b) are false statements.

Section 14.3

41. (a) Among the k-colorings of G using the colors $1, 2, \ldots, k$, let C_1 be the set of those colorings that color a maximum number of vertices 1. Among the colorings in C_1, let C_2 be the set of those colorings that color a maximum number of vertices 2. Continue in this manner.

(b) Let $H = K_{p+2,p+2}$ with partite sets $U = \{u_1, u_2, \ldots, u_{p+2}\}$ and $W = \{w_1, w_2, \ldots, w_{p+2}\}$. Let $G = H - \{u_i w_i : 1 \le i \le p+2\}$.

43. (a) See the graph below. (b) Yes. (c) No.



45. The upper bounds are 5, 2 and 3, respectively.
47. $s = 2$ and $t = 6$.
49. $\chi(G) = 3$.
51. $\frac{1}{1-k}$.
53. (a) $\chi(G) \ge 2$. This bound is sharp. (b) $\chi(G) \ge 3$. This bound is sharp.
55. Show that there is an orientation D of G such that $\ell(D) = 1$.

Chapter 15 Section 15.1

1. (a) $G = C_5$. (b) G is the Grötzsch graph.
3. The statement is true.
5. If H is a connected and noncomplete induced subgraph of the graph of the octahedron, then H is one of the three graphs. In each case, $\chi(H) = \omega(H)$.
7. (a) The statement is false. (b) The statement is true.
9. (a) Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Define an intersection graph H by $V(H) = \{S_1, S_2, \ldots, S_n\}$ such that S_i consists of the subscripts of the edges of G that are incident with v_i in G.

(b) Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $A = \{1, 2, \ldots, n\}$. Let $\mathcal{F}$ be the set of 2-element subsets of A such that $\{i, j\} \in \mathcal{F}$ if and only if $v_i v_j \in E(G)$.

11. Suppose that G is an interval graph defined by a set S of closed intervals. For an induced subgraph H of G, consider an appropriate subset of S.
13. (a) The interval graph G is the path $(J_1, I_1, J_2, I_2, J_3, I_3, J_4, I_4, J_5)$.

(b) The interval graph G is a tree obtained from the path $(J_1, I_1, J_2, I_2, J_4, I_3, J_5)$ by adding the vertex J_3 and the edge $I_2 J_3$.

(c) The interval graph G is the forest $2P_5$.

15. (a) Let u_i correspond to I_i for $1 \le i \le 5$. Then G is the graph with $V(G) = \{u_1, u_2, \ldots, u_5\}$ and $E(G) = \{u_1u_2, u_1u_3, u_2u_3, u_2u_4, u_3u_4, u_4u_5\}$.

(b) Let $J_1 = (0.9, 7.1)$, $J_2 = (5.9, 8.1)$, $J_3 = (5.9, 9.1)$, $J_4 = (6.9, 20.1)$ and $J_5 = (19.9, 22.1)$.
17. Assume, to the contrary, that there is an interval graph G that is not a chordal graph.

Section 15.2
19. For $n = 7$, let $G_7 = \overline{C}_7$.
21. $\overline{C}_8$ is perfect.
23. The converse is true.
25. $\chi(Shad(G)) = \chi(G)$ and $\omega(Shad(G)) = \omega(G)$.

Section 15.3
27. Let there be given any collection $\mathfrak{L} = \{L(v) : v \in V(G)\}$ of color lists of size $1 + \Delta(G)$ for the vertices of G, where the colors are positive integers. Suppose that the vertices of G are listed in the order $v_1, v_2, \ldots, v_n$ and the greedy algorithm is applied. At each step in the greedy algorithm, there is a color available in $L(v)$ for the current vertex v under consideration since $\deg v < 1 + \Delta(G)$.
29. Let $G = K_{2,3}$. Clearly, $\chi_\ell(G) \geq 2$. Let the partite sets of G be $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ and let there be given an arbitrary collection $\mathfrak{L} = \{L(v) : v \in V(G)\}$ of color lists of size 2 for the vertices of G. Consider two cases (1) $L(u_1) \cap L(u_2) \neq \emptyset$ and (2) $L(u_1) \cap L(u_2) = \emptyset$.
31. Show that the graph $K_{3,27}$ is not 3-choosable by exhibiting an appropriate list of size 3 for each vertex of $K_{3,27}$ that does not produce a list coloring.
33. First, show that $P_n \,\square\, K_2$ is not 2-choosable for $n \geq 4$ by exhibiting an appropriate list of size 2 for each vertex of $P_n \,\square\, K_2$ that does not produce a list coloring. Show by induction on $n \geq 1$ that $P_n \,\square\, K_2$ is 3-choosable.

Chapter 16 Section 16.2
1. Consider Brazil, Argentina, Bolivia and Paraguay.
3. Use Theorem 10.23 and the fact that there is a planar embedding of every maximal outerplanar graph of order at least 3 such that the boundary of every interior region is a triangle.

Section 16.3
5. Proceed by induction on the order of an outerplanar graph.

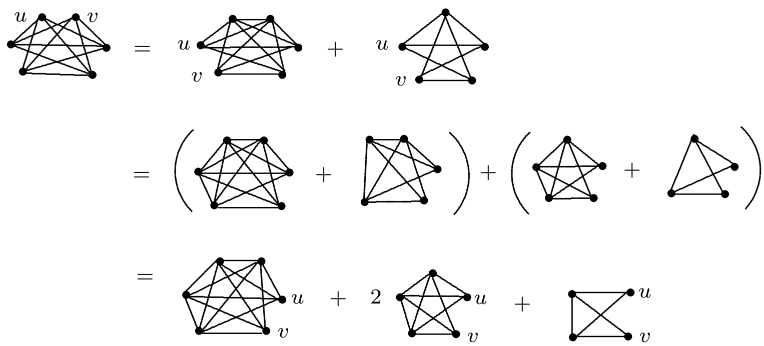
Section 16.4
7. Assume, to the contrary, that $K_{4,4}$ contains a subdivision F of K_5. Suppose that $V(F) = X \cup Y$ such that $\deg_F x = 4$ for each $x \in X$ and $\deg_F y = 2$ for each $y \in Y$.
9. Consider the graph G of order $n = 13$ consisting of three copies G_1, G_2, G_3 of K_3 and two copies G_4, G_5 of K_2 where each vertex of G_i is adjacent to each vertex of G_{i+1} for $i = 1, 2, \ldots, 5$, where $G_6 = G_1$.

Section 16.5

11. Recall that the chromatic polynomial of a graph G of order n can be expressed as

$$P(G,\lambda) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + (-1)^{n+1}a_{n-1}\lambda.$$

13. (a) See the figure below.



$$\begin{aligned} &= (K_6 + K_5) + 2(K_5 + K_4) + (K_4 + K_3) \\ &= K_6 + 3K_5 + 3K_4 + K_3 \end{aligned}$$

(b) $P(K_{2,2,2}, 3) = 6 = \chi(K_{2,2,2})!$

15. (a) Every two unicyclic graphs of the same order and containing a cycle of the same length are chromatically equivalent.

(b) $n - 2$.

17. (a) Assume that the statement is false. Then among all counterexamples, let S be the set of those of minimum order n and let G be a graph in S of minimum size m. Then $P(G,\lambda) = \lambda g(\lambda)$ where $g(0) = 0$. Then G is not a tree. Let $e = uv$ be an edge of G that is not a bridge and let H be the graph obtained by identifying u and v in $G - e$. Consider the coefficients of λ in $P(G,\lambda)$, $P(G - e,\lambda)$ and $P(H,\lambda)$.

(b) Apply Exercise 16 and (a).

19. Proceed by induction on the number of blocks in G and apply Exercise 16.

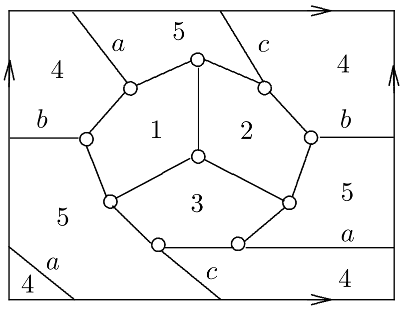
21. Since $K_{r,r}$ is the only bipartite graph of order $2r$ and size r^2, it is chromatically unique.

23. The vertices of G can be listed as $v_1, v_2, \ldots, v_n$ such that v_i $(2 \le i \le n)$ is adjacent to at least one vertex of $\{v_1, v_2, \ldots, v_{i-1}\}$. Consider the number of ways that the vertices of G can be colored using this ordering.

25. The statement is true.

Section 16.6

27. (a) See the figure below. (b) By the Generalized Euler Identity (Theorem 11.10), $r = 5$. (c) Five colors are needed.



29. Consider the graph K_8.

Chapter 17 Section 17.1

1. The chromatic index of this graph is 4.
3. Since $\Delta(G \,\Box\, K_2) = 1 + \Delta(G)$, it suffices to show that there exists a $(1 + \Delta(G))$-edge coloring of $G \,\Box\, K_2$. Because $\chi'(G) \leq 1 + \Delta(G)$, there exists a $(1 + \Delta(G))$-edge coloring of G.
5. (a) Observe that G contains G_1, G_2 and the complete bipartite graph K_{n_1,n_2} as subgraphs.
   (b) Consider $G_1 = K_2$ and $G_2 = C_5$.

Section 17.2

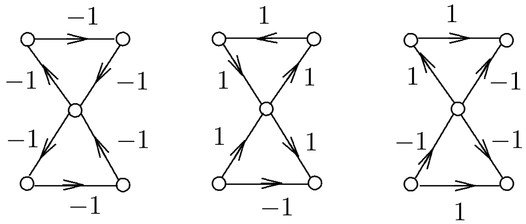
7. Show that G is overfull and apply Corollary 17.9.
9. Since G is 3-regular, it follows that G has even order. Consider a Hamiltonian cycle C of G.
11. The graph of each regular polyhedra is of Class one.
13. Show that there exists a $\Delta(G)$-regular bipartite graph containing G as an induced subgraph.
15. Apply Theorem 17.8.
17. Let $G = C_n \,\Box\, K_2$ where $n \geq 3$ is odd.
19. Apply Theorem 17.8.
21. Apply Corollary 17.9.
23. (a) France has two consecutive days off.
   (b) Player B only plays in one match on Day 4 and all other players play at least two matches every day.
25. The graph G is overfull.

Section 17.3

27. (a) No. (b) Yes.
29. (a) No. (b) No. (c) Yes.
31. (a) $\chi'(G) = 4$. (b) No.

Chapter 18 Section 18.1

1. See the figure below.



(a) D_1 (b) D_2 (c) D_3

3. Show that $\sigma^+(v; a\phi_1 + b\phi_2) = \sigma^-(v; a\phi_1 + b\phi_2)$ for integers a and b and for every vertex v of D.
5. Let G be a bridgeless graph containing an Eulerian trail and let u and v be the two odd vertices of G. Then G is 2-edge-connected and by Theorem 4.18, G contains two edge-disjoint $u - v$ paths P and P'. Begin by assigning flow values to the appropriately directed edges of these paths.
7. Let $V(K_6) = \{1, 2, 3, 4, 5, 6\}$. A cycle double cover of K_6 consisting of five Hamiltonian cycles is (1,2,3,4,5,6,1), (1,3,6,2,5,4,1), (1,4,2,6,5,3,1), (1,5,2,3,4,6,1), (1,2,4,6,3,5,1).

Section 18.2

9. Let $\Delta(G) = \Delta$ and for each edge e of G, let $L(e)$ be a list of $2\Delta - 1$ colors (positive integers). Let $e_1, e_2, \ldots, e_m$ be an ordering of the edges of G and give a greedy coloring of the edges.

Section 18.3

11. $\chi(G) = 3$, $\chi'(G) = 4$ and $\chi''(G) = 5$.
13. $\chi''(G) = 4$.
15. Since $\Delta(K_{s,t}) = t$, it follows that $\chi''(K_{s,t}) \geq t+1$. Describe a $(t+1)$-total coloring of $K_{s,t}$.

Chapter 19 Section 19.1

1. $T_{7,2} = K_{3,4}$, $t_{7,2} = 12$, $T_{7,3} = K_{2,2,3}$, $t_{7,3} = 16$,
$T_{7,4} = K_{1,2,2,2}$, $t_{7,4} = 18$, $T_{8,2} = K_{4,4}$, $t_{8,2} = 16$,
$T_{8,3} = K_{2,3,3}$, $t_{8,3} = 21$, $T_{8,4} = K_{2,2,2,2}$, $t_{8,4} = 24$.
3. $m = 13$.
5. Show that G contains a vertex of degree 2 or more.
7. For $n \geq 4$, the smallest integer m such that every graph of order n and size m contains $K_{1,3}$ as a subgraph is $n + 1$.
9. (a) Let $e = uv$ be an edge of G, let $A = N(u) - \{v\}$ and let $B = N(v) - \{u\}$. First show that $|A \cap B| \geq (n + 2)/3$.
(b) For an integer $r \geq 3$, let $G = K_{r,r,r}$.
11. $m = \lfloor n^2/4 \rfloor + 2$.

Section 19.2

13. Let $g = g(G)$. Consider the cases where g is even or g is odd.
15. Suppose that H is an (s, g)-graph and assume, to the contrary, that H is an (s, g)-cage. Since H contains 4-cycles, either $g = 3$ or $g = 4$.
17. (a) $n(4, 3, 4) = 5$. (b) $n(4, 3, 6) = 8$.

Chapter 20 Section 20.1

1. (a) Consider $H = G \vee K_1$. (b) Consider $F = G + K_1$.
3. To show that $R(s', t') \leq R(s, t)$, let G be a graph of order $R(s, t)$. Then apply the definition of Ramsey number.
5. To show that $R(4, 4) \leq 18$, consider a red-blue coloring of K_{18}. Each vertex of K_{18} is incident with nine edges of the same color.
7. $\min\{t_c\} = 2$.
9. No.

Section 20.2

11. $R(F, H) = 7$.
13. $R(K_3, C_5) = 9$.
15. Consider the cases when n is even and n is odd. Apply Theorems 13.5 and 13.2.
17. (a) Yes. (b) Let $n = R_\chi(s-1, t) + R_\chi(s, t-1)$. Let there be given a red-blue coloring of the complete graph K_n. Show that K_n contains a red subgraph with chromatic number at least s or a blue subgraph with chromatic number at least t.

(c) $R_\chi(3, 3) = 5$. (d) For $3 \leq s \leq t$, $R_\chi(s, t) = 1 + (s-1)(t-1)$.

19. Let $n = R(K_{n_1}, K_{n_2}, \ldots, K_{n_k}, T_s)$. To show that $n \leq 1 + (r-1)(s-1)$, apply Theorem 12.21. To show that $n \geq 1 + (r-1)(s-1)$, exhibit an edge coloring of $K_{(r-1)(s-1)}$ with the colors $1, 2, \ldots, k+1$ such that for each i with $1 \leq i \leq k$ there is no copy of K_{n_i} all of whose edges are colored i and there is no copy of T_s all of whose edges are colored $k+1$.
21. (a) $R(K_{1,3}, K_3) = 7$. (b) $MR(K_{1,3}, K_3) = 6$.
23. $MR(F, H) = 6$.
25. $RR(K_{1,3}, P_4) = 5$.
27. $BR(P_4, P_4) = 3$.

Chapter 21 Section 21.1

1. Follow the proof of Theorem 21.1.
3. Determine a bound on $\log_2 10$.
5. Consider the probability space S whose elements are the $2^{\binom{n}{2}}$ different labeled tournaments T with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. The probabilities are defined by $P[(v_i, v_j) \in E(T)] = P[(v_j, v_1) \in E(T)] = \frac{1}{2}$ for each pair v_i, v_j of distinct vertices of T and these events are then mutually independent. Consider, for each k-element subset $U \subseteq \{v_1, v_2, \ldots, v_n\}$, the probability of the event that no vertex in $V(G) - U$ is adjacent to every vertex of U.

Section 21.2

7. Apply Theorem 21.7 to property $Q_{2,0}$.

9. Let n be a positive integer such that $(n-3)(n-4)/12 > k$. By Theorem 21.9, almost all graphs contain K_n as a subgraph. By Theorem 11.18, $\gamma(K_n) > k$. Therefore, almost all graphs have genus exceeding k and consequently, almost no graphs have genus at most k.

11. Show that if $r(n)$ is a threshold function for some property, then so too is $2r(n)$. What about $cr(n)$ for any constant $c > 0$?

13. Consider the eccentricity sequence of a connected graph with diameter 2.

Bibliography

[1] B. M. Ábrego, S. Fernández-Merchant and G. Salazar, The rectilinear crossing number of K_n: Closing in (or are we?) *Thirty essays on geometric graph theory* 5-18, Springer, New York, 2013. [286]

[2] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdős and O. R. Oellermann, The ascending subgraph decomposition problem. *Congr. Numer.* **58** (1987) 7-14. [345]

[3] N. Alon, Choice numbers of graphs; a probabilistic approach, *Comb. Prop. Comput.* **1** (1992) 107-114. [409]

[4] B. Alspach, Research problems, Problem 3. *Discrete Math.* **36** (1981) 333. [349]

[5] B. Alspach, The wonderful Walecki construction. *Bull. Inst. Combin. Appl.* **52** (2008) 7- 20. [341]

[6] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$. *J. Combin. Theory Ser B.* **81** (2001) 77-99. [348, 350]

[7] V. I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices. (Russian) *Prikl. Mat. i Programmirovanie Vyp.* **11** (1974) 3-8. [325, 550]

[8] I. Anderson, Perfect matchings of a graph. *J. Combin. Theory Ser. B* **10** (1971) 183-186. [311]

[9] K. Appel and W. Haken, Every planar map is four colorable. *Bull. Amer. Math. Soc.* **82** (1976) 711-712. [427]

[10] D. Archdeacon and P. Huneke, A Kuratowski theorem for nonorientable surfaces. *J. Combin. Theory Ser. B* **46** (1989) 173-231. [300]

[11] A. T. Balaban, Trivalent graphs of girth nine and eleven, and relationships among cages. *Rev. Roumaine Math. Pures Appl.* **18** (1973) 1033-1043. [516]

[12] E. Bannai and T. Ito, On finite Moore graphs. *J. Fac. Sci. Univ. Tokyo, Sect. IA* **20** (1973) 191-208. [518]

[13] J. Battle, F. Harary and V. Kodama, Every planar graph with nine points has a nonplanar complement. *Bull. Amer. Math. Soc.* **68** (1962) 569-571. [293]

[14] D. Bauer, H. J. Broersma and H. J. Veldman, Not every 2-tough graph is Hamiltonian. *Discrete Appl. Math.* **99** (2000) 317-321. [138]

[15] M. Behzad, *Graphs and Their Chromatic Numbers.* Ph.D. Thesis, Michigan State University (1965). [496]

[16] M. Behzad, G. Chartrand and J. K. Cooper, Jr., The colour numbers of complete graphs. *J. London Math. Soc.* **42** (1967) 226-228. [497]

[17] L. W. Beineke, Characterizations of derived graphs. *J. Combin. Theory* **9** (1970) 129-135. [149]

[18] L. W. Beineke and F. Harary, The genus of the n-cube. *Canad. J. Math.* **17** (1965) 494-496. [295]

[19] L. W. Beineke and R. D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four. *J. Graph Theory* **4** (1980) 145-155. [282, 283]

[20] C. T. Benson, Minimal regular graphs of girths eight and twelve. *Canad. J. Math.* **18** (1966) 1091-1094. [516]

[21] C. Berge, Two theorems in graph theory. *Proc. Nat. Acad. Sci.* U.S.A. **43** (1957) 842-844. [307]

[22] C. Berge, *Théorie des Graphes et Ses Applications.* Dunod, Paris (1958). [99, 323]

[23] C. Berge, Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris* **247** (1958) 258-259. [317]

[24] C. Berge, Some classes of perfect graphs. *Six Papers on Graph Theory.* Indian Statistical Institute, Calcutta (1963) 1-21. [396]

[25] C. Berge, *Graphs and Hypergraphs.* North-Holland, Amsterdam-London (1973). [20, 326]

[26] H. Bielak and M. M. Syslo, Peripheral vertices in graphs. *Studia Sci. Math. Hungar.* **18** (1983) 269-275. [49]

[27] G. D. Birkhoff, A determinant formula for the number of ways of coloring a map. *Ann. of Math.* **14** (1912) 42-46. [438]

[28] D. Blanuša, Problem cetiriju boja. *Glasnik Mat. Fiz. Astr.* **1** (1946) 31-42. [474]

[29] A. Blass and F. Harary, Properties of almost all graphs and complexes. *J. Graph Theory.* **3** (1979) 225-240. [556, 558]

[30] J. Blažek and M. Koman, A minimal problem concerning complete plane graphs. *Theory of Graphs and its Applications.* Publ. House. Czechoslovak Acad. Sci. Prague (1964) 113-117. [277]

[31] B. Bollobás and E. J. Cockayne, The irredundance number and maximum degree of a graph. *Discrete Math.* **49** (1984) 197-199. [325]

[32] B. Bollobás and A. J. Harris, List-colourings of graphs. *Graphs Combin.* **1** (1985) 115-127. [492]

[33] J. A. Bondy, Pancyclic graphs. I. *J. Combin. Theory Ser. B* **11** (1971) 80-84. [145]

[34] J. A. Bondy and V. Chvátal, A method in graph theory. *Discrete Math.* **15** (1976) 111-136. [130, 131]

[35] O. Borůvka, O jistém Problému minimálním. *Práce Mor. Přírodověd. Spol. v Brně* (*Acta Societ. Scient. Natur. Moravicae*) **3** (1926) 37-58. [81]

[36] R. C. Brigham and R. D. Dutton, A compilation of relations between graph invariants. *Networks* **15** (1985) 73-107. [381]

[37] A. Brodsky, S. Durocher and E. Gethner, Toward the rectilinear crossing number of K_n: new drawings, upper bounds, and asymptotics. *Discrete Math.* **262** (2003) 59–77. [286]

[38] R. L. Brooks, On coloring the nodes of a network. *Proc. Cambridge Philos. Soc.* **37** (1941) 194-197. [378]

[39] D. Bryant, D. Horsley and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths. *Proc. London Math. Soc.* To appear. [349, 350, 491]

[40] F. Buckley, Z. Miller and P. J. Slater, On graphs containing a given graph as center. *J. Graph Theory* **5** (1981) 427-434. [48]

[41] P. Camion, Chemins et circuits hamiltoniens des graphes complets. *C. R. Acad. Sci. Paris* **249** (1959) 2151-2152. [181]

[42] P. A. Catlin, Hajós' graph-coloring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B* **26** (1979) 268-274. [435]

[43] A. Cayley, On the theory of analytical forms called trees. *Philos. Mag.* **13** (1857) 19-30. [64]

[44] A. Cayley, A theorem on trees. *Quart. J. Math.* **23** (1889) 376-378. [73]

[45] G. Chartrand, F. Fujie and P. Zhang On an extension of an observation of Hamilton. *J. Combin. Math. Combin. Comput.* To appear. [142]

[46] G. Chartrand and F. Harary, Graphs with prescribed connectivities, *Theory of Graphs.* Academic Press, New York (1968) 61-63. [96]

[47] G. Chartrand, A. M. Hobbs, H. A. Jung, S. F. Kapoor and C. St. J. A. Nash-Williams, The square of a block is Hamiltonian connected. *J. Combin. Theory Ser. B* **16** (1974) 290-292. [147]

[48] G. Chartrand, H. Jordon and P. Zhang, A cycle decomposition conjecture for Eulerian graphs. *Australas. J. Combin.* **58** (2014) 48-59. [351]

[49] G. Chartrand and J. Mitchem, Graphical theorems of the Nordhaus-Gaddum class. *Recent Trends in Graph Theory.* Springer, Berlin (1971) 55-61. [380]

[50] G. Chartrand, A. D. Polimeni and M. J. Stewart, The existence of 1-factors in line graphs, squares, and total graphs. *Nedrl. Akad. Wetensch. Proc. Ser. A* 76 *Indag. Math.* **35** (1973) 228-232. [120]

[51] G. Chartrand and C. E. Wall, On the Hamiltonian index of a graph. *Studia Sci. Math. Hungar.* **8** (1973) 43-48. [151]

[52] A. G. Chetwynd and A. J. W. Hilton, Star multigraphs with three vertices of maximum degree. *Math. Proc. Cambridge Philos. Soc.* **100** (1986) 303-317. [339, 466]

[53] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem. *Ann. of Math.* **164** (2006) 51-229. [405]

[54] V. Chvátal, On Hamilton's ideals. *J. Combin. Theory Ser. B* **12** (1972) 163-168. [131]

[55] V. Chvátal, Tough graphs and Hamiltonian circuits. *Discrete Math.* **5** (1973) 215-228. [136, 138]

[56] V. Chvátal, A combinatorial theorem in plane geometry. *J. Combin. Theory Ser. B* **18** (1975) 39-41. [285]

[57] V. Chvátal, Tree-complete graph Ramsey numbers. *J. Graph Theory* **1** (1977) 93. [534]

[58] V. Chvátal and P. Erdős, A note on hamiltonian circuits. *Discrete Math.* **2** (1972) 111-113. [133]

[59] A. Clebsch, Über die Flächen vierter Ordnung, welche eine Doppelcurve zweiten Grades besitzen. *J. für Math.* **69** (1868) 142-184. [532]

[60] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs. *Networks* **7** (1977) 247-261. [323]

[61] S. Cook, The complexity of theorem proving procedures. *Proceedings of the Third Annual ACM Symposium on Theory of Computing* (1971) 151-158. [140]

[62] R. Courant and H. Robbins, *What is Mathematics*? Oxford University Press, London (1941). [446]

[63] B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown, Proof of the 1-factorization and Hamiltonian Decomposition Conjectures. *Memoirs of the AMS.* To appear. [339, 343]

[64] R. M. Damerell, On Moore graphs. *Proc. Cambridge Philos. Soc.* **74** (1973) 227-236. [518]

[65] C. F. de Jaenisch, *Applications de l'Analyse Mathematique au Jeu des Echecs.* Petrograd (1862). [322]

[66] B. Descartes, A three colour problem. *Eureka* **9** (1947) 21. [394]

[67] B. Descartes, Network-colouring. *Math. Gazette* **32** (1948) 67-69. [474]

[68] B. Descartes, The expanding unicurse. *Proof Techniques in Graph Theory* (F. Harary, ed.). Academic Press, New York (1969) 25. [199]

[69] E. A. Dinic, An algorithm for the solution of the problem of maximal flow in a network with power estimation. *Soviet Math. Dokl.* **11** (1970) 1277-1280. [201]

[70] G. A. Dirac, Some theorems on abstract graphs. *Proc. London Math. Soc.* **2** (1952) 69-81. [129]

[71] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.* **27** (1952) 85-92. [434]

[72] G. A. Dirac, In abstrakten Graphen vonhandene vollständige 4-Graphen und ihre Unterteilungen. *Math. Nachr.* **22** (1960) 61-85. [107]

[73] G. A. Dirac, On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg* **25** (1961) 71-76. [400]

[74] J. Edmonds and R. A. Karp, Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of ACM.* **19** (1972) 248-264. [201, 202]

[75] E. Egerv́ary, On combinatorial properties of matrices (Hungarian). *Mat. Lapok* **38** (1931) 16-28. [320]

[76] R. B. Eggleton and R. K. Guy, The crossing number of the n-cube. *Notices Amer. Math. Soc.* **17** (1970) 757. [281]

[77] P. Elias, A. Feinstein and C. E. Shannon A note on the maximum flow through a network. *IRE Trans. on Inform. Theory* **IT 2** (1956) 117-119. [199]

[78] P. Erdős, Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.* **53** (1947) 292-294. [530]

[79] P. Erdős, Graph theory and probability II. *Canad. J. Math* **13** (1961) 346-352. [396, 551]

[80] P. Erdős, Extremal problems in graph theory. *A Seminar on Graph Theory.* Holt, Rinehart and Winston, New York (1967) 54-59. [505]

[81] P. Erdös and T. Gallai, Graphs with prescribed degrees of vertices (Hungarian). *Mat. Lapok* **11** (1960) 264-274. [23]

[82] P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs. *Congr. Numer.* **126** (1980) 125-157. [406, 428]

[83] P. Erdös and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. (German) *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **12** (1963) 251-257. [511]

[84] P. Erdös and G. Szekeres, A combinatorial problem in geometry. *Compositio Math.* **2** (1935) 463-470. [526]

[85] A. Errera, Une demonstration du theoreme de Petersen. *Mathesis* **36** (1922) 56-61. [315]

[86] L. Euler, Solutio problematis ad geometriam situs pertinentis. *Comment. Academiae Sci. I. Petropolitanae* **8** (1736) 128-140. [6, 115, 118]

[87] L. Euler, Elementa doctrinae solidorum. *Novi Comm. Acad. Sci. Imp. Petropol.* **4** (1752-1753) 109-140. [240]

[88] L. Euler, Demonstratio nonnullarum insignium proprietatum quibas solida hedris planis inclusa sunt praedita. *Novi Comm. Acad. Sci. Imp. Petropol.* **4** (1752-1753) 140-160. [240]

[89] I. Fáry, On straight line representation of planar graphs. *Acta Univ. Szeged Sect. Sci. Math.* **11** (1948) 229-233. [283]

[90] H. J. Finck, On the chromatic numbers of a graph and its complement. *Theory of Graphs.* Academic Press, New York (1968) 99-113. [381]

[91] S. Fisk, A short proof of Chvátal's watchman theorem. *J. Combin. Theory Ser. B* **24** (1978) 374. [285]

[92] H. Fleischner, The square of every two-connected graph is Hamiltonian. *J. Combin. Theory Ser. B* **16** (1974) 29-34. [147]

[93] L. R. Ford, Jr. and D. R. Fulkerson, Maximal flow through a network. *Canad. J. Math.* **8** (1956) 399-404. [198, 199]

[94] L. R. Ford, Jr. and D. R. Fulkerson, A simple algorithm for finding maximal network flows and an application to the Hitchcock problem. *Canad. J. Math.* **9** (1957) 210-218. [200]

[95] P. Franklin, The four color problem. *Amer. J. Math.* **44** (1922) 225-236. [252]

[96] O. Frink and P. A. Smith, Irreducible non-planar graphs. *Bull. Amer. Math. Soc.* **36** (1930) 214. [254]

[97] R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Gruppe. *Compositio Math.* **6** (1938) 239-250. [227]

[98] T. Gallai, Maximum-minimum Sätze über Graphen. *Acta Math. Acad. Sci. Hungar.* **9** (1958) 395-434. [396]

[99] T. Gallai, Über extreme Punkt- und Kantenmengen. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **2** (1959) 133-138. [319]

[100] T. Gallai, On directed paths and circuits. *Theory of Graphs: Proceedings of the Colloquium held at Tihany, Hungary*, 1969 (P. Erdös and G. Katona, eds). Academic Press, New York (1969) 115-118. [383]

[101] F. Galvin, The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B* **63** (1995) 153-158. [493]

[102] D. P. Geller, Problem 5713. *Amer. Math. Monthly* **77** (1970) 85. [379]

[103] A. Georgakopoulos, A short proof of Fleischner's theorem. *Discrete Math.* **309** (2009) 6632-6634. [147]

[104] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit Hamiltonien. *C. R. Acad. Sci. Paris* **251** (1960) 495-497. [169]

[105] E. N. Gilbert, Random graphs. *Ann. Math. Stat.* **30** (1959) 11411144. [556]

[106] S. W. Golomb, How to number a graph. *Graph Theory and Computing.* Academic Press, New York (1972) 23-37. [351, 356]

[107] R. L. Graham and P. Hell, On the history of the minimum spanning tree problem. *Annals of the History of Computing* **7** (1985) 43-57. [81]

[108] R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs. *Canad. J. Math.* **7** (1955) 1-7. [532]

[109] E. J. Grinberg, Plane homogeneous graphs of degree three without hamiltonian circuits. *Latvian Math. Yearbook* **4** (1968) 51-58. [265]

[110] R. P. Gupta, The chromatic index and the degree of a graph. *Notices Amer. Math. Soc.* **13** (1966) 719. [455, 459]

[111] R. K. Guy, A combinatorial problem. *Bull. Malayan Math. Soc.* **7** (1960) 68-72. [277]

[112] R. K Guy, Crossing numbers of graphs. *Graph Theory and Applications.* Springer-Verlag, New York (1972) 111-124. [276, 278, 286]

[113] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich* **88** (1943) 133-142. [437]

[114] R. Haggkvist and A. G. Chetwynd, Some upper bounds on the total and list chromatic numbers of multigraphs. *J. Graph Theory* **16** (1992) 503-516. [492]

[115] A. Hajnal and J. Surányi, Über die Auflösung von Graphen in vollständige Teilgraphen. *Ann. Univ. Sci. Budapest Eötvös. Sect. Math.* **1** (1958) 113-121. [400]

[116] G. Hajós, Über eine Konstruktion nicht n-färbbarer Graphen. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg. Math.-Nat. Reihe.* **10** (1961) 116-117. [435]

[117] S. L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph. *SIAM J. Appl. Math.* **10** (1962) 496-506. [21]

[118] P. Hall, On representation of subsets. *J. London Math. Soc.* **10** (1935) 26-30. [308]

[119] F. Harary, The maximum connectivity of a graph. *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962) 1142-1146. [100]

[120] F. Harary, P. C. Kainen and A. J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers. *Nanta Math.* **6** (1973) 58-67. [281]

[121] F. Harary and L. Moser, The theory of round robin tournaments. *Amer. Math. Monthly* **73** (1966) 231-246. [178, 183]

[122] F. Harary and C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs. *Canad. Math. Bull.* **8** (1965) 701-709. [149]

[123] F. Harary and R. Z. Norman, The dissimilarity characteristic of Husimi trees. *Ann. of Math.* **58** (1953) 134-141. [61]

[124] V. Havel, A remark on the existence of finite graphs (Czech.) *Časopis Pěst. Mat.* **80** (1955) 477-480. [21]

[125] P. J. Heawood, Map colour theorems. *Quart. J. Math.* **24** (1890) 332-338. [422, 426, 444, 445]

[126] P. J. Heawood, On the four-colour map theorem. *Quart. J. Pure Appl. Math.* **29** (1898) 270-285. [427]

[127] L. Heffter, Über das Problem der Nachbargebiete. *Math. Ann.* **38** (1891) 477-508. [446]

[128] C. Hierholzer, Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechnung zu umfahren. *Math. Ann.* **6** (1873) 30-32. [120]

[129] A. J. Hoffman and R. R. Singleton, On Moore graphs with diameter 2 and 3. *IBM J. Res. Develop.* **4** (1960) 497-504. [516, 518]

[130] I. Holyer, The NP-completeness of edge colouring. *SIAM J. Comput.* **4** (1981) 718-720. [460]

[131] J. Hopcroft and R. E. Tarjan, Efficient planarity testing. *J. Assoc. Comput. Mach.* **21** (1974) 549-568. [257]

[132] R. Isaacs, Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly* **82** (1975) 221-239. [474]

[133] H. Izbicki, Reguläre Graphen beliebigen Grades mit vorgegebenen Eigenschaften. *Monatsh. Math.* **64** (1960) 15-21. [228]

[134] B. Jackson, Hamilton cycles in regular 2-connected graphs. *J. Combin. Theory Ser. B* **29** (1980) 27-46. [132]

[135] F. Jaeger, A survey of the cycle double cover conjecture. *Cycles in Graphs.* North-Holland, Amsterdan (1985) 1-12. [491]

[136] V. Jarník, O jistém problému min imálním. (Czech) *Acta Societ. Scient. Natur. Moravicae* **6** (1930) 57-63. [83]

[137] T. Jensen and G. F. Royle, Small graphs with chromatic number 5: a computer search. *J. Graph Theory* **19** (1995) 107-116. [395]

[138] J. J. Karaganis, On the cube of a graph. *Canad. Math. Bull.* **11** (1968) 295-296. [146]

[139] R. M. Karp, Reducibility among combinatorial problems, in *Complexity Computer Computations* (R. E. Miller and J. W. Thatcher, eds.). Plenum, New York (1972) 85-103. [140]

[140] F. Kárteszi, Piani finiti ciclici come risoluzioni di un certo problema di minimo. *Boll. Un. Mat. Ital.* **15** (1960) 522-528. [510]

[141] J. B. Kelly and L. M. Kelly, Paths and circuits in critical graphs. *Amer. J. Math.* **76** (1954) 786-792. [394]

[142] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme gefürht wird. *Ann. Phys. Chem.* **72** (1847) 497-508. [64, 77]

[143] T. P. Kirkman, On a problem in combinatorics. *Cambridge and Dublin Math. J.* **2** (1847) 191-204. [347]

[144] D. J. Kleitman, The crossing number of $K_{5,n}$. *J. Combin. Theory* **9** (1970) 315-323. [279]

[145] W. L. Kocay, On Stockmeyer's nonreconstructible tournaments. *J. Graph Theory* **9** (1985) 473-476. [230]

[146] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.* **77** (1916) 453-465. [336, 461]

[147] D. König, Graphen und Matrizen. *Math. Lapok.* **38** (1931) 116-119. [416]

[148] D. König, *Theorie der endlichen und unendliehen Graphen.* Akademische Verlagsgesellschaft, Leipzig (1936). [103, 226]

[149] A. D. Korshunou, A solution of a problem of P. Erdős and A. Rényi about Hamiltonian cycles in non-oriented graphs, Metody Diskr. Anal. Teoriy, Upr. Syst., Sb. Trudov Novosibirsk. **31** (1977) 17-56 (in Russian). [560]

[150] J. B. Kruskal, On the shortest spanning tree of a graph and the traveling salesmen problem. *Proc. Amer. Math. Soc.* **7** (1956) 48-50. [82]

[151] K. Kuratowski, Sur le problème des courbes gauches en topologie. *Fund. Math.* **15** (1930) 271-283. [254]

[152] H. G. Landau, On dominance relations and structure of animal societies. III. The condition for a score structure. *Bull. Math. Biophys.* **15** (1953) 143-148. [176, 179]

[153] A.-M. Legendre, Élements de Géométrie. F. Didot, Paris (1794). [240]

[154] L. Lesniak, Results on the edge-connectivity of graphs. *Discrete Math.* **8** (1974) 351-354. [102]

[155] L. Lesniak, Eccentric sequences in graphs. *Period. Math. Hungar.* **6** (1975) 287-293. [75]

[156] S. A. J. Lhuilier, Mémoir sur la polyédrométrie, contenant une démonstration directe du théorème d'Euler sur les polyèdres, et un examen de diverses exceptions auxquelles ce théorème est assujetti. *Ann. Math. Pures et Appliquées* **3** (1812–1813) 169-189. [290]

[157] D. R. Lick, Characterizations of n-connected and n-line-connected graphs. *J. Combin. Theory Ser. B* **14** (1973) 122-124. [107]

[158] L. Lovász, On chromatic number of finite set-systems. *Acta. Math. Acad. Sci. Hungar.* **79** (1967) 59-67. [396, 405, 551]

[159] L. Lovász, A characterization of perfect graphs. *J. Combin. Theory Ser. B* **13** (1972) 95-98. [402]

[160] L. Lovász, Normal hypergraphs and the perfect graph conjecture. *Discrete Math.* **2** (1972) 253-267. [402]

[161] W. Mantel, Problem 28. *Wiskundige Opgaven* **10** (1907) 60-61. [15, 503]

[162] B. Manvel, On Reconstruction of Graphs. Doctoral thesis, Univerity of Michigan (1970). [232]

[163] M. M. Matthews and D. P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs. *J. Graph Theory* **8** (1984) 139-146. [137, 157]

[164] D. W. Matula, The cohesive strength of graphs. *The Many Facets of Graph Theory.* Springer, Berlin (1969) 215-221. [373]

[165] W. F. McGee, A minimal cubic graph of girth seven. *Canad. Math. Bull.* **3** (1960) 149-152. [514]

[166] B. D. McKay, Computer reconstruction of small graphs. *J. Graph Theory* **1** (1977) 281-283. [229]

[167] B. D. McKay, W. Myrvold and J. Nadon, Fast backtracking principles applied to find new cages. *Proceedings of the Ninth Annual ACM-SIAM Symposium on Discrete Algorithms.* ACM, New York (1998) 188-191. [516]

[168] T. A. McKee, Recharacterizing Eulerian: intimations of new duality. *Discrete Math.* **51** (1984) 237-242. [122]

[169] K. Menger, Zur allgemeinen Kurventheorie. *Fund. Math.* **10** (1927) 95-115. [102]

[170] M. Meyniel, Une condition suffisante d'existence d'un circuit Hamiltonien dans un graphe oriente. *J. Combin. Theory Ser. B* **14** (1973) 137-147. [168]

[171] M. Mirzakhani, A small non-4-choosable planar graph. *Bull. Inst. Combin. Appl.* **17** (1996) 15-18. [430]

[172] M. Molloy and B. Reed, A bound on the total chromatic number. *Combinatorica* **18** (1998) 241-280. [498]

[173] J. W. Moon, On subtournaments of a tournament. *Canad. Math. Bull.* **9** (1966) 297- 301. [183]

[174] J. Mycielski, Sur le coloriage des graphes. *Colloq. Math.* **3** (1955) 161-162. [394]

[175] C. St. J. A. Nash-Williams, Edge-disjoint Hamiltonian circuits in graphs with the vertices of large valency, in *Studies in Pure Mathematics.* Academic Press (1971) 157-183. [343]

[176] C. St. J. A. Nash-Williams, Hamiltonian arcs and circuits, in *Recent Trends in Graph Theory.* Springer Verlag (1971) 197-210. [343]

[177] A. Nijenhuis, Note on the unique determination of graphs by proper subgraphs. *Notices Amer. Math. Soc.* **24** (1977) A-290. [229]

[178] E. A. Nordhaus and J. W. Gaddum, On complementary graphs. *Amer. Math. Monthly* **63** (1956) 175-177. [380]

[179] O. Ore, Note on Hamilton circuits. *Amer. Math. Monthly* **67** (1960) 55. [129]

[180] O. Ore, *Theory of Graphs.* Amer. Math. Soc. Colloq. Pub., Providence, RI (1962). [323, 324]

[181] O. Ore, Hamilton connected graphs. *J. Math. Pures Appl.* **42** (1963) 21-27. [140]

[182] S. Pan and R. B. Richter, The crossing number of K_{11} is 100. *J. Graph Theory* **56** (2007) 128-134. [278]

[183] C. Payan, Sur le nombre d'absorption d'un graphe simple. (French) Colloque sur la Théorie des Graphes (Paris, 1974). Cahiers Centre Études Recherche Opér. **17** (1975) 307-317. [325, 550]

[184] C. Payan and N. H. Xuong, Domination-balanced graphs. *J. Graph Theory* **6** (1982) 23-32. [325]

[185] E. Pegg, Review of *The Colossal Book of Mathematics* by Martin Gardner. *Notices of the AMS.* **49** (2002) 1085-1086. [475]

[186] J. Petersen, Die Theorie der regulären Graphen. *Acta Math.* **15** (1891) 193-220. [13, 313, 339]

[187] J. Petersen, Sur le théoréme de Tait. *L' Intermédiaire des Mathematiciens* **5** (1898) 225-227. [314, 336]

[188] J. Plesník, Critical graphs of given diameter. *Acta Fac. Rerum Natur. Univ. Comenian. Math.* **30** (1975) 71-93. [101]

[189] L. Pósa, Hamiltonian circuits in random graphs. *Discrete Math.* **14** (1976) 359-364. [560]

[190] R. C. Prim, Shortest connection networks and some generalizations. *Bell Syst. Tech. J.* **36** (1957) 1389-1401. [83]

[191] H. Prüfer, Neuer Beweis eines Satzes über Permutationen. *Arch. Math. Phys.* **27** (1918) 142-144. [69]

[192] R. Rado, A theorem on independence relations. *Quart. J. Math. Oxford Ser.* **13** (1942) 83-89. [321]

[193] F. P. Ramsey, On a problem of formal logic. *Proc. London Math. Soc.* **30** (1930) 264-286. [523]

[194] F. P. Ramsey, *The Foundations of Mathematics, and Other Logical Essays.* Harcourt, Brace and Company, New York (1931). [523]

[195] D. K. Ray-Chaudhuri and R. M. Wilson, Solution of Kirkman's schoolgirl problem. *Combinatorics* (*Proc. Sympos. Pure Math.*, Vol. XIX, Univ. California, Los Angeles, Calif., 1968), Amer. Math. Soc., Providence, RI. (1971) 187-203. [348]

[196] R. C. Read, An introduction to chromatic polynomials. *J. Combin. Theory* **4** (1968) 52-71. [438, 440]

[197] L. Rédei, Ein kombinatorischer Satz. *Acta Litt. Szeged* **7** (1934) 39-43. [180, 181]

[198] B. Reed, ω, Δ, and χ. *J. Graph Theory* **27** (1998) 177-212. [374, 381]

[199] S. Řiha, A new proof of the theorem by Fleischner. *J. Combin. Theory Ser. B* **52** (1991) 117-123. [147]

[200] R. D. Ringeisen and L. W Beineke, The crossing number of $C_3 \times C_n$. *J. Combin. Theory Ser. B* **24** (1978) 134-136. [281]

[201] G. Ringel, Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter. *Abh. Math. Sem. Univ. Hamburg* **20** (1955) 10-19. [295]

[202] G. Ringel, Problem 25. *Theory of Graphs and its Applications.* Nakl. ČSAV, Prague (1964) 162. [353]

[203] G. Ringel, Das Geschlecht des vollständigen paaren Graphen. *Abh. Math. Sem. Univ. Hamburg* **28** (1965) 139-150. [294]

[204] G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. USA* **60** (1968) 438-445. [294, 447]

[205] H. E. Robbins, A theorem on graphs, with an application to a problem in traffic control. *Amer. Math. Monthly* **46** (1939) 281-283. [166]

[206] N. Robertson and P. D. Seymour, Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B* **92** (2004) 325-357. [300]

[207] N. Robertson, P. Seymour and R. Thomas, Hadwiger's conjecture for K_6-free graphs. *Combinatorica* **13** (1993) 279-361. [437]

[208] N. Robertson, P. D. Seymour and R. Thomas, Tutte's edge-colouring conjecture. *J. Combin. Theory Ser. B* **70** (1997) 166-183. [489]

[209] N. Robertson, P. D. Seymour and R. Thomas, Excluded minors in cubic graphs. arXiv:1403.2118v1. [489]

[210] N. Robertson, P. D. Seymour and R. Thomas, Cyclically 5-connected cubic graphs. arXiv:1503.02298. [489]

[211] A. Rosa, On certain valuations of the vertices of a graph. *Theory of Graphs.* Gordon and Breach, New York (1967) 349-355. [351, 352, 354, 355]

[212] B. Roy, Nombre chromatique et plus longs chemins d'un graph. *Rev AFIRO* **1** (1967) 127-132. [383]

[213] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles. *J. Combin. Des.* **10** (2002) 27-78. [348, 350]

[214] D. P. Sanders, P. D. Seymour and R. Thomas, Edge 3-coloring cubic doublecross graphs. Preprint. [489]

[215] D. P. Sanders and R. Thomas, Edge 3-coloring cubic apex graphs. Preprint. [489]

[216] D. P. Sanders and Y. Zhao, Planar graphs of maximum degree seven are class I. *J. Combin. Theory Ser. B* **83** (2001) 201-212. [473]

[217] S. Schuster, Interpolation theorem for the number of end-vertices of spanning trees. *J. Graph Theory* **7** (1983) 203-208. [75]

[218] M. Sekanina, On an ordering of the set of vertices of a connected graph. *Spisy Přío̒d Fak. Univ. Brno.* (1960) 137-141. [146]

[219] P. D. Seymour, Sums of circuits. *Graph Theory and Related Topics.* Academic Press, New York (1979) 341-355. [491]

[220] P. D. Seymour, Nowhere-zero 6-flows. *J. Combin. Theory Ser. B* **30** (1981) 130-135. [489]

[221] C. E. Shannon, A theorem on coloring the lines of a network. *J. Math. Physics* **28** (1949) 148-151. [459]

[222] T. Slivnik, Short proof of Galvin's theorem on the list-chromatic index of a bipartite multigraph. *Combin. Probab. Comput.* **5** (1996) 91-94. [493]

[223] J. Spencer, Ramsey's theorem - a new lower bound. *J. Combin. Theory* **18** A (1975) 108-115. [547]

[224] J. Spencer, Ten Lectures on the Probabilistic Method, 2ed. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM 1994. [547, 560]

[225] S. K. Stein, Convex maps. *Proc. Amer. Math. Soc.* **2** (1951) 464-466. [283]

[226] B. M. Stewart, On a theorem of Nordhaus and Gaddum. *J. Combin. Theory* **6** (1969) 217-218. [381]

[227] P. K. Stockmeyer, The falsity of the reconstruction conjecture for tournaments. *J. Graph Theory* **1** (1977) 19-25. [230]

[228] P. K. Stockmeyer, Erratum to: "The falsity of the reconstruction conjecture for tournaments." *J. Graph Theory* **62** (2009) 199-200. [230]

[229] G. Szekeres, Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.* **8** (1973) 367-387. [474, 491]

[230] G. Szekeres and H. S. Wilf, An inequality for the chromatic number of a graph. *J. Combin. Theory* **4** (1968) 1-3. [377]

[231] T. Szele, Kombinatorische Untersuchungen über gerichtete vollständige Graphen. *Mat. Fiz. Lapok* **50** (1943) 223-256. [181, 549]

[232] P. G. Tait, Remarks on the colouring of maps. *Proc. Royal Soc. Edinburgh* **10** (1880) 729. [468]

[233] C. Thomassen, Landau's characterization of tournament score sequences, in *The Theory and Applications of Graphs*, Wiley, New York (1981) 589-591. [176]

[234] C. Thomassen, Every planar graph is 5-choosable. *J. Combin. Theory Ser. B* **62** (1994) 180-181. [428]

[235] C. Thomassen, Some remarks on Hajós' conjecture. *J. Combin. Theory Ser. B* **93** (2005) 95-105. [436]

[236] S. Toida, Properties of an Euler graph. *J. Franklin Inst.* **295** (1973) 343-345. [122]

[237] P. Turán, Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** (1941) 436-452. [504]

[238] P. Turán, A note of welcome. *J. Graph Theory* **1** (1977) 7-9. [278]

[239] W. T. Tutte, On hamiltonian circuits. *J. London Math. Soc.* **21** (1946) 98-101. [266]

[240] W. T. Tutte, A family of cubical graphs. *Proc. Cambridge Philos. Soc.* **43** (1947) 459-474. [510, 514]

[241] W. T. Tutte, On the imbedding of linear graphs in surfaces. *Proc. London Math. Soc.* **51** (1949) 474-483. [489]

[242] W. T. Tutte, A short proof of the factor theorem for finite graphs. *Canad. J. Math.* **6** (1954) 347-352. [311]

[243] W. T. Tutte, A theorem of planar graphs. *Trans. Amer. Math. Soc.* **82** (1956) 99-116. [266]

[244] O. Veblen, An application of modular equations in analysis situs. *Ann. of Math.* **14** (1912) 86-94. [121]

[245] L. M. Vitaver, Determination of minimal coloring of vertices of a graph by means of Boolean powers of the incidence matrix. (Russian) *Dokl. Akad. Nauk. SSSR* **147** (1962) 758-759. [383]

[246] V. G. Vizing, On an estimate of the chromatic class of a p-graph. (Russian) *Diskret. Analiz.* **3** (1964) 25-30. [455, 459, 496]

[247] V. G. Vizing, Critical graphs with given chromatic class. *Metody Diskret. Analiz.* **5** (1965) 9-17. [473]

[248] V. G. Vizing, Coloring the vertices of a graph in prescribed colors (Russian) *Diskret. Analiz* **29** (1976) 3-10. [406, 428]

[249] M. Voigt, List colourings of planar graphs. *Discrete Math.* **120** (1993) 215-219. [428]

[250] K. Wagner, Bemerkungen zum Vierfarbenproblem. *Jber. Deutsch. Math-Verein.* **46** (1936) 26-32. [283]

[251] K. Wagner, Üer eine Eigenschaft der ebene Komplexe. *Math. Ann.* **114** (1937) 570-590. [260, 437]

[252] D. J. A. Welsh and M. B. Powell, An upper bound for the chromatic number of a graph and its application to timetabling problems. *Computer J.* **10** (1967) 85-86. [377]

[253] P. Wernicke, Über den kartographischen Vierfarbensatz. *Math. Ann.* **58** (1904) 413-426. [250]

[254] A. T. White, *Graphs of Groups on Surfaces, Interactions and Models.* North-Holland, Amsterdam (2001). [280]

[255] H. Whitney, Congruent graphs and the connectivity of graphs. *Amer. J. Math.* **54** (1932) 150-168. [98, 106, 149]

[256] H. Whitney, The coloring of graphs. *Ann. of Math.* **33** (1932) 688-718. [438, 441]

[257] J. E. Williamson, Panconnected graphs. II. *Period. Math. Hungar.* **8** (1977) 105-116. [144]

[258] R. M. Wilson, Decompositions of complete graphs into subgraphs isomorphic to a given graph. *Congr. Numer.* **15** (1976) 647-659. [345]

[259] D. R. Woodall, Sufficient conditions for circuits in graphs. *Proc. London Math. Soc.* **24** (1972) 739-755. [169]

[260] D. R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing-number conjecture. *J. Graph Theory* **17** (1993) 657-671. [279]

[261] J. W. T. Youngs, Minimal imbeddings and the genus of a graph. *J. Math. Mech.* **12** (1963) 303-315. [292]

[262] K. Zarankiewicz, On a problem of P. Turán concerning graphs. *Fund. Math.* **41** (1954) 137-145. [279]

[263] A. A. Zykov, On some properties of linear complexes (Russian). *Mat. Sbornik N. S.* **24** (1949) 163-188. [394, 440]

Supplemental References

- N. Alon and J. H. Spencer, *The Probabilistic Method, Third Edition.* Wiley, New York (2008).
- A. Benjamin, G. Chartrand and P. Zhang, *The Fascinating World of Graph Theory.* Princeton University Press, Princeton, NJ (2015).
- N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory*, 1736–1936. Oxford University Press, London (1976).
- J. A. Bondy and U. S. R. Murty, *Graph Theory.* Springer, New York (2008).
- G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton, FL (2009).
- R. Diestel, *Graph Theory, Third Edition.* Springer-Verlag, Berlin (2005).
- J. Gross, J. Yellen and P. Zhang (editors), *Handbook of Graph Theory, Second Edition.* CRC Press, Boca Raton, FL (2014).
- D. B. West, *Introduction to Graph Theory, Second Edition.* Prentice-Hall, Upper Saddle River, NJ (2001).
- R. Wilson, *Four Colors Suffice: How the Map Problem Was Solved.* Princeton University Press, Princeton, NJ (2002).

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